110. *On the Semi-ordered Ring and its Application to the Spectral Theorem.*

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(Comm. by T. TAKAGI, M.I.A., Nov. 12, 1942.)

This note deals with some remarks about semi-ordered rings and their application to the spectral theorem. Semi-ordered rings have been treated jointly by Messrs. I. Vernikoff, S. Krein and A. Tovbin[1]. We first observe that for their result the assumption of the associative law of multiplication is unnecessary; it follows, as the commutativity, from the other axioms, and this fact will be of use in applications. Further, their theorems may be obtained rather easily also from Clifford-Lorenzen's theorem concerning semi-ordered abelian groups[2] by considering operator-domains. As for application, we prove a spectral theorem in the semi-ordered rings without appealing to the spectral theorem in vector lattice but relying upon Baire's category theorem. We thus obtain a new approach to the spectral theorem for bounded self-adjoint operators in a Hilbert space.

1. *Elementary observations about semi-ordered abelian groups.* Let G be a semi-ordered abelian group, that is, an abelian group which possesses a semi-order $x \geqq y$ (equivalent to $x-y \geqq 0$) such that

(i) if $x \geqq 0$ and $y \geqq 0$ then $x+y \geqq 0$,

(ii) if $x \geqq 0$ and $-x \geqq 0$ then $x=0$.

We assume a further condition:

(iii) if $nx \geqq 0$ for a certain natural number n, then $x \geqq 0$.

Let moreover G possess an Archimedean unit e:

(iv) $\begin{cases} \text{for any } x \text{ there exists a natural number } n=n(x) \text{ such that} \\ -ne \leqq x \leqq ne. \end{cases}$

And we call the totality N of those elements x in G satisfying $-e \leqq tx \leqq e$ (for every $t=1, 2, \dots$) the radical of G. N is a normal subgroup[3] of G, and the factor group G/N is also a semi-ordered group.

In virtue of the condition (iii) G is, according to Clifford-Lorenzen's

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1) Sur les anneaux semi-ordonnés, C. R. URSS, **30** (1941). Cf. also H. Nakano Teilweise geordnete Algebra, Jap. J. Math., **17** (1941).

2) A. H. Clifford: Partially ordered abelian groups, Ann. of Math., **41** (1940). P. Lorenzen: Abstracte Begründung der multiplikativen Idealtheorie, Math. Zeitschr., **45** (1939).

3) Here we call a subgroup of G normal when it is a kernel of an order-homomorphism of G. Thus a subgroup H is normal if and only if $x \in H$, $0 \leqq y \leqq x$ implies $y \in H$.

theorem[4], order-isomorphically embedded in a direct sum of linearly ordered groups:

$$G \leqq \cdots G_\sigma + \cdots, \quad G \ni x \leftrightarrow (\dots, x_\sigma, \dots) \quad (x_\sigma \in G_\sigma);$$

here the order relation in the direct sum is explained, as usual, component-wise, and without losing generality we may suppose that when x runs over G its component x_σ exhausts G_σ. The image (component) e_σ of e is, for each σ, an Archimedean unit of G_σ, and we denote the radical of G_σ by N_σ. Evidently an element x belongs to N when and only when x_σ belongs to N_σ for every σ. Thus the group G/N is embedded isomorphically in the direct sum $\bar{G}_\sigma = G_\sigma/N_\sigma$:

$$x \bmod. N \leftrightarrow (\dots, x_\sigma \bmod. N_\sigma, \dots).$$

The order relation is preserved in the direction "$\rightarrow$". Further, each $\bar{G}_\sigma = G_\sigma/N_\sigma$ is, as an Archimedean linearly ordered group, a subgroup of the ordered group of real numbers, and the kernel M_σ of the homomorphism $G \rightarrow G_\sigma$ is a maximal normal subgroup of G. If, in particular, N consists of 0 only, that is, if the condition:

(v) $\qquad$ if $\quad -e \leqq tx \leqq e \quad (t=1, 2, \dots) \quad$ then $\quad x=0$,

is satisfied, *then G itself is mapped isomorphically into the direct sum of the Archimedean linearly ordered groups $\bar{G}_\sigma$, order being preserved in the direct direction "$\rightarrow$"*. Furthermore, if the stronger condition:

