26. On a Characterisation of Join Homomorphic Transformation-lattice

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(Comm. by T. Takagi, M.I.A., March 12, 1943.)

1. Introduction. A mapping $f$ of a lattice $L_1$ into a lattice $L_2$ is called join homomorphic, when for any elements $a, b$ of $L_1$ there exists the relation

$$f(a \lor b) = f(a) \lor f(b).$$

This mapping is order preserving, for, if $a > b$ in $L_1$, it follows $f(a) > f(b)$ in $L_2$.

If we define $f_1 > f_2$, when for any element $a$ of $L_1$, $f_1(a) > f_2(a)$ is satisfied, then the set of all join homomorphic transformations forms a partially ordered set $\{f\}$. If $L_2$ is complete and completely distributive, then $\{f\}$ is a complete lattice. For there exist the following relations for any element $a$ of $L_1$

$$(f_1 \lor f_2)(a) = f_1(a) \lor f_2(a),$$

$$(\bigvee_X (f_x | X))(a) = \bigvee_X (f_x(a) | X),$$

$$(f_1 \land f_2)(a) = \bigwedge_X (g_x(a) | X),$$

$$(\bigwedge_X (f_x | X))(a) = \bigwedge_Y (h_y(a) | Y),$$

where $\{g_x | x \in X\}$ is the set of all transformations such that $g_x < f_1, f_2$, and $\{h_y | y \in Y\}$ is the set of all transformations such that $h_y < f_x$ for all $x$ of $X$. This join $f_1 \lor f_2$, meet $f_1 \land f_2$, complete join $\bigvee_X f_x$ and complete meet $\bigwedge_X f_x$ are again clearly join homomorphic transformations.

In this paper we are concerned with the problem of a lattice-theoretic characterisation of this join homomorphic transformation-lattice for the case, when $L_2$ is the two-element lattice \{0, 1\}.

Lemma 1. All ideals in $L$ form a lattice, which is dual isomorphic with the join homomorphic transformation-lattice $\{f\}$ of $L$ into \{0, 1\}.

Proof. Let $f$ be a join homomorphic mapping of $L$ into \{0, 1\}. Then the set $f^{-1}(0)$ is an ideal in $L$. For if $a, b \in f^{-1}(0)$, then $f(a \lor b) = f(a) \lor f(b) = 0$; therefore $a \lor b \in f^{-1}(0).$ And if $a \in f^{-1}(0)$, $b < a$, then clearly $f(b) < f(a) = 0$. Hence $f^{-1}(0)$ includes $b$.

Conversely, let $\mathcal{A}$ be an ideal in $L$, then the transformation $f$ such that

$$f(a) = 0, \quad a \in \mathcal{A},$$

$$f(a) = 1, \quad a \notin \mathcal{A},$$

is clearly join homomorphic. Hence the correspondence between an ideal \( A \) in \( L \) and a join homomorphic transformation of \( L \) into \( \{0, 1\} \) is one to one.

Furthermore this correspondence is a dual lattice isomorphism. Let \( f_1, f_2 \) be any two such transformations, and let \( A_1, A_2 \) be respectively the ideals \( f_1^{-1}(0), f_2^{-1}(0) \). Now if \( (f_1 \cup f_2)(a) = f_1(a) \cup f_2(a) = 0 \), then \( a \) is included in the ideal \( A_1 \cap A_2 \). Conversely, if \( a \in A_1 \cap A_2 \), then \( f_1(a) = f_2(a) = 0 \); therefore

\[
(f_1 \cup f_2)(a) = 0.
\]

Hence

\[
(f_1 \cup f_2)^{-1}(0) = A_1 \cap A_2.
\]

And if \( (f_1 \cap f_2)(a) = 0 \), then \( a \) is included in all such ideals \( A \) that \( A \supset A_1, A_2 \), i.e. \( A \supset A_1 \cap A_2 \). When we denote by \( A \cup A \) the least ideal \( B \) such that \( B \supset A_1 \cup A_2 \), i.e. \( A \cup A = \bigwedge A \), then \( a \in A_1 \cup A_2 \). Conversely if \( a \in A_1 \cup A_2 \), then \( a \in B \) for any ideal \( B \). Hence for any transformation \( g_n \) such that \( g_n < f_1, f_2 \), we have \( g_n(a) = 0 \), i.e.

\[
(f_1 \cap f_2)(a) = \bigvee \{ g_n(a) \} = 0.
\]

Therefore we conclude

\[
(f_1 \cap f_2)^{-1}(0) = A_1 \cup A_2.
\]

2. Transformation-lattice.

Lemma 2. Every element \( f \) of \( \{f\} \) has at least one expression as the meet of some meet-irreducible elements.

