16. Boundary Values of Analytic Functions.

By Shunji KAMETANI.

Taga Higher Technical School, Ibaragi.

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I. Given a rectifiable simple Jordan curve C with length c in the Gaussian plane by the equation $t=t(s)=\xi(s)+i\eta(s)$ (ξ, η being real) where the parameter s is the arc length measured from a certain fixed point t_0 to t in the positive sense along C, so that it varies in the interval $[0, c]$, the functions $\xi(s)$ and $\eta(s)$ or $t(s)$, where we have $t(0)=t(c)$, satisfying Lipschitz condition on $[0, c]$, are absolutely continuous, and at almost every point s of that interval, have derivatives such that $\big(\xi'(s)\big)^2+\big(\eta'(s)\big)^2=|t'(s)|^2=1$.

Suppose that $f(t)=f\big(\xi(s)+i\eta(s)\big)=f\big(t(s)\big)$ is a measurable function of s defined on C. Then, by the Lebesgue integrals $\int_C f(t)dt$ and $\int_L f(t)dt=\int_{t_1}^{t_2} f(t)dt$, L being an arc $\underset{t}{E}(s_1\leqq s\leqq s_2)$ with end points $t_j=t(s_j)$ $(j=1, 2)$, we mean the Lebesgue integrals $\int_0^c f\big(t(s)\big)\, t'(s)ds$ and $\int_{s_1}^{s_2} f\big(t(s)\big)\, t'(s)ds$ respectively, where $t'(s)=\xi'(s)+i\eta'(s)$.

It may be needless to say that the integrals of this kind are the generalisation of the ordinary contour integrals of a complex variable. Let us remark here that, among others, the theorem concerning differentiation of an indefinite integral and the theorem of integration by parts remain valid also for our integrals, so that, for instance, writing $F(t)=\int_{t_0}^{t} f(t)dt$, where $t_0=t(0)$, we have, for almost all the values of s,

$$F'(t)=f(t)\,,$$

and also we have

$$\int_C \frac{f(t)dt}{t-z}=\left[\frac{F(t(s))}{t(s)-z}\right]_0^c+\int_C \frac{F(t)dt}{(t-z)^2}$$

$$=\int_C \frac{F(t)dt}{(t-z)^2}\,,$$

since $t(c)=t(0)$.

II. Let D and D' be the interior and the exterior of C respectively and we shall denote the points of D and D' by z and z' respectively. Now we shall consider the analytic function $\varphi(z)$ defined by the following integral:

$$(*)\qquad \varphi(z)=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt\,.$$

If $f(t)$ is given continuously on C, it is evidently necessary for $\varphi(z)$ to tend to $f(t)$ as $z\rightarrow t$ for every point t of C, that we should have

$$\int_C \frac{f(t)}{t-z'}dt=0 \quad \text{for every } z', \tag{1}$$

or the equivalent relations $\int_C f(t)t^m dt=0$ $(m=0, 1, 2, \dots)$.

Among other writers, Prof. Kakeya has obtained a condition to be satisfied by C for the sufficiency of (1)[1].

Priwaloff has shown that, in order that $\varphi(z)$ tend to $f(t)$ for almost all the points t of C (namely, for every point t, except perhaps points belonging to a set, which corresponds to a set of measure zero, in the interval of s), as z tends to t along any line not touching C, it is necessary and sufficient that the condition (1) should hold, provided only that $f(t)$ is integrable on C[2].

His proof the essential part of which is devoted to verify the sufficiency depends upon the idea of Cauchy's principal value and requires long calculations.

We shall now prove, by the elements of Lebesgue's theory and the method suggested by Prof. Kakeya's paper cited above, a theorem from which we shall easily obtain, almost as its corollaries, the essential part of Priwaloff's result and moreover Fatou's well known theorem.

III.

