51. *Vector Lattices and Additive Set Functions.*

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§ 1. *Introduction.* One of the fundamental theorems in the theory of the integral is that of Radon-Nikodym concerning the countably additive set functions. The proof given below (§ 2) of this theorem will be shorter than the standard one in S. Saks' book[1] or the one recently published by C. Carathéodory[2]. Our proof is carried out by a *maximal method,* by making use of a lemma which is a simple modification of H. Hahn's decomposition theorem[3]. The same method is also applied (§ 3 and § 4) to give lattice-theoretic formulations of Radon-Nikodym's theorem. Of these ten years, such formulations were given more or less explicitly by many authors, F. Riesz, H. Freudenthal, Garrett Birkhoff, S. Kakutani, F. Maeda and S. Bochner-S. R. Phillips[4]. Our maximal method also makes use of the ideas of Riesz and Freudenthal, but it seems to be more direct than those of the cited authors. Thus, without appealing to Freudenthal's spectral theorem, we may obtain Kakutani's lattice-theoretic characterisation of the Banach space (L) from our result in § 3. Moreover the result in § 4 will give a simplified proof and extension of Freudenthal's spectral theorem.

§ 2. *The concrete case.* A class $\mathfrak{X}$ of sets in a space X is called *countably additive* if (i) the empty set belongs to $\mathfrak{X}$, (ii) with E its complement CE also belongs to $\mathfrak{X}$ and (iii) the sum $\bigvee_{n=1}^{\infty} E_n$ of sequence $\{E_n\}$ of sets $\in \mathfrak{X}$ belongs to $\mathfrak{X}$. A real-valued finite function $F(E)$ defined on $\mathfrak{X}$ is called *countably additive* if $F(\bigvee_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} F(E_n)$ for any sequence $\{E_n\}$ of mutually disjoint sets $\in \mathfrak{X}$. Let $\varphi(E)$ be a *measure* on X, that is, $\varphi(E)$ be countably additive and non-negative on X. Without losing generality we assume that any subset of a set $\in \mathfrak{X}$ of φ-measure zero also belongs to $\mathfrak{X}$. A countably additive set function

1) Theory of the integral, Warsaw (1937), 36.

2) Ueber die Differentiation von Maszfunktionen, Math. Zeits., **42** (1940), 181–189. J. von Neumann also gave an interessant proof by making use of the Banach space (L_2). See his Rings of operators, III, Ann. of Math., **41** (1940), 126.

3) S. Saks: loc. cit., 32.

4) F. Riesz: Sur la décomposition des opérations linéaires, Bologna Congress, III (1928), 143–148. Sur quelques notions fondamentales dans la théorie générale des opérations linéaires, Ann. of Math., **41** (1940), 174–206. Sur la théorie ergodique des espaces abstraits, Acta Szeged, **10** (1941), 1–20, to be cited as [$R-3$]. H. Freudenthal: Teilweise geordnete Modulen, Proc. Acad. Amsterdam, **39** (1936), 641–651, to be cited as [$F-1$]. G. Birkhoff: Dependent probabilities and spaces (L), Proc. Nat. Acad., **24** (1938), 154–158. S. Kakutani: Mean ergodic theorem in abstract L-spaces, Proc., **15** (1939), 121–123. Concrete representations of abstract L-spaces and the mean ergodic theorem, Ann. of Math., **42** (1941), 523–537, to be cited as [$K-2$]. F. Maeda: Partially ordered linear spaces, J. Sci. Hirosima Univ., **10** (1940), 137–150. S. Bochner-S. R. Phillips: Additive set functions and vector lattices, Ann. of Math., **42** (1941), 316–324.

$F(E)$ is called *absolutely continuous* if $\varphi(E)=0$ implies $F(E)=0$. $F(E)$ is called *singular* if there exists a set E_0 of φ-measure zero such that $F(E)=0$ for all $E\subset CE_0$. A real-valued function $f(x)$ on X is called *measurable* if the set $\underset{x}{E}\big(f(x)>\alpha\big)\in\mathfrak{X}$ for all real number α. We may define the *integral* $\int\limits_X f(x)\varphi(dx)$ of Lebesgue's type of *integrable*, measurable function $f(x)$. It is well-known that the set function $F(E)=\int\limits_E f(x)\varphi(dx)$ on $\mathfrak{X}$, the *indefinite integral*, is countably additive and absolutely continuous.