Proof. Let \( f^{-1}(0) = \{x \mid X\} \), \( A_x = a \cap L \), and let \( f_x \) be the join homomorphic transformation such that

\[
f_x^{-1}(0) = A_x.
\]

Then \( f = \bigwedge f_x \). For from \( f^{-1}(0) \supset f_x^{-1}(0) \) it follows \( f < f_x \), i.e. \( f < \bigwedge f_x \).

And if \( g < \bigwedge f_x \), then \( g^{-1}(0) \supset f_x^{-1}(0) \), i.e. \( g^{-1}(0) \supset \bigwedge A_x = f^{-1}(0) \). Hence \( g < f \). Therefore it must be \( f = \bigwedge f_x \).

Every \( f_x \) is meet-irreducible or finite-meet-reducible into some meet-irreducible elements. For if

\[
f_x = \bigwedge \{ g_y \mid Y \} \quad g_y^{-1}(0) \text{ principal ideal},
\]

then \( f_x < g_y \); hence \( f_x^{-1}(0) = A_x > g_y^{-1}(0) \). If \( A_x = g_y^{-1}(0) \) for all \( y \), then \( A_x = (\bigwedge g_y^{-1}(0)) \). But \( A_x = (\bigwedge g_y^{-1}(0)) \) is the least ideal, which includes all the ideal \( g_y^{-1}(0) \). Whence for some finite elements \( b_x \in g_y^{-1}(0) \)

1) \( A_1 \sim A_2 \) means the set sum of \( A_1 \) and \( A_2 \).

2) \( a \) is said meet-irreducible, when, if \( a = \bigwedge \{ a_x \mid X \} \), then necessarily \( a = a_x \) for some \( x \). See A. Komatu: On a Characterisation of Order Preserving Transformation-lattice. Proc. 19 (1943), 27.

3) \( a \) is said finite-meet-reducible or finite-meet-reducible into meet-irreducible elements, when, if \( a = \bigwedge \{ a_x \mid X \} \) with meet-irreducible elements \( a_x \), then \( a = a_{x_1} \cdots a_{x_m} \) for some finite subset \( x, \ldots, x_n \) of \( X \).
(j=1, 2, ..., n) it must be \(a_x < b_{y_1} \cup \cdots \cup b_{y_n}\).

Therefore \(\mathbb{A}_x < (\bigcap_j g_{y_j})^{-1}(0)\), i.e. \(\mathbb{A}_x = \bigcup_j g_{y_j}(0)\).

This shows easily that \(f_x\) is finite-meet-reducible into some meet-irreducible elements.

**Lemma 3.** The subset \(L\)' of all meet-irreducible elements and all meet-finite-reducible elements in \(\{f\}\) forms a lattice, which is dual isomorphic with \(L\).

**Proof.** Let \(f\) be a meet-irreducible element or a finite-meet-reducible element, i.e. \(f \in L\)', and let \(f^{-1}(0) = \{a_x \mid X\} \) and \(a_x \cap L = \mathbb{A}_x\). Let \(f_{x_1}\) be the transformation such that \(f_{x_1}^{-1}(0) = \mathbb{A}_{x_1}\), then \(f = \bigcap f_{x_1}\) as in lemma 2.

From the finite-meet-reducibility of \(f\) we can prove easily

\[f = f_{x_1} \cap \cdots \cap f_{x_n}\]

Whence \(f^{-1}(0)\) is the least ideal which includes \(f_{x_i}^{-1}(0) = \mathbb{A}_{x_i}\) \((i = 1, 2, \ldots, n)\). Therefore \(f^{-1}(0)\) is the principal ideal

\[(a_{x_1} \cup \cdots \cup a_{x_n}) \cap L.\]

From lemma 1 and 2 we conclude that \(L\) is dually lattice isomorphic with \(L\).

**Lemma 4.** Join in \(\{f\}\) is continuous with respect to the generalizd \((\sigma)\) topology\(^1\) of \(\{f\}\). Meet is not necessarily continuous.

