14. *On the Uniform Distribution of Values of a Function mod. 1.*

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1. *Uniform distribution of values of $f(x)$ mod. 1.*

Let $f(x)$ be a continuous function defined for $0 \leqq x < \infty$ and $\big(f(x)\big) = f(x) - [f(x)]$, so that $0 \leqq \big(f(x)\big) < 1$. Let $\mathfrak{A} = [\alpha, \beta]$ $(0 \leqq \alpha < \beta \leqq 1)$ be an interval in $[0, 1]$ and $E(r, \mathfrak{A})$ be the set of points x on the x-axis, which lie in $[0, r]$, such that $\alpha \leqq \big(f(x)\big) \leqq \beta$ and $mE(r, \mathfrak{A})$ be its measure. If for any $\mathfrak{A}$,

$$\lim_{r\to\infty} \frac{mE(r, \mathfrak{A})}{r} = |\mathfrak{A}| = \beta - \alpha, \tag{1}$$

then we say that the values of $f(x)$ distribute uniformly mod. 1.

H. Weyl[1] proved that (I) the necessary and sufficient condition, that the values of $f(x)$ distribute uniformly mod. 1 is that

$$\int_0^r e^{2\pi a i f(x)} dx = o(r), \tag{2}$$

for any integer a $(\neq 0)$.

(II) Let $F(t)$ be periodic with period 1 and be integrable in Riemann's sense in $[0, 1]$. If the values of $f(x)$ distribute uniformly mod. 1, then

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r F\big(f(x)\big) dx = \int_0^1 F(t) dt. \tag{3}$$

We will prove

Theorem I. Let $f(x)$ be a positive continuous increasing convex function of $\log x$, *such that* $\lim\limits_{x\to\infty} \dfrac{f(x)}{\log x} = \infty$, *then the values of $f(x)$ distribute uniformly mod. 1.*

Proof. Let a $(\neq 0)$ be an integer and put $t = 2\pi a f(x) = \varphi(x)$ and $x = \psi(t)$ be its inverse function. We suppose that $a > 0$; the case $a < 0$ can be proved similarly. From the convexity of $f(x)$ as a function of $\log x$, $x\varphi'(x) = \dfrac{\psi(t)}{\psi'(t)}$ is an increasing function[2] of x. If $x\varphi'(x) < K$ for $0 \leqq x < \infty$, then $\varphi(x) = O(\log x)$, which contradicts the hypothesis. Hence $\lim\limits_{x\to\infty} x\varphi'(x) = \infty$, so that $\dfrac{\psi'(t)}{\psi(t)}$ is a decreasing function of t and

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1) H. Weyl: Über die Gleichverteilung von Zahlen mod. 1. Math. Ann. **77** (1916). In Weyl's paper (II) is not expressed explicitly, but (II) follows from (I) easily.

(2) $\varphi'(x)$ may cease to exist at an enumerable set of points, where we define $\varphi'(x)$ suitably.

$\lim_{t\to\infty}\frac{\psi'(t)}{\psi(t)}=0$. By (I), it suffices to prove $\int_0^r e^{i\varphi(x)}dx=o(r)$. We will first prove

$$I=\int_0^r \sin\varphi(x)dx=o(r)\,. \tag{4}$$

By the second mean value theorem, we have by putting $\rho=\varphi(r)$, $\rho_0=\varphi(o)$, $r=\psi(\rho)$,

$$\begin{aligned}I&=\int_{\rho_0}^{\rho}\sin t\cdot\psi'(t)dt=\int_{\rho_0}^{\tau}\sin t\cdot\psi'(t)dt+\int_{\tau}^{\rho}\psi(t)\sin t\cdot\frac{\psi'(t)}{\psi(t)}dt\\&=\int_{\rho_0}^{\tau}\sin t\cdot\psi'(t)dt+\frac{\psi'(\tau)}{\psi(\tau)}\int_{\tau}^{\tau_2}\psi(t)\sin t dt\\&=\int_{\rho_0}^{\tau}\sin t\cdot\psi'(t)dt+\frac{\psi'(\tau)}{\psi(\tau)}\psi(\tau_2)\int_{\tau_1}^{\tau_2}\sin t dt\\&=\int_{\rho_0}^{\tau}\sin t\cdot\psi'(t)dt+\frac{\psi'(\tau)}{\psi(\tau)}O\big(\psi(\rho)\big)\qquad(\rho_0<\tau<\tau_1<\tau_2<\rho)\,,\end{aligned}$$

