PAPERS COMMUNICATED

50. On Axioms of Linear Functions.

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1. Let a set G of elements $a, b, c, \dots$, satisfy the following axioms:

(1) There exists an operation in G which associates with each pair a, b of G an element c of G, i.e., $a \cdot b = c$.

(2) The operation satisfies the associative law:

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d).$$

(3) If a, b, c are any given elements, each of the equations $a \cdot x = c$ and $x \cdot b = c$ is uniquely soluble in G for x.

As an example, we show that a real linear function of two real variables x, y, i.e.,

$$x \cdot y = \lambda x + \mu y + \nu$$

satisfies the above axioms (1), (2), (3). Conversely, we shall prove

Theorem 1[1]. *The set G forms an abelian group with respect to the new operation $x+y=z$ which is defined by the equation*

$$a \cdot s + r \cdot b = a \cdot b,$$

where r and s denote two fixed elements in G.

Furthermore, the operation $x \cdot y$ of G is expressed as a linear function of x, y with respect to the new operation such that

$$x \cdot y = Ax + By + c,$$

where A and B denote the automorphisms of G and are mutually permutable, that is, $AB = BA$, and c is a fixed element in G.

Next, let us consider a set G^* of elements $a, b, c, \dots$, which satisfies the axioms (1), (2) and the axiom

(3*) There exists at least one unit element 0 in G^*, i.e., $0 \cdot 0 = 0$ and, if a is any given element, each of the equations $x \cdot 0 = a$ and $0 \cdot x = a$ has at least one solution in G^* for x.

As examples, we show that the sum (or product) of two sets a, b of points, i.e.,

$$a \cdot b = a \dot{+} b \dot{+} 0$$

and a linear differential expression of two real functions $x(t)$, $y(t)$ of a real variable t, i.e.,

1) K. Toyoda, On axioms of mean transformations and automorphic transformations of abelian groups, Tôhoku Math. Journal, **47** (1940), pp. 239–251.

K. Toyoda, On affine geometry of abelian groups, Proc. **16** (1940), 161–164.

D. C. Murdoch, Quasi-groups which satisfy certain generalized laws, American Journal of Math., **16** (1939), pp. 509–522.

$$x(t)\cdot y(t)=\Big(\sum_{k=1}^{n}a_k(t)\frac{d^k}{dt^k}\Big)x(t)+\Big(\sum_{k=1}^{m}b_k(t)\frac{d^k}{dt^k}\Big)y(t)\,,$$

$$\Big(\sum_{k=1}^{n}a_k(t)\frac{d^k}{dt^k}\Big)\Big(\sum_{k=1}^{m}b_k(t)\frac{d^k}{dt^k}\Big)=\Big(\sum_{k=1}^{m}b_k(t)\frac{d^k}{dt^k}\Big)\Big(\sum_{k=1}^{n}a_k(t)\frac{d^k}{dt^k}\Big),$$

satisfy the above three axioms (1), (2), (3*). Conversely, we have

Theorem 2[2]. *Let us introduce a new operation* $x+y\sim z$ *into* G^* *which is defined by the two conditions:*

(i) *If* $x\cdot 0=x'\cdot 0$, $x\sim x'$.

(ii) *If* $x=a\cdot 0$ *and* $y=0\cdot b$, $x+y=a\cdot b$.

Then, the new operation $x+y\sim z$ *is one-valued in* G^* *and the set* G^* *forms a commutative semi-group with respect to this operation.*

Furthermore, the operation $x\cdot y$ *of* G^* *is expressed as a linear function of* x, y *with respect to the new operation such that*

$$x\cdot y=Ax+By\,,$$

where A *and* B *denote the homomorphisms of* G^* *and are mutually permutable, i. e.,* $AB=BA$.

Finally, let us consider a set G of elements $a_1, a_2, \dots, b_1, b_2, \dots$, which satisfies the following axioms:

(1) There exists an operation in G which associates with each class of n elements $a_1, a_2, \dots, a_n$ of G an $(n+1)$-th element a_0 of G, i. e., $(a_1, a_2, \dots, a_n)=a_0$.

(2) The operation satisfies the associative law:

$$\big((a_1, a_2, \dots, a_n),\ (b_1, b_2, \dots, b_n),\ \dots\dots,\ (d_1, d_2, \dots, d_n)\big)$$
$$=\big((a_1, b_1, \dots, d_1),\ (a_2, b_2, \dots, d_2),\ \dots\dots,\ (a_n, b_n, \dots, d_n)\big)\,.$$

(3) If a, b, c are any given elements, each of the equations

$$(x, b, a, \dots, a)=c \quad\text{and}\quad (b, x, a, \dots, a)=c$$

is uniquely soluble in G for x.