**Proof.** Let a directed set of elements \(\{f_x \mid X\}\) converge to \(f\). Then there exist two directed sets of elements \(\{\varphi_x \mid X\}, \{\psi_x \mid X\}\) such that

\[
\begin{align*}
\varphi_{x_1} &< \varphi_{x_2}, \quad \text{for } x_1 < x_2 \text{ in } X, \\
\psi_{x_1} &> \psi_{x_2}, \\
\varphi_x &< f_x < \psi_x \quad \text{for any } x \in X,
\end{align*}
\]

and

\[
\bigcup_X \{\varphi_x \mid x \in X\} = \lim_f = \bigcap_X \{\psi_x \mid x \in X\}.
\]

Hence for any element \(g\) of \(\{f\}\)

\[
(1) \quad \left\{ \begin{array}{l}
\varphi_{x_1} \cup g < \varphi_{x_2} \cup g \\
\varphi_{x_1} \cup g > \varphi_x \cup g \\
\varphi_x \cup g < f_x \cup g < \psi_x \cup g
\end{array} \right\} \quad \text{for any } x_1 < x_2 \text{ in } X,
\]

\[
(2) \quad \left( \bigcup_X (\varphi_x \mid X) \right) \cup g = (\lim_f) \cup g = \left( \bigcap_X (\psi_x \mid X) \right) \cup g.
\]

It is clear that \(\bigcup_X \varphi_x \cup g = \bigcup_X (\varphi_x \cup g)\). Furthermore we can prove easily \(\bigcap_X \psi_x \cup g = \bigcap_X (\psi_x \cup g)\). For if \(a \in \left( \bigcap_X (\psi_x \cup g) \right)\), then \(a \in (\bigcap_X (\psi_x \cup g))^{-1}(0)\) and \(a \in g^{-1}(0)\); by the first relation it follows \(a < a_{x_1} \cup \cdots \cup a_{x_n}\) for some finite \(a_{x_i} \in \varphi_{x_i}^{-1}(0)\) \((i = 1, \ldots, n)\). Let \(x\) be an

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element of $X$ such that for every $x_i \geq x_0$, then $\varphi_x < \varphi_{x_0}$, i.e. $\varphi_x^{-1}(0) \geq \varphi_{x_0}^{-1}(0)$. Hence every $a_{x_i}$ is included in the ideal $\varphi_x^{-1}(0)$ and so is $a$. Therefore we conclude for this $a$ that $a \in (\varphi_x \cup g)^{-1}(0) < \bigcup_{x} \left( (\varphi_x \cup g)^{-1}(0) \right)$, i.e. $(\bigcap_{x} \varphi_x) \cup g \geq \bigcap_{x} (\varphi_x \cup g)$.

The inverse order is obvious from $\varphi_x \cup g \geq (\bigcap_{x} \varphi_x) \cup g$, hence

$$\bigcap_{x} \varphi_x) \cup g = \bigcap_{x} (\varphi_x \cup g).$$

The formula (2) now takes the form

$$\bigcup_{x} (\varphi_x \cup g) = (\lim f_x) \cup g = \bigcap_{x} (\varphi_x \cup g).$$

From (1) and (3) we see that $\lim (f_x \cup g) = (\lim f_x) \cup g$, i.e. $\{f_x \cup g \mid X\}$ converges to $f \cup g$.

3. Characterisation of the transformation-lattice.

Lemma 5. Let $L^*$ be a lattice with the following properties:

i) complete, ii) every element $a$ is a meet of meet-irreducible elements, iii) join is continuous with respect to the generalized $(o)$-topology of $L^*$.

Then, if $a = \bigcap_{x} a_x = \bigcap_{y} b_y$ are any two reductions of $a$ into infinite meet-irreducible components, we can select for every $y$ suitably some finite $x_i$ ($i=1, 2, \ldots, n$) such that

$$b_y \geq a_{x_1} \cap \cdots \cap a_{x_n}$$

and for every $x$ some finite $y_j$ ($j=1, 2, \ldots, m$) such that

$$a_x \geq b_{y_1} \cap \cdots \cap b_{y_m}.$$

Proof. Let $I'$ be the set of all finite subsets $\{a\}$ of $X$, then $I'$ is a directed set. If $a = \{x_1, x_2, \ldots, x_n\}$ and $a_a = a_{x_1} \cap \cdots \cap a_{x_n}$, then for $a < \beta$ in $I'$ we have $a_a > a_{x_i}$ in $L^*$.

Clearly $a < a_{x_i}$ for every $a \in I'$; hence

$$a < \bigcap_{x} a_x.$$ But if we select $a_{x_i} \in I'$ suitably for every $x \in X$ such that $x \in a_{x_i}$, then $a_x > a_{x_{x_i}}$ in $L^*$; hence

$$a = \bigcap_{x} a_x > \bigcap_{x} a_{x_{x_i}} > \bigcap_{a_x} a_x.$$ From (4) and (5) it follows that the directed set of elements $\{a_{x_i} \mid I'\}$ converges to $a$. From the property iii) of $L^*$

$$b_y = b_y \cup a = b_y \cup (\bigcap_{x} a_x) = \bigcap_{a_x} (a_{x} \cup b_y).$$

From the property ii)

$$a_x \cup b_y = \bigcap_{z_x} c_z,$$

i.e. $b_y = \bigcap_{a_x \in I'} (\bigcap_{z_x} c_z)$. But $b_y$ is meet-irreducible, hence $b_y = c_z > a_x \cup b_y$

for some $z \in Z_a$.