where

$$\big|O\big(\psi(\rho)\big)\big|\leqq 2\psi(\rho)=2r\,.$$

We take τ so large that $\left|\frac{\psi'(\tau)}{\psi(\tau)}\right|\leqq\varepsilon$ and then ρ so large that $\left|\int_{\rho_0}^{\tau}\sin t\psi'(t)dt\right|\leqq\varepsilon\psi(\rho)$. Then $|I_1|\leqq\psi(\rho)(\varepsilon+2\varepsilon)=3\varepsilon r$, so that $I=o(r)$. Similarly $\int_0^r\cos\varphi(x)dx=o(r)$. Hence $\int_0^r e^{i\varphi(x)}dx=o(r)$, q. e. d.

Since the Nevanlinna's characteristic function $T(r)$ of a transcendental meromorphic function for $|z|<\infty$ satisfies the condition of Theorem I, we have

Theorem II. The values of $T(r)$ distribute uniformly mod. 1.

2. *Uniform distribution mod. 1 of higher dimensions.*

Let $w_1=f_1(x_1,\dots,x_n),\dots,w_m=f_m(x_1,\dots,x_n)$ be continuous functions defined for $-\infty<x_i<\infty$ $(i=1,2,\dots,n)$ and $\mathfrak{A}$ be an interval: $0\leqq\alpha_i\leqq w_i\leqq\beta_i\leqq 1$ $(i=1,2,\dots,m)$. Let $S(r)$: $x_1^2+\dots+x_n^2\leqq r^2$ be a sphere and $E(r,\mathfrak{A})$ be the set of points $(x_1,\dots,x_n)$ in $S(r)$, such that $\Big(\big(f_1(x_1,\dots,x_n)\big),\dots,\big(f_m(x_1,\dots,x_n)\big)\Big)$ lie in $\mathfrak{A}$ and $|\mathfrak{A}|$, $mE(r,\mathfrak{A})$, $V(r)$ be the measure of $\mathfrak{A}$, $E(r,\mathfrak{A})$, $S(r)$ respectively. If for any $\mathfrak{A}$,

$$\lim_{r\to\infty}\frac{mE(r,\mathfrak{A})}{V(r)}=|\mathfrak{A}|\,, \tag{5}$$

then we say that the values of $(f_1,\dots,f_m)$ distribute uniformly mod. 1.

Similarly as (I), (II), we can prove that (I′) the necessary and sufficient condition, that the values of $(f_1,\dots,f_m)$ distribute uniformly mod. 1, is that

$$\int\cdots\cdots\cdots\cdots\int_{x_1^2+\dots+x_n^2\leqq r^2} e^{2\pi i(a_1f_1(x_1,\dots,x_n)+\dots+a_mf_m(x_1,\dots,x_n))}dx_1\dots dx_n=o(r^n) \tag{6}$$

for any integers a_i, such that $|a_1|+\cdots+|a_m|\neq 0$.

(II′) Let $F(t_1,\ldots,t_m)$ be periodic with respect to t_i with period 1 and be integrable in Riemann's sense as a function of $(t_1,\cdots,t_m)$ in $0\leqq t_i\leqq 1$ $(i=1,2,\ldots,m)$. If the values of $(f_1,\cdots,f_m)$ distribute uniformly mod. 1, then

$$\lim_{r\to\infty}\frac{1}{V(r)}\int_{x_1^2+\cdots+x_n^2\leq r^2}\cdots\cdots\cdots\int F\Big(f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)\Big)dx_1\ldots dx_n$$
$$=\int_0^1\cdots\int_0^1 F(t_1,\ldots,t_m)dt_1\ldots dt_m. \quad (7)$$

By (I′), (II′), we can prove that if the values of $(f_1,\cdots,f_m)$ distribute uniformly mod. 1, then (5) holds, when $\mathfrak{A}$ is any set, which is measurable in Jordan's sense and $|\mathfrak{A}|$ be its measure.

