2. *Closure in General Lattices.*

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1. *Introduction.* The concept "closure" was axiomatized by F. Riesz and Kuratowski[1] on the field of sets, and Terasaka[2] generalized it onto abstract Boolean algebras. The object of this note is to extend it onto general lattices. Incidentally "combinations of topologies" of G. Birkhoff[3] are treated from more general point of view.

By "closure" we mean a transformation α on a complete lattice L into itself, which satisfies

[*1.1*] 1° $a \leqq a\alpha$, 2° $(a \cup b)\alpha = a\alpha \cup b\alpha$,
3° $(a\alpha)\alpha = a\alpha$, 4° $0\alpha = 0$.

From this definition we can easily get

(1.2) 1° $a \leqq b$ *implies* $a\alpha \leqq b\alpha$,
2° $(a \cap b)\alpha \leqq a\alpha \cap b\alpha$.

As usual we define closedness of a by $a\alpha = a$, and denote the set of closed elements by C or C_α. Then[4]

(1.3) *A closure* α *determines a meet-complete sublattice* C *of* L.

Proof is trivial. If a closure α determines a meet-complete sublattice C, then we denote it by $\alpha \rightarrow C$.

2. *Meet-complete sublattices.* Conversely, if a meet-complete sublattice C which contains 0 and 1 is given and define a transformation β on L as

$$(2.1) \qquad a\beta = \bigwedge (x;\ x \geqq a,\ x \in C),$$

then β satisfies evidently 1°, 3°, 4° of [1.1] and furthermore we can prove β is a join-homomorphism: by (2.1) $a\beta \cup b\beta \geqq a \cup b$ implies $(a \cup b)\beta \leqq a\beta \cup b\beta$, and conversely $a, b \leqq (a \cup b)\beta$ implies $a\beta \cup b\beta \leqq (a \cup b)\beta$. Hence

(2.2) *Any meet-complete sublattice* C *determines a closure* β.

We denote this fact by $C \rightarrow \beta$. If $\alpha \rightarrow C$ and $C \rightarrow \beta$, then by (2.1) $a\alpha \geqq a\beta$, conversely $a \leqq a\beta$ implies $a\alpha \leqq a\beta\alpha = a\beta$ (because $a \in C$), hence if we denote by Γ the set of all closures on L, and by $\sum$ the set of all meet-complete sublattices which contain 0 and 1, then

(2.3) *There exists a one-to-one correspondence between* Γ *and* $\sum$.

3. *Combinations of topologies.* Now, let us define join and meet of two closures following G. Birkhoff's method. We assume C_α and

1) c. f. Kuratowski, *Topologie* I, Warsaw, 1933.
2) Terasaka, Theorie der topologischen Verbände, Proc. **13** (1937).
3) G. Birkhoff, On the combinations of topologies, Fund. Math., **26** (1936).
4) *A* lattice C is called *meet-complete*, if unrestricted meet operation is defined in C.

C_β are meet-complete sublattices containing 0 and 1, and combine them by

(1) $C_\alpha \frown C_\beta$ is meant set-theoretic product, and

(2) $C_\alpha \smile C_\beta$ is a sublattice generated by C_α and C_β.

Then we can evidently get

(3.1) *$\sum$ forms a lattice.*

If $C_\alpha \to \alpha$ and $C_\beta \to \beta$, then we define

$$(1)\quad C_\alpha \frown C_\beta \to \alpha \smile \beta\,, \qquad (2)\quad C_\alpha \smile C_\beta \to \alpha \frown \beta\,.$$

From this definition, (2.3) and (3.1), we can conclude evidently

(3.2) *Γ is a lattice and is dual isomorphic with $\sum$.*

When we define partially ordering by the terms of lattice operations, then we can conclude

$$(3.3)\quad \alpha \leqq \beta \textit{ if and only if } x\alpha \leqq x\beta\,.$$