Therefore it must be $b_y = a_x \cup b_y$, i.e.
Similarly we can prove for every $x$ with some finite $y_j$ ($j=1,2,...,m$) $a_x > b_y, \bigcap \cdots \bigcap b_{y_m}$.

**Theorem.** Let $L^*$ be a lattice with the following properties:

i) complete

ii) every element $a$ is a meet of meet-irreducible elements.

iii) join is continuous with respect to the generalized $(o)$-topology of $L^*$.

iv) the set $L$ of all meet-irreducible elements and all finite-meet-reducible elements forms a lattice with the (relative) order of $L^*$. Then $L^*$ is isomorphic with the join homomorphic transformation-lattice of $L'$ into \{0,1\}, where $L'$ is dual isomorphic to the lattice $L$.

**Proof.** (1) One to one Correspondence.

Let $a = \bigcap X a_x$ be an expression of $a$ with meet-irreducible elements \{$a_x \mid X$\}. Let $a'_x \in L'$ be the element which corresponds to $a_x \in L$, and let $f_x$ be the join homomorphic mapping of $L'$ into \{0,1\} such that

$$f_x^{-1}(0) = a'_x \bigcap L' = \forall x.'$$

Let $f$ be the mapping of $L'$ into \{0,1\} such that

$$f^{-1}(0) = \bigvee X \forall x.$$ 

Now we consider the correspondence $a \rightarrow f$. Clearly $a_x \rightarrow f_x$. This correspondence is uniquely determined. For if $a = \bigcap X a_x = \bigcap Y b_y$, then from lemma 5 for every $y$ with some $x \in X$ ($i=1,2,...,n$)

$$b_y > a_{x_i} \bigcap \cdots \bigcap a_{x_n}.$$ 

Hence $b_y$ is included in the ideal $\forall a_{x_i} \bigcap L' = \forall a_{x_i}'$, i.e.

$$\forall y' = b_y \bigcap L' \subset \bigvee_i \forall a_{x_i}'.$$

Similarly for every $x' \forall x' < \bigvee \forall y'$, whence

$$\forall x' = \forall y'.$$

This correspondence is one to one. For if $a = \bigcap X a_x, b = \bigcap Y b_y$, $a = b$, then at least for one $a_x$ (or $b_y$) there exist no finite subsets $y_1,\ldots,y_m$ (or $x_1,\ldots,x_n$) such that

$$a_x > b_{y_1} \bigcap \cdots \bigcap b_{y_m}.$$ 

Hence in $L'$ $a_x \notin \bigvee Y \forall y'$, therefore

$$f_x^{-1}(0) = f_y^{-1}(0), \text{ i.e. } f_x = f_y.$$ 

(2) Let $f$ be a join homomorphic transformation of $L'$ into \{0,1\}, and let $f^{-1}(0) = \forall x' = \forall a_{x'} \bigcap L'$. Clearly

$$\forall x' = \bigvee X \forall x' = \bigvee Y (a_{x'} \bigcap L').$$

From completeness of $L^*$ there exists an element $a$ such that
Hence $a \rightarrow f$.

(3) Meet homomorphism.
Let $a = \bigwedge_x a_x$, $b = \bigwedge_y b_y$, then $a \land b = (\bigwedge_x a_x) \land (\bigwedge_y b_y)$: Let $f_a, f_b$ and $f_{a \land b}$ be respectively the following mappings of $L'$ into $\{0, 1\}$ such that $f_a^{-1}(0) = \bigvee_x (a' \land L')$, $f_b^{-1}(0) = \bigvee_y (b' \land L')$, $f_{a \land b}^{-1}(0) = \bigvee_{x,y} \{(a_x \land L'), (b'_y \land L')\}$, then clearly $f_{a \land b} = f_a \land f_b$.

The last formula follows from the relation $\bigvee_{x,y} \{(a_x \land L'), (b'_y \land L')\} = \left(\bigvee_x (a'_x \land L')\right) \cup \left(\bigvee_y (b'_y \land L')\right)$.

We can easily prove from 1)-3) that this correspondence is isomorphic.

Corollary. The lattice $L$ of all join homomorphic transformations of finite lattice $L'$ into $\{0, 1\}$ is dual isomorphic to $L'$.