Then, we have

Theorem 3[3]. *Let us introduce a new operation into* G *such that*

$$(a, e_0, e_0, \dots, e_0)+(e_0, b, e_0, \dots, e_0)=(a, b, e_0, \dots, e_0)\,,$$

where $e_0=(e_1, e_2, \dots, e_n)$ *and* $e_1, e_2, \dots, e_n$ *denote* n *fixed elements in* G.

Then, the set G *forms an abelian group with respect to the new operation* $x+y=z$ *and the operation* $(x_1, x_2, \dots, x_n)$ *of* G *is expressed as a linear function of elements* $x_1, x_2, \dots, x_n$ *with respect to the new operation such that*

$$(x_1, x_2, \dots, x_n)=A_1x_1+A_2x_2+\cdots\cdots+A_nx_n+c\,,$$
$$A_iA_k=A_kA_i\,,\qquad (i, k=1, 2, \dots, n)\,,$$

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2) G. Birkhoff, Lattice Theory, 1940.

3) K. Toyoda, On linear functions of abelian groups, Proc. **16** (1940), 524–528.

M. Nagumo, Über eine Klasse der Mittelwerte, Japanese Journal of Math., **7** (1930), pp. 71–79.

A. Kolmogoroff, Sur la notion de la moyenne, Atti della Reale Academia Nazionale dei Lincei, **12** (1930), pp. 388–391.

where A_1, A_2 *are the automorphisms of* G *and* $A_3, A_4, \dots, A_n$ *are the homomorphisms of* G *and* c *denotes a fixed element in* G.

2. *Proof of Theorem 1.* Analogously as my previous papers[1), 3)], we shall proceed as follows.

Lemma 1. If the two equations $x\cdot a=b\cdot y$ *and* $z\cdot a=b\cdot w$ *hold, then* $x\cdot w=z\cdot y$.

Lemma 2.

$$(x+r)\cdot s+(y+r)\cdot s=(x+y+r)\cdot s\,,$$
$$r\cdot(x+s)+r\cdot(y+s)=r\cdot(x+y+s)\,.$$

Proof. Let us put

$$x=a\cdot s\,,\quad y=r\cdot b\,,\quad a\cdot b=p\cdot s\,,$$
$$r=r'\cdot s=r\cdot r''=r'''\cdot r\,,$$
$$s=s'\cdot s=r\cdot s''=s\cdot s'''.$$

Then, we have

$$\begin{aligned}(x+r)\cdot s+(y+r)\cdot s&=(a\cdot r'')\cdot s+(r'\cdot b)\cdot(s\cdot s''')\\&=(a\cdot r'')\cdot s+(r'\cdot s)\cdot(b\cdot s''')\\&=(a\cdot r'')\cdot s+r\cdot(b\cdot s''')\\&=(a\cdot r'')\cdot(b\cdot s''')\end{aligned}$$

and

$$\begin{aligned}(x+y+r)\cdot s&=(a\cdot b+r)\cdot s=(p\cdot r'')\cdot(s\cdot s''')\\&=(p\cdot s)\cdot(r''\cdot s''')=(a\cdot b)\cdot(r''\cdot s''')\\&=(a\cdot r'')\cdot(b\cdot s''')\,,\end{aligned}$$

which shows that the first equation holds.

Similarly, we can prove the second equation by the identities

$$\begin{aligned}r\cdot(x+s)+r\cdot(y+s)&=(r'''\cdot a)\cdot(s'\cdot b)\\&=r\cdot(x+y+s)\,,\end{aligned}$$

for any two elements x, y.

Proof of Theorem 1. By means of Lemma 1, we know that G forms an abelian group with respect to the new operation $x+y=z$ and that, if we put

$$(x+r)\cdot s=Ax \quad\text{and}\quad r\cdot(y+s)=By\,,$$

then we get, by means of Lemma 2,

$$\begin{aligned}x\cdot y=x\cdot s+r\cdot y&=A(x-r)+B(y-s)\\&=Ax+By+c\,,\end{aligned}$$

where $-c=Ar+Bs$.

3. *Proof of Theorem 2.* By the definitions, wo have

Lemma 1.

(i) *If* $x\sim x'$ *and* $y\sim y'$, *then* $x\cdot y\sim x'\cdot y'$.

(ii) *If* $x\cdot 0=b\cdot y$ *and* $z\cdot 0=b\cdot w$, *then* $x\cdot w\sim z\cdot y$.