(vi) $\qquad$ if $\quad tx \leqq e \quad (t=1, 2, \dots) \quad$ then $\quad x \leqq 0$,

is fulfilled, the order relation is preserved in the both directions "$\rightarrow$" and "$\leftarrow$", that is, *G is order-isomorphically embedded in the direct sum of $\bar{G}_\sigma$*. For, if $x_\sigma \leqq 0_\sigma \bmod. N_\sigma$ then $x_\sigma \leqq z_\sigma$ for a certain element z_σ in N_σ, whence $tx_\sigma \leqq tz_\sigma \leqq e_\sigma$ for every natural number t. When this is the case for every σ, the assumption in (vi) is fulfilled and we have $x \leqq 0$. Now, suppose that G possesses a domain of operators $\Omega = \{A\}$, which is by itself a semi-ordered abelian group, satisfying the axioms (i), (ii) and such that

(vii) if $\quad x \geqq 0 \quad$ (in G), $\quad A \geqq 0 \quad$ (in Ω) $\quad$ then $\quad Ax \geqq 0 \quad$ (in G),

4) See 2). As the proof suggested by Clifford shows, we may take as G_σ linearly ordered groups which are (not order-but) group-isomorphic with G. Namely: consider subsets P in G which satisfy the conditions: i) if $x > 0$ then $x \in P$, ii) if $x \in P$, $y \in P$ then $x+y \in P$, iii) $0 \bar{\in} P$, iv) if $mx \in P$ $(m > 0)$ then $x \in P$. When P is such a subset (for instance, the set of all the positive elements of G), we can re-order G by calling the elements of P positive; denote the semi-ordered abelian group thus obtained by $G(P)$. Further, if there is an element x in G such that neither x nor $-x$ is contained in P, then there exists a second P which contains P and x both. We may, for example, take the set $\underset{z}{\mathfrak{E}}(z;\, kz = mp + nx,\ k > 0,\ m \geqq 0,\ n \geqq 0,\ m+n > 0)$ for a new P. From this follows that if P is a maximal such subset then $G(P)$ is linearly ordered. Moreover any non-zero and non-negative element is contained in at least one maximal such subset P, whence positive in $G(P)$. Thus G is order-isomorphically embedded in the direct sum of $G(P)$'s, P running over all the maximal such subsets in G, by letting x correspond for each P to x itself in $G(P)$.

(viii) $(A+B)x=Ax+Bx$, $A(x+y)=Ax+Ay$.

Let moreover

(ix) $\left\{\begin{array}{l}\Omega \text{ possess an Archimedean unit } I \text{ which satisfies } Ix=x \\ \text{for every } x\in G.\end{array}\right.$

Then we have the *Lemma: Every normal subgroup of G is allowable with respect to Ω. Proof:* Let H be a normal subgroup in G, and let $x\in H$, $A\in\Omega$. There is a natural number n such that $-nI\leqq A\leqq nI$ whence $-nIx\leqq Ax\leqq nIx$. Thus $-nx\leqq Ax\leqq nx$ and Ax belongs to H too. In particular the maximal normal subgroups M_σ are Ω-allowable, and the above isomorphisms are operator-isomorphisms with respect to Ω.

2. *Semi-ordered rings.* Let R be a ring with a unit element e and real multipliers. Neither the commutativity nor the associativity of the multiplication is assumed. Let in R be defined a semi-order such that

(I) if $x\geqq 0$ and $y\geqq 0$ than $x+y\geqq 0$ and $xy\geqq 0$,

(II) if $x\geqq 0$ and $-x\geqq 0$ then $x=0$,

(III) if $x\geqq 0$ and α (real number) $\geqq 0$ then $\alpha x\geqq 0$.

Further we assume that the ring unit e is an Archimedean unit:

(IV) $\left\{\begin{array}{l}\text{for any } x \text{ there is a positive number } \alpha=\alpha(x) \text{ such} \\ -\alpha e\leqq x\leqq \alpha e.\end{array}\right.$

Then R satisfies, considered as an abelian group possessing R itself as an (either left or right) operator domain, the conditions (i)–(iv), (vii)–(ix) of the preceding section. And, every residue class mod. an $M=M_\sigma$ is represented by a multiple αe. Hence, we have, expressing the condition (v), (vi) in a modified fashion,

Theorem 1. Let R satisfy, besides the above conditions,

(V) *if* $-\frac{1}{t}e\leqq x\leqq \frac{1}{t}e$ $(t=1,2,\dots)$ *then* $x=0$.