Radon-Nikodym's theorem states that any countably additive set function may uniquely be expressed as the sum of an indefinite integral and a singular set function. In particular, countably additive and absolutely continuous set functions may be identified with the indefinite integrals.

For the proof we need a

Lemma 1. Let $G(E)$ be a non-negative, countably additive set function and suppose that $G(E)$ is not identically zero on $\mathfrak{X}$. Then either $G(E)$ is singular or there exist a rational number $\alpha>0$ and a set E_α of φ-measure >0 such that $G(E)\geqq\alpha\varphi(E)$ for all $E\subset E_\alpha$.

Proof. By Hahn's decomposition theorem, there exists, for any (rational) number $\alpha>0$, a set $E_\alpha\subset\mathfrak{X}$ such that $G(E)\geqq\alpha\varphi(E)$ for $E\subset E_\alpha$ and $G(E)\leqq\alpha\varphi(E)$ for $E\subset CE_\alpha$. Assume that $\varphi(E_\alpha)=0$ for all rational number $\alpha>0$, then the sum $\bigvee\limits_{\alpha>0}E_\alpha=E_0$ is of φ-measure zero and $G(E)=0$ for all $E\subset CE_0$.

Proof of Radon-Nikodym's theorem. It will be sufficient to prove the case of non-negative, countably additive set function $F(E)$. Let μ be the supremum of the numbers $\int\limits_X f(x)\varphi(dx)$, when $f(x)$ runs through the set $[F]$ of all the non-negative, integrable functions such that $\int\limits_E f(x)\varphi(dx)\leqq F(E)$ all over $\mathfrak{X}$. Then there exists a sequence $\{f_n(x)\}$ of functions $\in[F]$ such that $\lim\limits_{n\to\infty}\int\limits_X f_n(x)\varphi(dx)=\mu$. We have, for any $E\in\mathfrak{X}$,

$$\int\limits_E \sup_{m\leq n}\big(f_m(x)\big)\varphi(dx)=\sup_{E=\bigvee\limits_{m=1}^{n}E_m}\Big(\sum_{m=1}^{n}\int\limits_{E_m}f_m(x)\varphi(dx)\Big)$$

$$\leqq\sup_{E=\bigvee\limits_{m=1}^{n}E_m}\Big(\sum_{m=1}^{n}F(E_m)\Big)=F(E),$$

where the sup on the second and the third terms are to be taken for all the disjoint decomposition of E. Hence, by putting $f(x)=\sup\limits_{1\leq n}f_n(x)$, we have $f(x)\in[F]$ and $\int\limits_X f(x)\varphi(dx)=\mu$. Next put $G(E)=F(E)-\int\limits_E f(x)\varphi(dx)$, then our task is to show that $G(E)$ is either singular or $G(E)=0$ on $\mathfrak{X}$. Let $G(E)$ be not singular and $G(E)\not\equiv 0$ on $\mathfrak{X}$, then there exist, by the lemma 1, a rational number $\alpha>0$ and a set E_α of φ-measure >0 such that $G(E)\geqq\alpha\varphi(E)$ for all $E\subset E_\alpha$. Let $c_\alpha(x)/\alpha$ be the characteristic function of E_α, then we would obtain $\big(f(x)+c_\alpha(x)\big)\in[F]$ and $\int\limits_X\big((f(x)+c_\alpha(x)\big)\varphi(dx)>\mu$, contrary to the definition of μ. The uniqueness of the decomposition may easily be proved. For a countably additive set function is identically zero if it is absolutely continuous and singular simultaneously.

§ 3. *The metrical case.* The space (A) of all the countably additive set functions $F(E)$ on $\mathfrak{X}$ is a semi-ordered linear space if we call $F\in(A)$ *non-negative* (written $F\geqq 0$) when $F(E)\geqq 0$ for all $E\in\mathfrak{X}$:

(1) If $F\geqq 0$ and $\alpha\geqq 0$, then $\alpha F\geqq 0$,

(2) If $F\geqq 0$ and $-F\geqq 0$, then $F=0$,

(3) If $F\geqq 0$ and $G\geqq 0$, then $F+G\geqq 0$,

(4) (A) is a lattice by the semi-order relation $\geqq$.