Theorem. *If a bounded measurable function $F(t)$ defined on C satisfies the following conditions:*

(2) *at $t_1=t(s_1)$ where, we assume, $t'(s_1)$ $(|t'(s_1)|=1)$ exists, $F(t)$ has a finite differential coefficient $F'(t_1)=\lim\limits_{t\to t_1}[\{F(t)-F(t_1)\}/(t-t_1)]$,*

$$\int_C \frac{F(t)}{(t-z')^2}dt=0 \quad \textit{for every } z'\in D', \tag{3}$$

then the analytic function $f(z)$ defined by the integral

$$f(z)=\frac{1}{2\pi i}\int_C \frac{F(t)}{(t-z)^2}dt, \qquad z\in D,$$

tends to $F'(t_1)$ as $z\to t_1$ along any line not touching C.

Proof. From (3), we have

$$f(z)=\frac{1}{2\pi i}\int_C \frac{F(t)}{(t-z)^2}dt-\frac{1}{2\pi i}\int_C \frac{F(t)}{(t-z')^2}dt$$
$$=\frac{(z-z')}{2\pi i}\int_C \frac{F(t)\{2t-(z+z')\}}{(z-z)^2(t-z')^2}dt.$$

We may choose z' symmetric to z with respect to t_1 if $|z-t_1|=r$ is sufficiently small, since at t_1, C has a tangent on account of the existence of $t'(s_1)$, so that we have $z+z'=2t_1$, from which follows

$$f(z)=\frac{z-z'}{\pi i}\int_C \frac{F(t)(t-t_1)dt}{(t-z)^2(t-z')^2}. \tag{4}$$

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1) S. Kakeya, On the boundary Values of analytic Functions, Proc. **13** (1937).

2) I. Priwaloff, Sur quelques propriétés métriques des fonctions analytiques, Ann. Ec. Polyt., (1925).

If F is regular in D and on C, then it is evident that (4) holds good and the integral on the right hand-side coincides with $F'(z)$. In particular, we have

$$1=\frac{z-z'}{\pi i}\int_C \frac{(t-t_1)^2 dt}{(t-z)^2(t-z')^2}$$

or

$$F'(t_1)=\frac{z-z'}{\pi i}\int_C \frac{(t-t_1)^2 F'(t_1)}{(t-z)^2(t-z')^2}dt \tag{5}$$

and

$$0=\frac{z-z'}{\pi i}\int_C \frac{(t-t_1)dt}{(t-z)^2(t-z')^2}. \tag{6}$$

We obtain from (4) and (5)

$$\big(f(z)-F'(t_1)\big)\pi i=(z-z')\int_C \frac{(t-t_1)[F(t)-(t-t_1)F'(t_1)]}{(t-z)^2(t-z')^2}dt. \tag{7}$$

We may clearly suppose that the point $t_1=t(s_1)$ does not coincide with $t_0=t(0)=t(c)$. Then, by the existence of $F'(t_1)$, there exists, to each positive ε, a small arc $C_\varepsilon=\underset{t}{E}(|s-s_1|\leqq\sigma_1)$, which contains t_1 and does not contain t_0, such that we have for every $t\in C_\varepsilon$

$$F(t)=F(t_1)+(t-t_1)F'(t_1)+(t-t_1)\lambda \quad\text{and}\quad |\lambda|<\varepsilon. \tag{8}$$

From (7), (8) and then (6), we have

$$\begin{aligned}\big(f(z)-F'(t_1)\big)\pi i&=(z-z')\int_C \frac{(t-t_1)[F(t_1)+\lambda(t-t_1)]}{(t-z)^2(t-z')^2}dt\\ &=(z-z')\int_C \frac{(t-t_1)^2\lambda}{(t-z)^2(t-z')^2}dt=(z-z')\left\{\int_{C_\varepsilon}+\int_{C-C_\varepsilon}\right\}.\end{aligned} \tag{9}$$