Then R is ring-isomorphic to a certain ring $R(\mathfrak{M})$ of (real-valued bounded functions over a certain space $\mathfrak{M}=\{M\}$: $x\leftrightarrow x(M)$, such that e is represented by 1: $e(M)\equiv 1$. In particular, R is both associative and commutative. The order is preserved in the direction $x\rightarrow x(M)$, when we order $R(\mathfrak{M})$ in the usual manner.

Theorem 2. If R satisfies the stronger condition:

(VI)[5] $\inf\limits_{t\to 0}\frac{1}{t}e$ *exists and is equal to* 0,

then the order is preserved in the both directions, so that R is ring-order-isomorphic to $R(\mathfrak{M})$.

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5) Whence for every x in R an order-limit $\frac{1}{t}x$ exists and $=0$.

Here, according to our construction, $\mathfrak{M}$ is a certain set of maximal normal[6] ideals of R, but not necessarily all of them. However, the theorems are still the more true if $\mathfrak{M}$ represents the totality of the maximal normal ideals of R. So, assume this be the case. Then $\mathfrak{M}$ is a bicompact Hausdorff space by the so-called *weak topology*, under which the functions in R are continuous. Furthermore, since 1 is contained in $R(\mathfrak{M})$ and since there exists for any two distinct points M, M' in $\mathfrak{M}$ an element x in R such that $x(M) \neq x(M')$, the ring $R(M)$ is dense in the ring of all the continuous functions on $\mathfrak{M}$ with respect to the metric defined by the greatest absolute value taken by a function as its norm[7] Hence

Theorem 3. Let R satisfy besides (I)–(VI) *the condition:*

(VII) *R is a Banach space by the norm* $\|x\| = \inf(-\alpha e \leqq x \leqq \alpha e)$.

Then R is ring-order-isomorphic to the ring $R(\mathfrak{M})$ of all the continuous functions over a bicompact Huasdorff space $\mathfrak{M}$. In particular, R is a vector lattice, viz. lattice-ordered abelian group with real multipliers.

Remark. Let, conversely, R be a vector lattice which satisfies (I)$'$–(VII):

(I) if $x \geqq 0$ and $y \geqq 0$ then $x+y \geqq 0$.

Such a vector lattice R is called, by S. Kakutani[8], an abstract (M) space. We may, following after F. Riesz and Y. Kawada[9], define a multiplication xy in R by

$$4xy = (x+y)^2 - (x-y)^2, \quad x^2 = \sup_{\lambda > 0}(2\lambda|x| - \lambda^2 e), \quad x = \sup(x, 0) - \inf(x, 0).$$

It is easy to see that R now satisfies the axioms (I)–(VIII). In this way, *the equivalence of the semi-ordered ring and the abstract (M) space may be proved appealing neither to the spectral theorem of H. Freudenthal*[10] *nor to the representation theorem*[11] *of the abstract (M) space.*

Another method of reducing the above theorems of our semi-ordered rings to the known results is, in case the condition (vi) is satisfied, to complete by cuts and apply the representation theory of vector lattices and lattice-ordered rings[12]. When we have only the

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6) Defined similarly as in 3). "Fundamental" in the sense of Vernikoff-Krein-Tovbin, loc. cit.

7) See H. Nakano: 連續函數ノ ring 及ビ vector lattice, 全國紙上數學談話會, **218** (1941).

8) Weak topology, bicompact set and the principle of duality, Proc. **16** (1940). See also the literatures referred to in K. Yosida: On the representation of the vector lattice, Proc., **18** (1942).

9) F. Riesz: Sur la théorie ergodique des espaces abstraits, Acta Szeged, **10** (1941), 1. ,Y. Kawada: 抽象 M-空間ノ表現ニ就テ, 全國紙上數學談話會, **227** (1941).

10) Teilweise geordnete Moduln, Proc. Amsterdam Acad., **39** (1936).

11) See 8).

12) B. Vulich: Une définition du produit dans les espaces semiordonnés linéaires, C. R. URSS., **26** (1940). H. Nakano: loc. cit. in 1). T. Ogasawara: Ring lattice 公理系及ビ表現論, 全國紙上數學談話會, **230** (1942).

condition (v), then we have first to re-order the ring by Vernikoff-Krein-Tovbin's procedure so as to have (vi).