In fact $F^+=F\vee 0=\sup(F,0)$, $F^-=F\wedge 0=\inf(F,0)$ are respectively defined by $F^+(E)=\sup\limits_{E'\subset E}F(E')$, $F^-(E)=\inf\limits_{E'\subset E}F(E')$. As is well-known, $F=F^++F^-$ and $|F|=F^+-F^-=\sup(F,-F)$. By putting $\|F\|=|F|(X)=$the total variation of F on $\mathfrak{X}$, we have

(5) (A) is a Banach space by the norm $\|F\|=\|(|F|)\|$ such that $F\geqq 0$, $G\geqq 0$ imply $\|F+G\|=\|F\|+\|G\|$.

Let (A) be any abstract space satisfying (1)–(5). Choose any *positive* element $\varphi\in(A)$ (written $\varphi>0$), that is, an element φ satisfying $\varphi\geqq 0$, $\varphi\not\equiv 0$. Call φ a *unit* of (A) and write 1 for φ; we also write α for $\alpha\cdot 1$ when there occur no ambiguities. An element G is called *singular* if $|G|\wedge 1=0$. A non-negative element E is called a *quasi-unit* if $E\wedge(1-E)=0$. A finite linear combination $\sum\limits_{i=1}^n\alpha_iE_i$ of quasi-units E_i is called a *step-element* and we call *absolutely continuous* the element which can be expressed as the strong limit of step-elements.

We will show that any element $\in(A)$ may uniquely be expressed as the sum of an absolutely continuous element and singular element.

For the proof we need four lemmas.

Lemma 2. A *vector lattice* satisfying (1)–(4) is a *distributive* lattice, viz. we have $(A\vee B)\wedge C=(A\wedge C)\vee(B\wedge C)$, $(A\wedge B)\vee C=(A\vee C)\wedge(B\vee C)$ for any three elements A, B and C.

Proof. See [$F-1$], 642.

Lemma 3. Let $0\leqq F_1\leqq F_2\leqq\cdots$ and let the sequence of numbers $\{\|F_n\|\}$ be bounded. Then the $\sup(F_1,F_2,\ldots)=F$ exists and we have $\lim\limits_{n\to\infty}\|F-F_n\|=0$.

Proof[1]. The set of all the non-negative elements is strongly closed. For we have $\|F\|-\|G\|\leqq\|(|F|-|G|)\|\leqq\|F-G\|$ from the well-known inequality $|F|-|G|\leqq|F-G|$. Since $\|F_n-F_m\|=\|F_n\|-$

1) due to [$R-3$], 7. Cf. also [$K-2$], 526.

$\|F_m\|$, $n \geqq m$, tends to 0 as m tends to ∞, F_n converges in norm to an element F. From $F_n - F_m \geqq 0$, $n \geqq m$, and $\lim_{n\to\infty} \|(F_n - F_m) - (F - F_m)\| = 0$ we obtain $F \geqq F_m$ $(m=1, 2, \dots)$ by the strong closure of the non-negative part of (A). Let $G \geqq F_m$ $(m=1, 2, \dots)$, then, in the same way, we obtain $G \geqq F$. Thus we have $F = \sup(F_1, F_2, \dots)$.

Lemma 4. The set $\mathfrak{T}$ of all the step-elements constitutes a sublattice of (A).

Proof. Since the symmetric relation $E \wedge (1-E) = 0$ is equivalent to $2E \wedge 1 = E$, the set $\mathfrak{E}$ of all the quasi-units forms a Boolean algebra. For $E_1, E_2 \in \mathfrak{E}$ implies $2(E_1 \vee E_2) \wedge 1 = (2E_1 \wedge 1) \vee (2E_2 \wedge 1) = E_1 \vee E_2$, $2(E_1 \wedge E_2) \wedge 1 = (2E_1 \wedge 1) \wedge (2E_2 \wedge 1) = E_1 \wedge E_2$. Therefore any two elements $F, G \in \mathfrak{T}$ may be expressed as $F = \sum_{i=1}^{n} \alpha_i E_i$, $G = \sum_{i=1}^{n} \beta_i E_i$, $E_i \wedge E_j = 0$ $(i \neq j)$, $E_i \in \mathfrak{E}$ $(i=1, 2, \dots, n)$, and hence $F \vee G = \sum_{i=1}^{n} \max(\alpha_i, \beta_i) E_i$ $F \wedge G = \sum_{i=1}^{n} \min(\alpha_i, \beta_i) E_i$ both $\in \mathfrak{T}$. Cf. [$K-2$], 531.