To prove our theorem, we have only to show, by (9), that there exists a constant N, independent of z and z' which are on any fixed line not touching C at t_1, such that

$$\left|\int_{C_\varepsilon}\right|\leqq\frac{N}{|z'-z|}\cdot\varepsilon,\quad\text{if}\quad |z'-z|=2r\quad\text{is sufficiently small,}$$

since the other integral $\int_{C-C_\varepsilon}$ is obviously bounded as $|z-z'|=2r\to 0$. We denote by $\theta_1(0\leqq\theta_1<\pi)$ the angle made by the tangent of C at t_1 and the positive real axis. Suppose further the points z and z' are on the line that makes a fixed angle $\psi_1\left(|\psi_1|<\frac{\pi}{2}\right)$ with the normal line of C at t_1, so that

$$z=t_1\mp rie^{i(\theta_1+\psi_1)},\quad z'=t_1\pm rie^{i(\theta_1+\psi_1)},\quad\text{where}\quad |z-t_1|=|z'-t_1|=r.$$

Since $t(s)$ has a differential coefficient of modulus 1 at s_1, we may write $t-t_1=(s-s_1)\big(t'(s_1)+\mu\big)=\sigma(e^{i\theta_1}+\mu)$, where $\sigma=s-s_1$, $e^{i\theta_1}=t'(s_1)$ and

$$\mu\to 0\quad\text{as}\quad s\to s_1. \tag{10}$$

We have, then

$$(t-z)(t-z')=\{(t-t_1)\pm rie^{i(\theta_1+\psi_1)}\}\cdot\{(t-t_1)\mp rie^{i(\theta_1+\psi_1)}\}$$
$$=\sigma^2(e^{i\theta_1}+\mu)^2+r^2e^{i2(\theta_1+\psi_1)}=e^{i2(\theta_1+\psi_1)}\{\sigma^2e^{-i2\psi_1}(1+\mu e^{-i\theta_1})^2+r^2\}\,.$$

Hence, writing $1+\mu e^{-i\theta_1}=(1+\delta)e^{-i\alpha}$, so that $\delta\to 0$ and $\alpha\to 0$ as $\mu\to 0$, we have

$$(11)\quad |(t-z)(t-z')|^2=\left(\sigma^2(1+\delta)^2e^{-i2(\psi_1+\alpha)}+r^2\right)\left(\sigma^2(1+\delta)^2e^{i2(\psi_1+\alpha)}+r^2\right)$$
$$=\sigma^4(1+\delta)^4+2\cos\{2(\psi_1+\alpha)\}\cdot\sigma^2(1+\delta)^2r^2+r^4\,.$$

We have, by (10), $\mu\to 0$ as $\sigma\to 0$, and therefore $\cos\{2(\psi_1+\alpha)\}\to\cos(2\psi_1)$ as $\sigma\to 0$. Hence, if $\psi_1\neq 0$ on one hand, we have always $|\cos\{2(\psi_1+\alpha)\}|\leqq k<1$, taking σ sufficiently small, since $|2\psi_1|<\pi$, and so, from (11)

$$|(t-z)(t-z')|^2\geqq\sigma^4(1-\delta_1)^4-2k\sigma^2(1+\delta_1)^2r^2+r^4\,,\quad (|\delta|\leqq\delta_1<1)$$

whose right hand-side will be of the positive definite form by choosing σ again so small that $k^2(1+\delta_1)^4<(1-\delta_1)^4$, which is possible because $\delta\to 0$ as $\sigma\to 0$. If, on the other hand, $\psi_1=0$, then we have always $\cos(2\alpha)\geqq 0$ for sufficiently small σ, so that, by (11)

$$|(t-z)(t-z')|^2\geqq\sigma^4(1+\delta)^4+r^4\geqq\sigma^4(1-\delta_1)^4+r^4\,,\quad (|\delta|\leqq\delta_1<1)$$

whose right hand-side is also positive definite.

Writing, for brevity, $H(\sigma,r)$ for $\sigma^4(1-\delta_1)^4-2k\sigma^2(1+\delta_1)^2r^2+r^4$ or $\sigma^4(1-\delta_1)^4+r^4$, we notice that $H(\sigma,r)=r^4H\left(\frac{\sigma}{r},1\right)=r^4H(\tau,1)$ where $\sigma/r=\tau$, and that $H(\tau,1)$ has a positive lower bound. Also we notice $\{H(\tau,1)\}^{-1}=O(\tau^{-4})$ as $\tau\to\pm\infty$.