3. *An abstract spectral theorem.* Let R be a semi-ordered ring, again neither associativity nor commutativity being assumed, satisfying the conditions (I)-(IV), (VII) and, furthermore,

(VIII) $\begin{cases} \text{for any increasing sequence } \{x_n\} \text{ bounded from above} \\ (x_1 \leqq x_2 \leqq \cdots \leqq y), \quad \sup x_n = \text{order-limit } x_n \text{ exists in } R. \end{cases}$

Then we have the

Theorem 4. There exists, for any $x \in R$, a resolution of the identity $\{e_\lambda\}$ with the properties:

(1) $e_\lambda^2 = e_\lambda \leqq e_\mu = e_\mu^2$ *if* $\lambda \leqq \mu$,

(2) *if* $\lambda_n \downarrow \lambda$, *then order-limit* $e_{\lambda_n} = e_\lambda$,

(3) $e_\lambda = e$ *for* $\lambda \geqq \|x\|$ *and* $e_\lambda = 0$ *for* $\lambda < -\|x\|$,

(4) $\begin{cases} \textit{for ang } \varepsilon > 0, \quad x = \int_{-\|x\|-\varepsilon}^{\|x\|} \lambda de_\lambda \quad \textit{(Riemann-Stieltjes integral} \\ \textit{in semi-order sense),} \end{cases}$

(5) $\{e_\lambda\}$ *is determined uniquely by the properties* (1)-(4).

Proof. By the theorem 3, there exists a bicompact Hausdorff space $\mathfrak{M}$ such that R is ring-order isomorphic to the ring $R(\mathfrak{M})$ of all the continuous functions on $\mathfrak{M}$. Let the isomorphism be given by $x \leftrightarrow x(M)$. We will prove the following property of the representation $R \to R(\mathfrak{M})$. Let $x_1 \leqq x_2 \leqq \cdots \leqq y$ and let order-limit $x_n = x$. By Baire's theorem, the discontinuities of the function $\bar{x}(M) = \lim_{n\to\infty} x_n(M)$ constitute a set of first category, viz. enumerable sum of non-dense sets. We have surely $x(M) \geqq \bar{x}(M)$. *In the truth, the set* $\mathop{\mathfrak{E}}_{M}\big(x(M) - \bar{x}(M) > 0\big)$ *is of first category. Proof:* If otherwise, we would have a point M_0 such that $x(M)$ is continuous at M_0 and $x(M_0) > \bar{x}(M_0)$, and thus we would obtain a continuous function $x^*(M)$ such that $x(M_0) > x^*(M_0)$ and $x(M) \geqq x^*(M) \geqq \bar{x}(M)$ on $\mathfrak{M}$. This contradicts to the isomorphism $R \leftrightarrow R(\mathfrak{M})$ and the definition of x as order-limit x_n. Here use is made of the fact that a bicompact Hausdorff space is not of first category.

Next consider the set $R'(\mathfrak{M})$ of all the bounded functions $x'(\mathfrak{M})$ on $\mathfrak{M}$ such that $x'(M)$ is different from a continuous function $x(M)$ only on a set of first category. We then identify two functions from $R'(\mathfrak{M})$ if they differ on a set of first category. Thus $R'(\mathfrak{M})$ is divided into classes, each class x' containing exactly one continuous function $x(M)$ which corresponds to an element $x \in R$ by the isomorphism $R \leftrightarrow R(\mathfrak{M})$. This results from the fact that the complementary to a set of first category is dense on the bicompact Hausdorff space $\mathfrak{M}$.