Lemma 1′. Let $G > 0$ and let $G \wedge 1 \neq 0$. Then there exists a rational number $\alpha > 0$ and a quasi-unit $E_\alpha > 0$ such that $G \geqq \alpha E_\alpha$.

Proof. There exists a rational number $\alpha > 0$ such that $(G-\alpha)^+ \wedge 1 \neq 0$. Assume the contrary, then $0 \leqq (G-\alpha)^+ \wedge 1 = \big((G-\alpha) \wedge 1\big)^+ = \big((G \wedge (1+\alpha) - \alpha\big)^+ = 0$ and hence $0 \leqq G \wedge (1+\alpha) \leqq \alpha$ for all rational number $\alpha > 0$. Thus we would have $G \wedge 1 = 0$, contrary to the hypothesis. Let now $(G-\alpha)^+ \wedge 1 \neq 0$ and hence $(G/\alpha - 1)^+ \wedge 1 \neq 0$. Then, by the lemma 3, $E_\alpha = \sup_{1 \leqq n} \big(n(G/\alpha - 1)^+ \wedge 1\big)$ exists and > 0. Since $2E_\alpha \wedge 1 = \Big(\sup_{1 \leqq n} \big(2n(G/\alpha - 1)^+ \wedge 2\big)\Big) \wedge 1 = E_\alpha$, E_α is a quasi-unit. Next from $n(G/\alpha - 1)^+ \wedge 1 = \Big(\big(n(G/\alpha - 1)\big) \wedge 1\Big)^+ = \Big(\big(nG/\alpha \wedge (n+1)\big) - n\Big)^+$ $\leqq \big((n+1)(G/\alpha \wedge 1) - n\big)^+$ we obtain $E_\alpha \leqq G/\alpha \wedge 1$. Thus $\alpha E_\alpha \leqq G$.

The decomposition of (A). Let $F \in (A)$ be positive, and denote by $[F]$ the set of all the non-negative step-elements T such that $T \leqq F$. Put $\mu = \sup_{T \in [F]} \|T\|$, then there exists a sequence $\{T_n\}$ of elements $\in [F]$ such that $\lim_{n\to\infty} \|T_n\| = \mu$. By the lemma 4, $F_n = \sup_{m \leqq n} T_m \in [F]$. We have, by the lemma 3, $\|\bar{F}\| = \mu$ and $\lim_{n\to\infty} \|\bar{F} - F_n\| = 0$, where $\bar{F} = \sup_{1 \leqq n} F_n$. We next prove that $G = F - \bar{F}$ is either singular or $= 0$. Assume the contrary, then, by the lemma 1′, there exists a quasi-unit $E_\alpha > 0$ such that $G \geqq \alpha E_\alpha$, $\alpha > 0$. Thus $\bar{F} + \alpha E_\alpha$ is a strong limit of elements $\in [F]$ and hence we must have $\|\bar{F} + \alpha E_\alpha\| = \|\bar{F}\| + \alpha \|E_\alpha\| \leqq \mu$, contrary to $\|\bar{F}\| = \mu$, $\|E_\alpha\| > 0$.

The uniqueness of the decomposition. It will be sufficient to show than an absolutely continuous, singular element G is $= 0$. Since G is absolutely continuous, we have $\lim_{n\to\infty} \|G - G_n\| = 0$, $G_n \in \mathfrak{T}$. As $0 \leqq E \leqq 1$ for any $E \in \mathfrak{E}$ we have $|G_n| \leqq \alpha_n$ $(n=1, 2, \dots)$ and hence, by the singularity of G, $|G| \wedge |G_n| = 0$ $(n=1, 2, \dots)$. As is well-known, we have

$|A \wedge B - A' \wedge B| + |A \vee B - A' \vee B| = |A - A'|$ in any vector lattice. Thus we have $\|G\| = \|(|G|)\| = \|(|G| \wedge |G|)\| \leqq \|(|G| - |G_n|)\| \leqq \|G - G_n\|$ $(n=1, 2, \ldots)$, proving $G=0$.

§ 4. *The general case.* The abstract space (A) satisfying (1)-(5) is, by lemma 3, *σ-complete*[1], viz.

