68. *Note on Banach Spaces (IV): On a Decomposition of Additive Set Functions.*

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This paper is devoted to prove an abstract decomposition theorem from which flow three types of decompositions. The first is the decomposition theorem concerning cardinal number which is due to R. S. Phillips[1], the second concerns with category, and the last concerns with Lebesgue measure which is due to H. Hahn[2]. The second type seems to be new. In the proof of the theorem of the third type, Pettis' theorem is used. Since the complete proof is not yet published, we give it in the last section.

Throughout this paper, we denote by L an abstract Boolean algebra and by $x(e)$ a completely additive function from L to a Banach space. And finally we suppose I is a σ-ideal in L. Obviously, in the real valued case, our proof is also applicable to bounded, finitely additive set functions.

1. Let $\{e_\alpha\}$ be a set in L such that $\{e_\alpha\}$ is a disjoint system in I, and $x(e_\alpha) \neq 0$ for all e_α[3]. Such $\{e_\alpha\}$ form evidently a system Γ with finite character, thus by the use of Zorn's lemma, Γ contains a maximal collection $\{e_\alpha^1\}$.

Since $\{e_\alpha^1\}$ is at most countable, $a = \bigvee_\alpha e_\alpha^1$ exists and belongs to I. If we put

$$x'(e) = x(a \frown e) \quad \text{and} \quad x''(e) = x(a' \frown e),$$

then $x = x' + x''$. We will now prove the unicity of decomposition. Let $\{e_\alpha^2\}$ be another maximal collection and put $b = \bigvee_\alpha e_\alpha^2$. By the identity $(a \frown e) \smile \{(b-a) \frown e\} = (b \frown e) \smile \{(a-b) \frown e\}$ and $x\{(b-a) \frown e\} = x\{(a-b) \frown e\} = 0$ we have $x(a \frown e) = x(b \frown e)$ for all e.

Summing up above results we get

Theorem 1. For any σ-ideal I in L. We can find an $a \in I$ such that the decomposition

$$x(e) = x(a \frown e) + x(a' \frown e) \tag{1}$$

is unique and the second part vanishes for all elements of the ideal.

2. We will now give applications of Theorem 1.

Let L be a Borel field of subsets of a space, and I be the family of all sets whose cardinal numbers do not exceed an infinite $\aleph$. Then Theorem 1 reads as

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1) R. S. Phillips, Bull. of A. M. S., **46** (1940), 274–277. Idea of our proof is essentially due to him.

2) H. Hahn, *Theorie der reellen Fnnktionen*, 1. Band, Berlin 1921, p. 422.

3) In the proof of Phillips, the last restriction is dropped.

Theorem 2 (of Phillips). *If $\aleph$ is an infinite cardinal number, then completely additive set function is decomposed into two parts as in* (1), *one of which vanishes on a set with cardinal number not greater than $\aleph$.*

Secondly let T be a topological space and L be a Borel field in T including all sets of the first category. Taking I as the system of all sets of the first category, Theorem 1 becomes

Theorem 3. *$x(e)$ can be decomposed into two parts uniquely as in* (1), *one of which vanishes on all sets of the first category.*

Finally, let L be the family of all measurable sets in a space T with respect to a completely additive measure[1], and I be an ideal of null sets, then we have by virtue of Theorem 1 the following

Theorem 4 (of Hahn). *Any completely additive set function is decomposed into the singular and absolutely continuous parts uniquely.*

Theorem 1 proves the vanishing of $x''(e)$ on $|e|=0$. If real valued, x'' becomes absolutely continuous and this is Hahn's decomposition. If B-space valued, we need Pettis' theorem whose proof is given in the next article.

3. Let T be a space with a completely additive measure[1]. We will now prove the following

Theorem 5 (of Pettis). *If $x(e)$ is B-space valued, completely additive and weakly absolutely continuous, then $x(e)$ becomes absolutely continuous in the strong sense.*

Suppose the contrary. Then there exists a sequence of measurable sets $\{e_n\}$ and $\varepsilon > 0$, such that $|e_n| \to 0$ and $|x(e)| > \varepsilon$ $(n=1, 2, \ldots)$. We will now show that there exists a sequence of disjoint sets $\{d_n\}$ such that $|d_n| \to 0$ and $|x(d_n)| > \varepsilon/2$[2]. This proves theorem since $x(e)$ is completely additive[3].

