# 35. *On an Extension of Bloch's Theorem.*

By Masatsugu TSUJI.

Mathematical Institute, Tokyo Imperial University.

(Comm. by S. KAKEYA, M.I.A., April 13, 1942.)

Though the following extension of Bloch's theorem is an immediate consequence of a general theorem due to L. Ahlfors[1], it will not be of no interest to prove it directly, by modifying Landau's proof[2] of Bloch's theorem.

*Theorem. Let $w=f(z)=z+\cdots$ be meromorphic in $|z|\leqq 1$ and $K$ be the Riemann sphere of diameter* 1, *which touches the $w$-plane at $w=0$ and $F$ be the Riemann surface of the inverse function $\varphi(w)$ of $w=f(z)$, spread upon $K$. Then $F$ contains a schlicht circle of radius $\geqq d>0$, where $d$ is a numerical constant.*

*Proof.* We put

$$\operatorname*{Max.}_{|z|\leqq 1} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2)=M, \tag{1}$$

then, since $f(z)=z+\cdots$ is meromorphic in $|z|\leqq 1$, $1\leqq M<\infty$.

Let $\dfrac{|f'(z_0)|}{1+|f(z_0)|^2}(1-|z_0|^2)=M$, then $|z_0|<1$.

By $Z=\dfrac{z-z_0}{1-\bar{z}_0 z}$, $f(z)$ becomes $F(Z)$, so that

$$\frac{|F'(Z)|}{1+|F(Z)|^2}(1-|Z|^2)=\frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2)\leqq M, \tag{2}$$

$$\frac{F'(0)}{1+|F(0)|^2}=Me^{i\theta}. \tag{3}$$

We rotate $K$ so that $F(0)=f(z_0)$ becomes $w=0$, then

$$G(Z)=e^{-i\theta}\frac{F(Z)-F(0)}{1+\bar{F}(0)F(Z)}=MZ+\cdots$$

is meromorphic in $|Z|\leqq 1$. Since

$$\frac{|G'(Z)|}{1+|G(Z)|^2}=\frac{|F'(Z)|}{1+|F(Z)|^2}\leqq\frac{M}{1-|Z|^2},$$

we have by integrating along a radius of $|Z|\leqq 1$,

$$\int_0^Z \frac{|G'(Z)|}{1+|G(Z)|^2}|dZ|\leqq\frac{M}{2}\log\frac{1+|Z|}{1-|Z|}. \tag{4}$$

---

1) L. Ahlfors: Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant a une region donnée. Théorème VI. Acta Societatis Scientiarum Fennicae. Nova Series A. Tom II. (1933).

2) E. Landau: Über die Blochschen Konstante und zwei verwandte Weltkonstanten. Math. Z. **30** (1929).

Since the left hand side of (4) is the length of the curve on $K$, which corresponds to a radius of $|Z|\leqq 1$, if $\frac{M}{2}\log\frac{1+|Z|}{1-|Z|}\leqq\frac{\pi}{4}$ or $|Z|\leqq\frac{e^{\frac{\pi}{2M}}-1}{e^{\frac{\pi}{2M}}+1}$, then $G(Z)$ lies on the lower semi-sphere of $K$, so that $G(Z)$ is regular and $|G(Z)|\leqq 1$ for $|Z|\leqq\frac{e^{\frac{\pi}{2M}}-1}{e^{\frac{\pi}{2M}}+1}$.

Since for $M\geqq 1$,

$$\frac{e^{\frac{\pi}{2M}}-1}{e^{\frac{\pi}{2M}}+1}\geqq\frac{k}{M}\qquad(k=\text{const.}),$$

$G(Z)=MZ+\cdots$ is regular for $|Z|\leqq\frac{k}{M}$ and $|G(Z)|\leqq 1$.

Hence if we put

$$x=\frac{MZ}{k},\qquad v=H(x)=\frac{G(Z)}{k},$$

then

$$v=H(x)=x+\cdots$$

is regular for $|x|\leqq 1$ and $|H(x)|\leqq\frac{1}{k}$, so that $x=v+\cdots$ is regular for $|v|\leqq d_1$ ($d_1=$const.). Returning to $w=f(z)$, we see that the Riemann surface $F$ of the inverse function $\varphi(w)$ of $f(z)$ contains a schlicht circle of radius $\geqq d>0$ ($d=$const.), q. e. d.