The proof of the theorem is now immediate. We have only to put $e_\lambda =$ the element $\in R$ which corresponds to the class containing the characteristic function $e'_\lambda(M)$ of the set $\mathop{\mathfrak{E}}_{M}\big(x(M) \geqq \lambda\big)$. For then we would have $|x(M)-\sum_{i=1}^{n}\lambda_i e'_{\lambda_i}(M)|\leqq\varepsilon$ and thus $|x(M)-\sum_{i=1}^{n}\lambda_i e_{\lambda_i}(M)|\leqq\varepsilon$, viz. $-\varepsilon e\leqq x-\sum_{i=1}^{n}\lambda_i e_{\lambda_i}\leqq\varepsilon e$ for $\lambda_1=-\|x\|-\varepsilon<\lambda_2<\lambda_3<\cdots<\lambda_n=\|x\|$, $\max_i(\lambda_{i+1}-\lambda_i)\leqq\varepsilon$. Perhaps the fact that $e'_\lambda(M)\in R'(\mathfrak{M})$ will demand proof. However the function $e'_\lambda(M)$ defined by $1-\sup_{n\geqq1}\Big(\inf\big(1,n\big(x(M)-\lambda\big)^+\big)\Big)$ belongs to $R'(\mathfrak{M})$ by the above property of the representation $R\rightarrow R(\mathfrak{M})$.

4. *Application to the Hilbert space.* Let (T) be a set of mutually commutative, bounded self-adjoint operators in Hilbert space $\mathfrak{H}$, and denote by $(T)'$ the totality of the bounded self-adjoint operators commutative with every operator $\in(T)$. Similarly we-define $(T)''=\big((T)'\big)'$, $(T)'''=\big((T)''\big)'$ etc. $R=(T)''$ is a ring with unit operator (=the identiy operator) I and is commutative, since from $(T)\leqq(T)'$ we obtain $(T)'\geqq(T)''$ and hence $(T)'''\geqq(T)''$. We define a semi-order in R by writing $T\geqq0$ if and only if $(T\cdot f,f)\geqq0$ for all $f\in\mathfrak{H}$. Then R satisfies (I)–(IV), (VIII). Hence the theorem 4 is directly applicable to R. Only the proof of the axioms (I) and (VIII) would be non-trivial. However these may be proved following after F. Riesz's idea[13].

Remark. The above procedure also gives a simultaneous resolutions $T=\int\lambda dE_\lambda(T)$, $S=\int\lambda dE_\lambda(S)$ such that $E_\lambda(T)E_\mu(S)=E_\mu(S)E_\lambda(T)$, if T and S and mutually commutative bounded self-adjoint operators. Hence our method also gives the spectral theorem of the bounded *normal* (and of course *unitary*) operators, for such operators are of the form $T=\sqrt{-1}\,S$, where T and S are mutually commutative, bounded self-adjoint operators.

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13) Über die linearen Transformationen des komplexen Hilbertschen Raumes, Acta Szeged, **5** (1930). Namely: *Ad. (I).* It will be sufficient to show that $TS\geqq0$, if $I\geqq T$, $S\geqq0$. Put $T_1=T$, $T_{n+1}=T_n-T_n^2$ $(n\geqq1)$. Then we obtain $I\geqq T_n\geqq0$ $(n\geqq1)$ by induction, because of the identities $T_{n+1}=T_n^2\ (I-T_n)+T_n\ (I-T_n)^2$, $I-T_{n+1}=(I-T_n)+T_n^2$. Hence $T\geqq\sum_{m=1}^{n}T_m^2$ $(n\geqq1)$ and thus $\lim\|T_n\cdot f\|^2=\lim(T_n^2\cdot f,f)=0$, proving $T=\sum_{m=1}^{\infty}T_m^2$. Similarly we have $S=\sum_{m=1}^{\infty}S_m^2$ and thus $TS=\sum_{i,j}T_i^2S_j^2=\sum_{i,j}(T_iS_j)^2\geqq0$. *Ad. (VIII).* We will prove the existence of the order-limit $T_n=T$ from $0\leqq T_1\leqq T_2\leqq\cdots\leqq S$. By (I), $\{(T_n^2\cdot f,f)\}$ is a bounded increasing sequence for any f, and hence $\lim(T_n^2\cdot f,f)$ exists. We have, again by (I), $T_{n+k}^2\geqq T_{n+k}T_n\geqq T_n^2$. Thus $\lim(T_{n+k}^2\cdot f,f)=\lim(T_n^2\cdot f,f)=\lim(T_{n+k}T_n\cdot f,f)$ and hence $\lim\big((T_n-T_m)^2\cdot f,f\big)=\lim\|T_n\cdot f-T_m\cdot f\|^2=0$. Therefore the *strong limit* $T_n\cdot f=T\cdot f$ exists. T is surely the order-limit T_n.