87. On the Function whose Imaginary Part on the Unit Circle Changes its Sign only Twice.

By Sôichi KAKEYA, M.I.A.
Mathematical Institute, Faculty of Science, Tokyo Imperial University.
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I. We are going to consider the function

$$f(z)=\sum_{n=1}^{\infty} c_n z^n = c_1 z + c_2 z^2 + \cdots \tag{1}$$

which is *regular within the unit circle and is continuous*, for simplicity, *to the boundary*. Putting

$$z=re^{i\theta}, \qquad f(z)=u(r,\theta)+iv(r,\theta) \tag{2}$$

we confine ourselves to the function which *satisfies one of the following two conditions*:

$$\left.\begin{aligned} v(1,\theta)=v(\theta) &\geqq 0 \quad \text{for} \quad \sigma_1 \leqq \theta \leqq \sigma_2 \\ &\leqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq \sigma_1 \quad \text{and} \quad \sigma_2 \leqq \theta \leqq 2\pi \end{aligned}\right\} \tag{3}$$

or

$$\left.\begin{aligned} v(\theta) &\leqq 0 \quad \text{for} \quad \sigma_1 \leqq \theta \leqq \sigma_2 \\ &\geqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq \sigma_1 \quad \text{and} \quad \sigma_2 \leqq \theta \leqq 2\pi \end{aligned}\right\} \tag{4}$$

namely the imaginary part of $f(z)$ on the unit circle $|z|=1$ may change its sign only at two points $e^{i\sigma_1}$ and $e^{i\sigma_2}$. $(0 \leqq \sigma_1 < \sigma_2 \leqq 2\pi)$.

It is easily to be seen that the function

$$g(z)=e^{-i\frac{\sigma_1+\sigma_2}{2}} \times \frac{(e^{i\sigma_1}-z)(e^{i\sigma_2}-z)}{z} \tag{5}$$

becomes positive on the unit circle for $\sigma_1 < \theta < \sigma_2$ and negative for the remaining arc. Hence the function

$$\begin{aligned} F(z)=\varepsilon f(z)g(z) &= \sum_{n=0}^{\infty} C_n z^n = C_0 + C_1 z + C_2 z^2 + \cdots \\ &= U(r,\theta)+iV(r,\theta) \end{aligned} \tag{6}$$

which is evidently continuous in the closed unit circle, must have the property

$$V(1,\theta)=V(\theta) \geqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq 2\pi \tag{7}$$

if ε denotes $+1$ or -1 according as $f(z)$ satisfies the condition (3) or (4).

By the actual multiplication of $F(z)$ and

$$\frac{1}{g(x)} = e^{i\frac{\sigma_1+\sigma_2}{2}} \times \frac{z}{(e^{i\sigma_1}-z)(e^{i\sigma_2}-z)} = \frac{1}{2i \sin \dfrac{\sigma_2-\sigma_1}{2}} \sum_{n=1}^{\infty} (e^{-in\sigma_1} - e^{-in\sigma_2}) z^n \tag{8}$$

we obtain

$$c_n=\frac{\varepsilon}{2i\sin\frac{\sigma_2-\sigma_1}{2}}\{(e^{-i\sigma_1}-e^{-i\sigma_2})C_{n-1}+(e^{-2i\sigma_1}-e^{-2i\sigma_2})C_{n-2}+\cdots+(e^{-ni\sigma_1}-e^{-ni\sigma_2})C_0\} \tag{9}$$

$$n=1,2,3.\dots$$

On the other hand, if we put

$$C_n=\alpha_n+i\beta_n\,,\qquad n=0,1,2,\dots \tag{10}$$

we get, by the well known formulas

$$\beta_0=\frac{1}{2\pi}\int_0^{2\pi}V(\theta)d\theta \tag{11}$$

$$\alpha_n=\frac{1}{\pi}\int_0^{2\pi}\sin n\theta\, V(\theta)d\theta \qquad n=1,2,\dots \tag{12}$$

$$\beta_n=\frac{1}{\pi}\int_0^{2\pi}\cos n\theta\, V(\theta)d\theta \qquad n=1,2,\dots \tag{13}$$

We now *assume*, for simplicity, *that*

$$c_1=1 \tag{14}$$

which infers, from (9),

$$\alpha_0=\varepsilon\cos\frac{\sigma_1+\sigma_2}{2}\,,\qquad \beta_0=\varepsilon\sin\frac{\sigma_1+\sigma_2}{2} \tag{15}$$

so that

$$\varepsilon\sin\frac{\sigma_1+\sigma_2}{2}=\frac{1}{2\pi}\int_0^{2\pi}V(\theta)d\theta\,. \tag{16}$$