Proof. Since we have

$$(x\cdot y)\cdot 0=(x\cdot 0)\cdot(y\cdot 0)=(x'\cdot 0)\cdot(y'\cdot 0)=(x'\cdot y')\cdot 0\,,$$

(i) holds. Also, (ii) is proved by

$$\begin{aligned}(x\cdot w)\cdot 0&=(x\cdot 0)\cdot(w\cdot 0)=(b\cdot y)\cdot(w\cdot 0)\\&=(b\cdot w)\cdot(y\cdot 0)=(z\cdot 0)\cdot(y\cdot 0)\\&=(z\cdot y)\cdot 0\,.\end{aligned}$$

Lemma 2.

(i) *The operation* $x+y\sim z$ *determines uniquely a third element* z *in* G^* *and, if* $x\sim x'$ *and* $y\sim y'$, *then* $x+y\sim x'+y'$.

(ii) $x\cdot 0+y\cdot 0=(x+y)\cdot 0$, *for any elements* x, y.

Proof. If we put

$$x=a\cdot 0=a'\cdot 0=0\cdot c\,,$$
$$y=d\cdot 0=0\cdot b=0\cdot b',$$

then, by means of Lemma 1 and the definition,

$$x+y=a\cdot b\sim d\cdot c\sim a'\cdot b',$$

which shows that the operation $x+y\sim z$ is one-valued.

Also, we have, by means of Lemma 1,

$$\begin{aligned}x\cdot 0+y\cdot 0&=(a\cdot 0)\cdot 0+(0\cdot b)\cdot 0\\&=(a\cdot 0)\cdot 0+0\cdot(b\cdot 0)\\&=(a\cdot 0)\cdot(b\cdot 0)=(a\cdot b)\cdot 0\\&=(a'\cdot b')\cdot 0=(x+y)\cdot 0\,.\end{aligned}$$

Consequently, (i) is proved by

$$(x+y)\cdot 0=x\cdot 0+y\cdot 0=x'\cdot 0+y'\cdot 0=(x'+y')\cdot 0\,,$$

that is

$$x+y\sim x'+y'.$$

Lemma 3.

(i) $x+y\sim y+x$.

(ii) $x+0\sim 0+x\sim x$.

(iii) $(x+y)+z\sim x+(y+z)$.

Proof. By means of Lemma 1 and the definitions, it is evident that (i) and (ii) hold.

In order to prove (iii), let us put

$$x=0\cdot a\,,\quad y=b\cdot 0\,,\quad z=0\cdot c$$
$$b\cdot a=p\cdot 0\,,\quad b\cdot c=q\cdot 0\,.$$

Then, by means of Lemmas 1 and 2, we get

$$(x+y)+z\sim(y+x)+z\sim b\cdot a+0\cdot c\sim p\cdot c$$

and

$$x+(y+z)\sim(y+z)+x\sim b\cdot c+0\cdot a\sim q\cdot a\,,$$

whence

$$(x+y)+z\sim p\cdot c\sim q\cdot a\sim x+(y+z)\,.$$

Lemma 4.

(i) If $0\cdot x=y\cdot b$ *and* $0\cdot z=w\cdot b$, *then* $0\cdot(w\cdot x)=0\,(y\cdot z)$.

(ii) $0\cdot x+0\cdot y=0\cdot(x+y)$, *for any elements* x, y.

Proof. Analogously as Lemma 1, we show that (i) holds.

In order to prove (ii), let us put

$$x=a\cdot 0=a'\cdot 0=0\cdot c\,,$$
$$y=d\cdot 0=0\cdot b=0\cdot b'.$$

Then, we get

$$\begin{aligned}0\cdot x+0\cdot y&=0\cdot(a\cdot 0)+0\cdot(0\cdot b)=(0\cdot a)\cdot 0+0\cdot(0\cdot b)\\&=(0\cdot a)\cdot(0\cdot b)=0\cdot(a\cdot b)=0\cdot(d\cdot c)\\&=0\cdot(a'\cdot b')=0\cdot(x+y)\,.\end{aligned}$$

Proof. of Theorem 2. By means of Lemma 3, we know that G^* forms an abelian semi-group with respect to the new operation $x+y\sim z$ and that, by means of Lemmas 1, 2 and 4, the operation $x\cdot y$ of G^* becomes a linear function of x, y, that is,

$$x\cdot y=x\cdot 0+0\cdot y=Ax+By\,.$$

4. *Proof of Theorem 3.* Analogously as Theorems 1 and 2, we apply the following lemmas.