(5)′ Any sequence $\{F_n\}$, $n=1, 2, \ldots$, bounded from above (below) admit the supremum (infimum) in (A).

Let now (A) be any abstract space satisfying (1)-(5)′, then we can extend the maximal method to obtain the decomposition of (A). In this case we call absolutely continuous the element which can be expressed as the order-limit of an enumerable number of step-elements. We first prove a

Lemma 1″. Let $G \geqq 0$, $G \geqq \alpha E$, $E \in \mathfrak{E}$ and $\alpha > 0$. Then, for any β, $0 < \beta < \alpha$, $E_\beta = \sup\limits_{1 \leqq n} \left(n(G/\beta - 1)^+ \wedge 1\right) \geqq E$ and $G \geqq \beta E_\beta$.

Proof. That $G \geqq \beta E_\beta$ was already proved in lemma 1′. We will prove $E_\beta \geqq E_\alpha$ in the special case $\alpha = 1$. The proof reads as follows. Let $0 < \delta < 1$, then we have $G - (1-\delta) = G - (1-\delta)E - (1-\delta)(1-E) \geqq \delta E - (1-\delta)(1-E)$ and $\left(\delta E - (1-\delta)(1-E)\right)^+ = \left(\delta E \vee (1-\delta)(1-E) - (1-\delta)(1-E)\right) = \delta E + (1-\delta)(1-E) - (1-\delta)(1-E) = \delta E$ by $E \in \mathfrak{E}$. Thus $n\left(G/(1-\delta) - 1\right)^+ \wedge 1 \geqq n\delta E \wedge 1 \geqq n\delta E \wedge E$ for $n=1, 2, \ldots$ and hence $E_{1-\delta} \geqq E$.

The decomposition of (A). Let $F > 0$ and put

$$\bar{F} = \sup_{\alpha > 0} (\alpha E_\alpha), \qquad E_\alpha = \sup_{1 \leqq n} \left(n(F/\alpha - 1)^+ \wedge 1\right),$$

where the supremum of the first term is to be taken over all the rational numbers $\alpha > 0$. If $G = F - \bar{F}$ is not singular, then there exist, by the lemma 1′, a rational number $\beta > 0$ and a quasi-unit $E > 0$ such that $G \geqq \beta E$. We have, by the lemma 1″, $E_{\beta^{(1)}} \geqq E$ if $0 < \beta^{(1)} < \beta$. By $F \geqq \bar{F}$, $\bar{F} \geqq \beta^{(1)} E_{\beta^{(1)}}$ and $F - \bar{F} \geqq \beta E$ we obtain $F \geqq 2\beta^{(1)} E$. Again by the lemma 1″ we have $E_{2\beta^{(2)}} \geqq E$ if $0 < \beta^{(2)} < \beta^{(1)}$. Thus by $F \geqq \bar{F}$, $\bar{F} \geqq 2\beta^{(2)} E_{2\beta^{(2)}}$ and $\bar{F} - F \geqq \beta E$ we obtain $F \geqq 3\beta^{(2)} E$. In this way we would have $F \geqq (n+1)\beta^{(n)} E$ $(n=1, 2, \ldots)$ if $0 < \beta^{(n)} < \beta$, which is a contradiction. Thus G must be singular.

The uniqueness of the decomposition. Let G be absolutely continuous and singular. As in § 3 we have $|G| \wedge |G_n| = 0$ $(n=1, 2, \ldots)$, where $G = \underset{n \to \infty}{\text{order-lim}}\, G_n$, $G_n \in \mathfrak{T}$. Thus $|G| = |G| \wedge |G| = \underset{n \to \infty}{\text{order-lim}}\, (|G| \wedge |G_n|) = 0$. This prove the uniqueness of decomposition.

Remark. The above result is fairly general and it may be applied to the theory of probability or to the operator theory in Hilbert or Banach spaces.

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1) In truth, it may be proved ([R—3], 7) that the vector lattice (A) satisfying (1)-(5) is *complete*, viz.

(5)″ any set bounded from above (below) admits the supremum (infimum) in (A). The second paper of Riesz and those of Maeda, Bochner-Phillips treat the vector lattice satisfying (1)-(5)″. However, the decomposition theorem would be more general if we assume the weaker condition (5)′ instead of (5)″.