Substituting (12), (13), (15) and (16) to (9), it follows

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(\theta,\sigma_1,\sigma_2)V(\theta)d\theta\,,\qquad n=1,2,\dots \tag{17}$$

where

$$\varphi_n(\theta,\sigma_1,\sigma_2)=\frac{\varepsilon}{\sin\frac{\sigma_2-\sigma_1}{2}}\left\{e^{-i(\sigma_1+\theta)}\times\frac{e^{-i(n-1)\sigma_1}-e^{-i(n-1)\theta}}{e^{-i\sigma_1}-e^{-i\theta}}-e^{-i(\sigma_2+\theta)}\times\frac{e^{-i(n-1)\sigma_2}-e^{-i(n-1)\theta}}{e^{-i\sigma_2}-e^{-i\theta}}+\frac{e^{-in\sigma_1}-e^{-in\sigma_2}}{1-e^{-i(\sigma_1+\sigma_2)}}\right\} \tag{18}$$

From (7), (16) and (17), we see that the domain D_n within which c_n should lie is the smallest convex *open* domain containing the curve described by

$$\varepsilon\sin\frac{\sigma_1+\sigma_2}{2}\varphi_n(\theta,\sigma_1,\sigma_2)\,,\qquad 0\leqq\theta\leqq 2\pi^{1)} \tag{19}$$

σ_1, σ_2 being fixed. Especially the upper limit of $|c_n|$ is equal to the maximum of

1) See the author's paper "On some integral equations—II," Proc. Math. Phys. Soc. Tôkyô, Ser. 2, **8** (1915).

$$\left| \sin \frac{\sigma_1+\sigma_2}{2} \varphi_n(\theta, \sigma_1, \sigma_2) \right| \tag{20}$$

with respect to θ. Let it be $G_n(\sigma_1, \sigma_2)$.

If we put
$$e^{-i\theta}=t \tag{21}$$

the expression (19) becomes

$$\frac{1-e^{-i(\sigma_1+\sigma_2)}}{e^{-i\sigma_1}-e^{-i\sigma_2}} \left\{ te^{-i\sigma_1} \frac{t^{n-1}-e^{-i(n-1)\sigma_1}}{t-e^{-i\sigma_1}} - te^{-i\sigma_2} \frac{t^{n-1}-e^{-i(n-1)\sigma_2}}{t-e^{-i\sigma_2}} + \frac{e^{-in\sigma_1}-e^{-in\sigma_2}}{1-e^{-i(\sigma_1+\sigma_2)}} \right\} \tag{22}$$

and $G_n(\sigma_1, \sigma_2)$ is the maximum magnitude of (22) with respect to $|t|=1$.

Thus we get

Theorem 1. If the function

$$f(z)=z+c_2z^2+\cdots \tag{23}$$

which is continuous in the closed unit circle, has the imaginary part $v(\theta)$ for $z=e^{i\theta}$, satisfying either the condition (3) *or* (4), *then we must have*

$$|c_n| < G_n(\sigma_1, \sigma_2) \tag{24}$$

For example, if we assume

$$\sigma_1=0, \qquad \sigma_2=\pi \tag{25}$$

namely that both of $|z|=1$ and its image of $f(z)$ are divided into two corresponding arcs by the real axes, then we get

$$\begin{aligned} G_n(\sigma_1, \sigma_2) &= G_n(0, \pi) \\ &= \operatorname*{Max}_{|t|=1} \left| t\frac{t^{n-1}-1}{t-1} + t\frac{t^{n-1}-(-1)^{n-1}}{t-(-1)} + \frac{1-(-1)^n}{1-(-1)} \right| = n \end{aligned} \tag{26}$$

This is a result once obtained by Mr. Ozaki[1].

If we let σ_1 and σ_2 vary themselves, then the maximum G_n of $G_n(\sigma_1, \sigma_2)$ is the absolute upper limit of $|c_n|$ in our case. Putting

$$e^{-i\sigma_1}=tx, \qquad e^{-i\sigma_2}=ty \tag{27}$$

we get, from (22),

$$\begin{aligned} G_n &= \operatorname*{Max}_{|x|, |y|, |t|=1} \left| \left\{ x\frac{1-x^{n-1}}{1-x} - y\frac{1-y^{n-1}}{1-y} \right\} \frac{1-t^2xy}{x-y} + \frac{x^n-y^n}{x-y} \right| \\ &= \operatorname{Max} |1+(x+y)+(x^2+xy+y^2)+\cdots+(x^{n-1}+x^{n-2}y+\cdots+y^{n-1}) \\ &\qquad -t^2xy\{1+(x+y)+\cdots+(x^{n-2}+x^{n-3}y+\cdots+y^{n-2})\}| \\ &= (1+2+3+\cdots+n)+\big(1+2+\cdots+(n-1)\big)=n^2 \end{aligned} \tag{28}$$

Hence the following theorem has been proved.