Lemma 1. If the simultaneous equations

$$(x, a, a, \dots, a)=(b, y, a, \dots, a)\,,$$
$$(z, a, a, \dots, a)=(b, w, a, \dots, a)$$

hold, then it follows that

$$(x, w, a, \dots, a)=(z, y, a, \dots, a)\,.$$

Proof. Putting $a'=(a, a, \dots, a)$, we get

$$\begin{aligned}&\big((x, w, a, \dots, a)\,,\quad a', a', \dots, a'\big)\\&=\big((x, a, a, \dots, a)\,,\quad (w, a, a, \dots, a)\,,\quad a', a', \dots, a'\big)\\&=\big((b, y, a, \dots, a)\,,\quad (w, a, a, \dots, a)\,,\quad a', a', \dots, a'\big)\\&=\big((b, w, a, \dots, a)\,,\quad (y, a, a, \dots, a)\,,\quad a', a', \dots, a'\big)\\&=\big((z, a, a, \dots, a)\,,\quad (y, a, a, \dots, a)\,,\quad a', a', \dots, a'\big)\\&=\big((z, y, a, \dots, a)\,,\quad a', a', \dots, a'\big)\,.\end{aligned}$$

Hereafter, we apply

Notation.

(i) $(e_1, e_2, \dots, e_n)=e_0$ *and* $(e_k, e_k, \dots, e_k)=e'_k$, *for* $k=1, 2, \dots, n$.

(ii) $(a, e_0, e_0, \dots, e_0)+(e_0, b, e_0, \dots, e_0)=(a, b, e_0, \dots, e_0)$.

Lemma 2.

$$\begin{aligned}&(x, a, a, \dots, a)+(a, y, a, \dots, a)\\&=(x, y, a, \dots, a)+(a, a, a, \dots, a)\,.\end{aligned}$$

Proof. If we put

$$x=(x_1, e_1, e_1, \dots, e_1)\,,$$
$$y=(y_2, e_2, e_2, \dots, e_2)=(e_2, y'_2, e_2, \dots, e_2)$$

and

$$a=(a_k, e_k, e_k, \dots, e_k)\,, \quad \text{for} \quad k=1, 2, \dots, n\,,$$
$$=(e_k, a'_k, e_k, \dots, e_k)\,,$$

then, it follows that

$$(x, a, a, \dots, a)+(a, y, a, \dots, a)$$
$$=\big((x_1, a_2, a_3, \dots, a_n)\,,\quad e_0, e_0, \dots, e_0\big)+\big(e_0, (a'_1, y'_2, a'_3, \dots, a'_n)\,,\quad e_0, e_0, \dots, e_0\big)$$
$$=\big((x_1, a_2, a_3, \dots, a_n)\,,\quad (a'_1, y'_2, a'_2, \dots, a'_n)\,,\quad e_0, e_0, \dots, e_0\big)$$
$$=\big((x_1, a'_1, e_1, \dots, e_1)\,,\quad (a_2, y'_2, e_2, \dots, e_2)\,,\quad (a_3, a'_3, e_3, \dots, e_3)\,, \dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots,\quad (a_n, a'_n, e_n, \dots, e_n)\big)$$

and

$$(x, y, a, \dots, a)+(a, a, a, \dots, a)$$
$$=\big((x_1, y_2, a_3, \dots, a_n)\,,\quad e_0, e_0, \dots, e_0\big)+\big(e_0, (a'_1, a'_2, a'_3, \dots, a'_n)\,,\quad e_0, e_0, \dots, e_0\big)$$
$$=\big((x_1, y_2, a_3, \dots, a_n)\,,\quad (a'_1, a'_2, a'_3, \dots, a'_n)\,,\quad e_0, e_0, \dots, e_0\big)$$
$$=\big((x_1, a'_1, e_1, \dots, e_1)\,,\quad (y_2, a'_2, e_2, \dots, e_2)\,,\quad (a_3, a'_3, e_3, \dots, e_3)\,, \dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots,\quad (a_n, a'_n, e_n, \dots, e_n)\big)\,.$$

On the other hand, we have, by means of Lemma 1,

$$(a_2, y'_2, e_2, \dots, e_2)=(y_2, a'_2, e_2, \dots, e_2)\,,$$

which shows that Lemma 2 holds.

Lemma 3.