Now let us return to consideration of the integral $\int_{C_\varepsilon}$ in (9). Choosing C_ε or σ_1 sufficiently small from the first, we have, by what have been discussed,

$$\left|\int_{C_\varepsilon}\frac{(t-t_1)^2\lambda}{(t-z)^2(t-z')^2}dt\right|\leqq\varepsilon\int_{C_\varepsilon}\frac{|t-t_1|^2|dt|}{|(t-z)(t-z')|^2}\leqq\varepsilon\int_{-\sigma_1}^{\sigma_1}\frac{(2\sigma)^2d\sigma}{H(\sigma,r)}$$
$$=\frac{4\varepsilon}{r}\int_{-\frac{\sigma_1}{r}}^{\frac{\sigma_1}{r}}\frac{\tau^2d\tau}{H(\tau,1)}\leqq\frac{8\varepsilon}{|z-z'|}\int_{-\infty}^{+\infty}\frac{\tau^2d\tau}{H(\tau,1)}\,.$$

The integral appearing on the right is surely convergent, since the integrand is continuous and $O(\tau^{-2})$ as $\tau\to\pm\infty$, which proves our theorem completely.

IV. *Some applications.* If an integrable function $f(t)$ defined on C satisfies the condition (1), then, by the remarks given in Section I, we have $\int_C\{F(t)/(t-z')^2\}dt=0$ where $F(t)=\int_{t_0}^t f(t)dt$. Moreover this indefinite integral, being continuous, is bounded and has a differential coefficient $F'(t)=f(t)$ almost everywhere on C. Since $t'(s)$ with modulus

1 exists almost everywhere, the function $f(z)$ in our Theorem tends to $F'(t)=f(t)$ for almost every point t of C, as $z \to t$ along any line not touching C. But integrating by parts, we find easily

$$f(z)=\frac{1}{2\pi i}\int_C \frac{F(t)}{(t-z)^2}dt=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt,$$

which proves the following theorem of Priwaloff.

Theorem of Priwaloff. *If an integrable function $f(t)$ defined on C satisfies the condition* (1), *then the analytic function $\varphi(z)$ defined by the integral* (*) *tends to $f(t)$ for almost every point t of C, as $z \to t$ along any line not touching C.*

Suppose now that C is a unit circle. Given a regular, bounded function $f(z)$ defined in the interior of C, we consider the integrated function $F(z)=\int_0^z f(z)dz$. Evidently $F(z)$ is not only regular in $|z|<1$, but also is continuous up to the boundary, and moreover, it satisfies the Lipschitz condition on C, from which we ascertain the absolute continuity of F on C, so that $F(t)$ is an indefinite integral of its differentiated function $F'(t)=g(t)$. But by Cauchy's integral formula, we have $f(z)=F'(z)=\frac{1}{2\pi i}\int_C \frac{F(t)}{(t-z)^2}dt$, and integrating by parts, we find this equal to $\frac{1}{2\pi i}\int_C [g(t)/(t-z)]dt$. Since $\int_C [F(t)/(t-z')]dt=0$ is obvious, it follows also that the condition (3) is fulfilled by $F(t)$. Hence, applying our Theorem, we have $f(z) \to F'(t)=g(t)$ as $z \to t$, for almost every t of C, along any line not touching C, which proves the following:

Theorem of Fatou. *If $f(z)$ is regular and bounded in $|z|<1$, then for almost every point t of the circle $|t|=1$, $f(z)$ tends to a limit as $z \to t$ along any line not touching the circle. Moreover, if we denote the almost everywhere existing limits by $g(t)$, we obtain the formula* $f(z)=\frac{1}{2\pi i}\int_C \frac{g(t)}{t-z}dt.$