3. *Note on the Fundamental Domain of a General Fuchsian Group.*

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Let R be the general Poincaré-space $\left(Z;\ N(Z)\leqq 1\right)$[1] and $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$ be a displacement of a general fuchsian group $\mathfrak{G}$; namely

$$U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \qquad U'S\bar{U}=S, \qquad S=\begin{pmatrix} E_m & 0 \\ 0 & -E_n \end{pmatrix}.$$

Definition. When $U_3\neq 0$, we denote the domain $(Z;\ \|U_3Z+U_4\|\geqq 1)$ by the symbol $M(U)$. A point Z is an inner or a boundary or an outer point of $M(U)$ according as $\|U_3Z+U_4\|>1$ or $=1$ or <1.

Theorem 1. 1°. *If* $U_3\neq 0$, $|E-\bar{W}'W|$ *is greater than, equals to, or smaller than* $|E-\bar{Z}'Z|$ *according as* Z *is an outer or a boundary or an inner point of* $M(U)$; *the converse also holds.*

2°. *If however* $U_3=0$, $|E-\bar{W}'W|=|E-\bar{Z}'Z|$.

Proof. 1°. It comes from the equality

$$\|U_3Z+U_4\|^{-2}=|E-\bar{W}'W|\,|E-\bar{Z}'Z|^{-1},$$

which I deduced in another place[2].

2. In this case $W=U_1ZU_4^{-1}$, where U_1 and U_4 are unitary, so that the result is evident.

We denote the intersection of R and all the domains $M(U)$ by M.

Theorem 2. Let Z_0 *be a point which gives the maximum value of* $|E-\bar{Z}'Z|$ *among the points equivalent to* Z_0 *and let* D *be the set of such points as* Z_0, *then* $D=M$.

Proof. Let Z_1 be an inner point of the space R, the number of such points Z equivalent to Z_1 that give $|E-\bar{Z}'Z|\geqq|E-\bar{Z}_1'Z_1|$ is finite, because the series

$$\sum_{U\subset\mathfrak{G}} |U_3Z+U_4|^{-k(n+m)}, \qquad k\geqq 2^{3)},$$

is absolutely convergent at an inner point of R.

As a point equivalent to an inner point of R is also an inner point of R, the maximum value of the expression $|E-\bar{Z}'Z|$ is really taken at an inner point of R equivalent to Z_1.

On the other hand, a point equivalent to a boundary point of R is also a boundary point of R and it is always $|E-\bar{Z}'Z|=0$, when $N(Z)=1$. Now let $Z\in M$ and let Y be a point equivalent to Z, namely $Y=(U_1Z+U_2)(U_3Z+U_4)$, then $|E-\bar{Z}'Z|\geqq|E-\bar{Y}'Y|$, if $U_3\neq 0$, and

1) $N(Z)$ means the norm of Z.
2) M. Sugawara. On the general Zetafuchsian Functions,
3) In the case of symmetrical matrices the exponent is $-k(n+1)$.

$|E-\bar{Z}'Z|=|E-\bar{Y}'Y|$, if $U_3=0$. Hence we get $Z\in D$. Conversely let $Z\in D$, then $|E-\bar{Z}'Z|\geqq|E-\bar{Y}'Y|$ for every point Y equivalent to Z. Hence we get $Z\in M$. Therefore we have at last $M=D$.

The transformations, in which $U_3=0$, make a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ in all. As the group of unitary transformations is compact, $\mathfrak{H}$ is finite of the order k, because, if otherwise, it would contain an infinitesimal transformation.

Now we introduce the euclidean metric in R, namely as the distance between two points A and B we take the expression $\sqrt{\mathrm{Sp.}\,\overline{(A-B)}'(A-B)}$ invariant under the transformation $W=U_1ZU_4^{-1}$. Let Z_1 be such a point of M that the points $Z_1, Z_2, \ldots, Z_k$ equivalent to Z_1, which are of course points of M, are all different to one another. We denote the set of points of M each of which has the shortest distance in this metric from Z_1 among the distances from the other Z_i by the symbol M_0, then we have

Theorem 3. M_0 is a fundamental domain of the fuchsian group $\mathfrak{G}$.

Proof. It is evident that every point Z of R has at least one image in M_0. Let Z_1 and Z_2 be two points of M_0, and Z_1 is equivalent to Z_2 by the transformation U of the group $\mathfrak{G}$. If $U_3\neq 0$, we have $|E-\bar{Z}_1'Z_1|\geqq|E-\bar{Z}_2'Z_2|$ as Z_1 is in $M(U)$ and $|E-\bar{Z}_2'Z_2|\geqq|E-\bar{Z}_1'Z_1|$ as Z_2 is in $M(U^{-1})$. Hence we get $|E-\bar{Z}_1'Z_1|=|E-\bar{Z}_2'Z_2|$. Therefore Z_1 and Z_2 lie on the boundary of $M(U)$ and $M(U^{-1})$; so that they are boundary points of M, thus of M_0.

If however $U_3=0$, they are also boundary points of M_0 by the definition of M_0. Therefore M_0 is a fundamental domain of $\mathfrak{G}$.

Remark. An inner point of M_0 can not be equivalent to a boundary point and two equivalent boundary points give the same value of the expression $|E-\bar{Z}'Z|$.