(i) $(x_1, x_2, \dots, x_{k-1}, e'_k, x_{k+1}, \dots, x_n)+(e'_1, e'_2, \dots, e'_{k-1}, x_k, e'_{k+1}, \dots, e'_n)$
$=(x_1, x_2, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n)\,.$

(ii) $(e'_1, e'_2, \dots, e'_{k-1}, x_k, e'_{k+1}, \dots, e'_n)+(e'_1, e'_2, \dots, e'_{k-1}, y_k, e'_{k+1}, \dots, e'_n)$
$=(e'_1, e'_2, \dots, e'_{k-1}, x_k+y_k-e'_k, e'_{k+1}, \dots, e'_n)\,.$

Proof. If we put

$$x_i=(a_i, e_i, e_i, \dots, e_i)\,, \quad \text{for} \quad i=1, 2, \dots, n\,,$$
$$=(e_i, a'_i, e_i, \dots, e_i)\,, \quad \text{for} \quad i=k\,,$$
$$y_i=(e_i, b'_i, e_i, \dots, e_i)\,, \quad \text{for} \quad i=k\,,$$

then it follows that

$$(x_1, \dots, x_{k-1}, e'_k, x_{k+1}, \dots, x_n)+(e'_1, \dots, e'_{k-1}, x_k, e'_{k+1}, \dots, e'_n)$$
$$=\big((a_1, \dots, a_{k-1}, e_k, a_{k+1}, \dots, a_n)\,,\quad e_0, e_0, \dots, e_0\big)$$
$$+\big(e_0, (e_1, \dots, e_{k-1}, a'_k, e_{k+1}, \dots, e_n)\,,\quad e_0, \dots, e_0\big)$$
$$=\big((a_1, \dots, a_{k-1}, e_k, a_{k+1}, \dots, a_n)\,,\quad (e_1, \dots, e_{k-1}, a'_k, e_{k+1}, \dots, e_n)\,,\quad e_0, \dots, e_0\big)$$

$$=\big((a_1, e_1, \cdots, e_1),\quad (a_2, e_2, \cdots, e_2),\quad \cdots\cdots,\quad (a_{k-1}, e_{k-1}, \cdots, e_{k-1}),$$
$$(e_k, a'_k, e_k, \cdots, e_k),\quad (a_{k+1}, e_{k+1}, \cdots, e_{k+1}),\quad \cdots\cdots,\quad (a_n, e_n, \cdots, e_n)\big)$$
$$=(x_1, x_2, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n)$$

and, by means of Lemma 2,

$$(e'_1, \cdots, e'_{k-1}, x_k, e'_{k+1}, \cdots, e'_n)+(e'_1, \cdots, e'_{k-1}, y_k, e'_{k+1}, \cdots, e'_n)$$
$$=\big((e_1, \cdots, e_{k-1}, a_k, e_{k+1}, \cdots, e_n),\quad e_0, \cdots, e_0\big)$$
$$+\big(e_0, (e_1, \cdots, e_{k-1}, b_k, e_{k+1}, \cdots, e_n),\quad e_0, \cdots, e_0\big)$$
$$=\big((e_1, \cdots, e_{k-1}, a_k, e_{k+1}, \cdots, e_n),\quad (e_1, \cdots, e_{k-1}, b'_k, e_{k+1}, \cdots, e_n),\quad e_0, \cdots, e_0\big)$$
$$=\big(e'_1, \cdots, e'_{k-1},\quad (a_k, b'_k, e_k, \cdots, e_k),\quad e'_{k+1}, \cdots, e'_n\big)$$
$$=\big(e'_1, \cdots, e'_{k-1},\quad (a_k, e_k, e_k, \cdots, e_k)+(e_k, b'_k, e_k, \cdots, e_k)-e'_k,\quad e'_{k+1}, \cdots, e'_n\big)$$
$$=(e'_1, \cdots, e'_{k-1},\quad x_k+y_k-e'_k,\quad e'_{k+1}, \cdots, e'_n)$$

Proof of Theorem 3. By means of Lemma 1 and the definition, we know that G forms an abelian group with respect to the new operation $x+y=z$ and that the operation $(x_1, x_2, \cdots, x_n)$ of G becomes a linear function of $x_1, x_2, \cdots, x_n$.

Because, we obtain, by means of Lemma 3,

$$(x_1, x_2, \cdots, x_n)=\sum_{k=1}^{n}(e'_1, \cdots, e'_{k-1}, x_k, e'_{k+1}, \cdots, e'_n)$$
$$=\sum_{k=1}^{n} A_k(x_k-e'_k)$$
$$=\sum_{k=1}^{n} A_k x_k+c,$$

where A_1, A_2 denote the automorphisms of G and $A_3, A_4, \cdots, A_n$ denote the homomorphisms of G and $c=-\sum_{k=1}^{n} A_k e'_k$.