70. On the Theory of Hypersurfaces in the Path-space of the Third Order.

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(Comm. by S. KAKEYA, M.I.A., July, 12, 1943.)

§ 0. The theory of path-space of the third order has been developed by Prof. H. Hombu[1]. In the present note, we shall deal with the theory of hypersurfaces in such a space. In an n-dimensional manifold V_n referred to a coordinate system x^λ $(\lambda=1,2,\dots,n)$, let us consider a system of paths, defined by the differential equations of the third order,

$$T^\lambda = x^{(3)\lambda} + H^\lambda(x, x^{(1)}, x^{(2)}) = 0. \tag{0.1}$$

In order that our system of paths admits of projective parameters, it is necessary that

$$\text{(a)}\quad H^\lambda_{(1)\nu} x^{(1)\nu} + 2H^\lambda_{(2)\nu} x^{(2)\nu} = 3H^\lambda, \qquad \text{(b)}\quad H^\lambda_{(2)\nu} x^{(1)\nu} = -3x^{(2)\lambda}. \tag{0.2}$$

The base connections of our V_n are defined by

$$\text{(a)}\quad \delta x^{(1)\lambda} = dx^{(1)\lambda} + \frac{1}{3} H^\lambda_{(2)\nu} dx^\nu, \tag{0.3}$$

$$\text{(b)}\quad \delta x^{(2)\lambda} = dx^{(2)\lambda} + \frac{2}{3} H^\lambda_{(2)\nu} dx^{(1)\nu} + \frac{1}{3} H^\lambda_{(1)\nu} dx^\nu.$$

We see that $\frac{\delta x^{(1)\lambda}}{dt} = 0$ (along any curve) and $\frac{\delta x^{(2)\lambda}}{dt} = 0$ (along paths).

The covariant derivative of a vector v^λ in V_n is given by

$$\delta v^\lambda = dv^\lambda + w^\lambda_\mu v^\mu, \tag{0.4}$$

where

$$w^\lambda_\mu = \overset{*}{\underset{(0)}{\Gamma}}{}^\lambda_{\mu\nu} dx^\nu + \overset{*}{\underset{(1)}{\Gamma}}{}^\lambda_{\mu\nu} \delta x^{(1)\nu},$$

$$\text{(a)}\quad \overset{*}{\underset{(0)}{\Gamma}}{}^\lambda_{\mu\nu} = \frac{1}{3} H^\lambda_{(2)\mu(1)\nu} - \frac{2}{9} H^\lambda_{(2)\mu(2)\sigma} H^\sigma_{(2)\nu}, \qquad \text{(b)}\quad \overset{*}{\underset{(1)}{\Gamma}}{}^\lambda_{\mu\nu} = \frac{2}{3} H^\lambda_{(2)\mu(2)\nu}. \tag{0.5}$$

The equation (0.4) can be also written as follows:

$$\delta v^\lambda = \nabla^{(0)}_\nu v^\lambda \cdot dx^\nu + \nabla^{(1)}_\nu v^\lambda \cdot \delta x^{(1)\nu} + \nabla^{(2)}_\nu v^\lambda \cdot \delta x^{(2)\nu},$$

where

$$\nabla^{(0)}_\nu v^\lambda = \bar{\nabla}^{(0)}_\nu v^\lambda + \overset{*}{\underset{(0)}{\Gamma}}{}^\lambda_{\mu\nu} v^\mu, \quad \nabla^{(1)}_\nu v^\lambda = \bar{\nabla}^{(1)}_\nu v^\lambda + \overset{*}{\underset{(1)}{\Gamma}}{}^\lambda_{\mu\nu} v^\mu, \quad \nabla^{(2)}_\nu v^\lambda = \bar{\nabla}^{(2)}_\nu v^\lambda, \tag{0.6}$$

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1) H. Hombu: Projektive Transformation eines Systems der gewöhnlichen Differentialgleichungen dritter Ordnung. Proc. **13** (1937), 187–190, Die projektive Theorie der "paths" 3-ter Ordnung. Proc. **14** (1938), 36–40.