1) Science Reports, Tokyo Bunrika Daigaku. **4** (1941), p. 79.

Theorem 2. If the function (23), *which is continuous in the closed unit circle, has the imaginary part* $v(\theta)$ *for* $z=e^{i\theta}$ *which may change its sign at most twice in the interval* $0 \leqq \theta \leqq 2\pi$, *then we must have*

$$|c_n| < n^2 \tag{29}$$

II. Some remarks are to be mentioned.

From (7) and (16), we must have

$$\varepsilon \sin \frac{\sigma_1+\sigma_2}{2} \geqq 0 \tag{30}$$

The equality sign should occur only when $V(\theta)\equiv 0$, so that

$$\alpha_0 = \pm 1, \quad \beta_0 = 0, \quad \alpha_n = \beta_n = 0 \quad (n>0) \tag{31}$$

namely $F(z)=\pm 1$ or

$$f(z) \equiv \frac{1}{g(z)} e^{i\frac{\sigma_1+\sigma_2}{2}} \qquad \left(\frac{\sigma_1+\sigma_2}{2}=0 \text{ or } \pi\right) \tag{32}$$

In this case, $v(\theta)$ becomes discontinuous. Hence we see that, *under our condition, it is necessary that*

$$\varepsilon \sin \frac{\sigma_1+\sigma_2}{2} > 0 \tag{33}$$

which was tacitly assumed in the preceding discussion.

We have also assumed previously that $c_1=1$. But we can apply the result to the general case, under *the only condition*

$$c_1 \neq 0 \tag{34}$$

In this case, we are to put

$$c_1 = \rho e^{i\omega}, \qquad e^{i\omega} z = \xi \tag{35}$$

so that the function (1) can be written in the form

$$\begin{aligned} f(z) &= \rho e^{i\omega} z + c_2 z^2 + \cdots \\ &= \rho\left\{\xi + \frac{c_2}{\rho e^{2i\omega}}\xi^2 + \cdots\right\} = \rho\varphi(\xi) \end{aligned} \tag{36}$$

Then $\varphi(\xi)$ is of the form (23) and its imaginary part on the unit circle may change its sign only at the points $e^{i(\sigma_1+\omega)}$ and $e^{i(\sigma_2+\omega)}$. Hence the theorem 1 shows that

$$\left|\frac{c_n}{\rho e^{ni\omega}}\right| = \left|\frac{c_n}{c_1}\right| < G_n(\sigma_1+\omega, \sigma_2+\omega) \tag{37}$$

and the theorem 2 shows that

$$\left|\frac{c_n}{c_1}\right| < n^2 \tag{38}$$

By the direct multiplication of the series (5) and (23), we get

$$C_0 = e^{i\frac{\sigma_1+\sigma_2}{2}}, \qquad C_1 = c_2 e^{i\frac{\sigma_1+\sigma_2}{2}} - (e^{i\frac{\sigma_1-\sigma_2}{2}} + e^{i\frac{\sigma_2-\sigma_1}{2}}) \tag{39}$$

$$C_n = c_{n+1} e^{i\frac{\sigma_1+\sigma_2}{2}} - c_n(e^{i\frac{\sigma_1-\sigma_2}{2}} + e^{i\frac{\sigma_2-\sigma_1}{2}}) + c_{n-1} e^{-i\frac{\sigma_1+\sigma_2}{2}} \tag{40}$$

$$n = 2, 3, \dots \qquad (c_1 = 1)$$

On the other hand (12), (13) and (16) show that the point $C_n = \alpha_n + i\beta_n$, $(n > 0)$ should lie within the circle described by

$$2\varepsilon \sin\frac{\sigma_1+\sigma_2}{2}(\sin n\theta + i\cos n\theta) \qquad 0 \leqq \theta \leqq 2\pi \tag{41}$$

So we get $$|C_n| < 2\left|\sin\frac{\sigma_1+\sigma_2}{2}\right|, \qquad n = 1, 2, \dots \tag{42}$$

Substituting in the place of C_n the right hand member of (39) or (40), we obtain *a set of inequalities satisfied by* $c_2, c_3, \dots$

The constant $G_n(\sigma_1, \sigma_2)$ of theorem 1, so also n^2 of theorem 2, is *the smallest possible number satisfying the said inequality.* For we can so take the imaginary part $v(\theta)$ of $f(z)$, hence the function $f(z)$ itself, that the imaginary part $V(\theta)$ of $F(z)$ should correspond to a constant as near to $G_n(\sigma_1, \sigma_2)$ as we please. Such $V(\theta)$ can be same for all n in the case of theorem 2, so that any finite number of $|c_n|$'s can be, at the same time, as near to n^2's respectively as we please.