We will prove[1)]

Theorem III. Let $P_1(x_1,\ldots,x_n),\ldots,P_m(x_1,\ldots,x_n)$ *be polynomials, such that*

$$a_1P_1(x_1,\ldots,x_n)+\cdots+a_mP_m(x_1,\ldots,x_n)\neq\text{const.},$$

for any integers a_i, *such that* $|a_1|+\cdots+|a_m|\neq 0$, *then the values of* $(P_1,\ldots,P_m)$ *distribute uniformly mod. 1.*

Proof. We suppose $n=m=2$; the other case can be proved similarly. Let a_1, a_2 be integers, such that $|a_1|+|a_2|\neq 0$ and put

$$x_1=r\cos\theta\,,\qquad x_2=r\sin\theta\,,$$

$$2\pi\Big(a_1P(x_1,x_2)+a_2P(x_1,x_2)\Big)=a_0(\cos\theta,\sin\theta)r^n+a_1(\cos\theta,\sin\theta)r^{n-1}$$
$$+\cdots+a_n=\Phi(r,\theta)\,,$$

where $a_i(\cos\theta,\sin\theta)$ are polynomials in $\cos\theta,\sin\theta$ and $n\geqq 1$ by the hypothesis. By (I′) it sufficies to prove

$$\int_0^{2\pi}\int_0^r e^{i\Phi(r,\theta)}rdrd\theta=o(r^2)\,. \quad (8)$$

Let $a_0(\cos\theta,\sin\theta)=0$ for $\theta=\theta_i$ $(i=1,2,\ldots,k)$, then $|a_0(\cos\theta,\sin\theta)|\geqq\eta>0$ for $|\theta-\theta_i|\geqq\delta>0$. Let $I_2=\sum_{i=1}^{k}(\theta_i-\delta,\theta_i+\delta)$ and I_1 be the remaining part of $[0,2\pi]$, so that $[0,2\pi]=I_1+I_2$. Then $|a_0(\cos\theta,\sin\theta)|\geqq\eta>0$ in I_1. Hence $\lim_{r\to\infty}|\Phi(r,\theta)|=\infty$ uniformly in I_1 and $\Phi'(r,\theta)\neq 0$ for $r\geqq r_0$, where $\Phi'(r,\theta)$ means $\dfrac{\partial\Phi(r,\theta)}{\partial r}$.

We put $x=\Phi(r,\theta)$ in I_1, then

$$i\int_{r_0}^r e^{i\Phi(r,\theta)}rdr=\int_{r_0}^r\frac{ire^{ix}}{\Phi'(r,\theta)}dx=\left[\frac{e^{ix}r}{\Phi'(r,\theta)}\right]_{r_0}^r-\int_{r_0}^r e^{ix}\left[\frac{1}{\Phi'(r,\theta)}-\frac{r\Phi''(r,\theta)}{[\Phi'(r,\theta)]^2}\right]dr\,.$$

Since in I_1, $\dfrac{r}{\Phi'(r,\theta)}=O\Big(\dfrac{1}{r^{n-2}}\Big)$, $\dfrac{1}{\Phi'(r,\theta)}-\dfrac{r\Phi''(r,\theta)}{[\Phi'(r,\theta)]^2}=O\Big(\dfrac{1}{r^{n-1}}\Big)$ uniformly, we have

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1) The case $n=1$ is proved in Weyl's paper l. c. (1).

$$\int_{I_1} d\theta \int_0^r e^{i\Phi(r,\theta)} r dr = O\left(\frac{1}{r^{n-2}}\right) = o(r^2) . \tag{10}$$

Since $2k\delta$ is the measure of I_2,

$$\int_{I_2} d\theta \int_0^r e^{i\Phi(r,\theta)} r dr = O(\delta r^2) . \tag{11}$$

Since δ can be taken arbitrarily small, we have from (10), (11),

$$\int_0^{2\pi} \int_0^r e^{i\Phi(r,\theta)} r dr d\theta = o(r^2) , \qquad \text{q. e. d.}$$

Let $w=P(z)=P_1(x, y)+iP_2(x, y)$ be a polynomial in $z=x+iy$, which is not a constant, then we see by the Cauchy-Riemann's differential equation, that $\alpha P_1(x, y)+\beta P_2(x, y) \neq$ const. for any constants α, β ($|\alpha|+|\beta| \neq 0$).

Hence we have

Theorem IV. Let $w=P(z)$ be a polynomial in $z=x+iy$, which is not a constant, then the values of $P(z)$ distribute uniformly mod. 1.