Let $e_1 = c_1$. Evidently $|x(c_1)| > \varepsilon$. By the resonance theorem we can find a linear functional f_0 such that

$$|f_0|=1, \qquad |f_0x(c_1)| = |x(c_1)|.$$

By the absolute continuity of $f_0x(e)$, there is an η_0 such that $|e| < \eta_0$ implies $|f_0x(e)| < \varepsilon/2$. We can find a c_2 in $\{e_n\}$ such as $|c_1 \frown c_2| < \eta_0$. We have $|f_0x(c_1 \frown c_2)| < \varepsilon/2$. By the equalities $(c_1 \frown c_2) \frown (c_1 - c_2) = 0$ and $(c_1 \frown c_2) \smile (c_1 - c_2) = c_1$,

$$f_0x(c_1) = f_0x(c_1 \frown c_2) + f_0x(c_1 - c_2),$$

and thus

1) If the measure of T is infinite, we suppose that T can be covered by an enumerable sequence of measurable sets of finite measure.

2) This is the point of the proof. Kakutani presupposed this in Kunisawa, Proc. **16** (1940), 68–72.

3) Cf. Pettis, Trans. of A. M. S., **44** (1938), 277–304.

$$\varepsilon < |x(c_1)| = |f_0x(c_1)| \leqq |f_0x(c_1 \frown c_2)| + |f_0x(c_1 - c_2)|$$
$$< \varepsilon/2 + |f_0| \cdot |x(c_1 - c_2)| \leqq \varepsilon/2^2 + |x(c_1 - c_2)|.$$

Thus we have

$$|x(c_1 - c_2)| > \varepsilon - \varepsilon/2^2 > \varepsilon/2 \quad \text{and} \quad |x(c_2)| > \varepsilon.$$

Again by the resonance theorem, there exist linear functionals f_1' and f_1'' with norm 1 such

$$|f_1'x(c_1 - c_2)| = |x(c_1 - c_2)|, \qquad |f_1''x(c_2)| = |x(c_2)|.$$

And there exists $\eta_1 > 0$ such that $|e| < \eta_1$ implies $|f_1'x(e)| < \varepsilon/2^3$ and $|f_1''x(e)| < \varepsilon/2^3$. We can also find c_3 in $\{e_n\}$ such as $|c_1 \frown c_3| < \eta_1$ and $|c_1 \frown c_2| < \eta_1$. This implies

$$|f_1'x(c_1 \frown c_2 - c_3)| < \varepsilon/2^3, \qquad |f_1''x(c_2 \frown c_3)| < \varepsilon/2^3.$$

Thus

$$\varepsilon - \varepsilon/2^2 < |x(c_1 - c_2)| = |f_1'x(c_1 - c_2)|$$
$$\leqq |f_1'x(c_1 - c_2 \smile c_3)| + |f_1'x(c_1 \frown c_3 - c_2)| \leqq |x(c_1 - c_2 \smile c_3)| + \varepsilon/2^3.$$

and also

$$\varepsilon < |x(c_2)| \leqq |x(c_2 - c_3)| + \varepsilon/2^3.$$

Hence

$$|x(c_1 - c_1 \smile c_2)| > \varepsilon - \varepsilon/2^2 - \varepsilon/2^3 > \varepsilon/2,$$
$$|x(c_2 - c_3)| > \varepsilon - \varepsilon/2^3 > \varepsilon/2, \qquad |x(c_3)| > \varepsilon.$$

Continuing this process, we obtain the sequence $\{c_n\}$ with the property:

$$|x(c_1 - \bigvee_{i=2}^{n} c_i)| > \varepsilon/2,$$
$$|x(c_2 - \bigvee_{i=3}^{n} c_i)| > \varepsilon/2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|x(c_n)| > \varepsilon/2.$$

Let us put $d_n = c_n - \bigvee_{i=n+1}^{\infty} c_i$ $(n = 1, 2, \ldots)$. Then $\{d_n\}$ is a disjoint system with the required property.