(0.7) (a) $\bar{\nabla}^{(0)}_{\nu}=\frac{\partial}{\partial x^{\nu}}-\frac{1}{3}H^{\mu}_{(2)\nu}\frac{\partial}{\partial x^{(1)\mu}}-\left(\frac{1}{3}H^{\mu}_{(1)\nu}-\frac{2}{9}H^{\mu}_{(2)\sigma}H^{\sigma}_{(2)\nu}\right)\frac{\partial}{\partial x^{(2)\mu}}$,

(b) $\bar{\nabla}^{(1)}_{\nu}=\frac{\partial}{\partial x^{(1)\nu}}-\frac{2}{3}H^{\mu}_{(2)\nu}\frac{\partial}{\partial x^{(2)\mu}}$,

(c) $\bar{\nabla}^{(2)}_{\nu}=\frac{\partial}{\partial x^{(2)\nu}}$.

Then, the curvature and torsion tensors of our V_n are given by

(0.8) $$\underset{21}{\delta\delta}v^{\lambda}-\underset{12}{\delta\delta}v^{\lambda}=v^{\mu}\{(R^{\lambda(0)(0)}_{\mu\sigma\pi}+\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\tau}S^{(0)(0)\tau}_{\sigma\pi(1)})\underset{1}{dx^{\sigma}}\underset{2}{dx^{\pi}}+R^{\lambda(1)(1)}_{\mu\sigma\pi}\underset{1}{\delta x^{(1)\sigma}}\underset{2}{\delta x^{(1)\pi}}$$
$$+R^{\lambda(0)(1)}_{\mu\sigma\pi}[\underset{1}{dx^{\sigma}}\underset{2}{\delta x^{(1)\pi}}]+R^{\lambda(0)(2)}_{\mu\sigma\pi}[\underset{1}{dx^{\sigma}}\underset{2}{\delta x^{(2)\pi}}]$$
$$+R^{\lambda(1)(2)}_{\mu\sigma\pi}[\underset{1}{\delta x^{(1)\sigma}}\underset{2}{\delta x^{(2)\pi}}]\}\,,$$

where

(0.9) (a) $R^{\lambda(0)(0)}_{\mu\sigma\pi}=\bar{\nabla}^{(0)}_{\pi}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}-\bar{\nabla}^{(0)}_{\sigma}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\pi}+\underset{(0)}{\overset{*}{\Gamma}}{}^{\tau}_{\mu\sigma}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\tau\pi}-\underset{(0)}{\overset{*}{\Gamma}}{}^{\tau}_{\mu\pi}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\tau\sigma}$,

(b) $R^{\lambda(1)(1)}_{\mu\sigma\pi}=\bar{\nabla}^{(1)}_{\pi}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}-\bar{\nabla}^{(1)}_{\sigma}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\pi}+\underset{(1)}{\overset{*}{\Gamma}}{}^{\tau}_{\mu\sigma}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\tau\pi}-\underset{(1)}{\overset{*}{\Gamma}}{}^{\tau}_{\mu\pi}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\tau\sigma}$,

(c) $R^{\lambda(0)(1)}_{\mu\sigma\pi}=\bar{\nabla}^{(1)}_{\pi}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}-\bar{\nabla}^{(0)}_{\sigma}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\pi}-\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\varepsilon\sigma}\underset{(1)}{\overset{*}{\Gamma}}{}^{\varepsilon}_{\mu\pi}+\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\varepsilon\pi}\underset{(0)}{\overset{*}{\Gamma}}{}^{\varepsilon}_{\mu\sigma}+\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\varepsilon}\underset{(0)}{\overset{*}{\Gamma}}{}^{\varepsilon}_{\sigma\pi}$,

(d) $R^{\lambda(0)(2)}_{\mu\sigma\pi}=\bar{\nabla}^{(2)}_{\pi}\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}+\frac{1}{3}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\tau}\bar{\nabla}^{(2)}_{\pi}H^{\tau}_{(2)\sigma}$,

(e) $R^{\lambda(1)(2)}_{\mu\sigma\pi}=\bar{\nabla}^{(2)}_{\pi}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}$,

and

(0.10) (a) $\underset{21}{\delta d}x^{\lambda}-\underset{12}{\delta d}x^{\lambda}=S^{(0)(0)\lambda}_{\mu\sigma(0)}\underset{1}{dx^{\mu}}\underset{2}{dx^{\sigma}}+S^{(0)(1)\lambda}_{\mu\sigma(0)}[\underset{1}{dx^{\mu}}\underset{2}{\delta x^{(1)\sigma}}]$,

(b) $\underset{21}{\delta\delta}x^{(1)\lambda}-\underset{12}{\delta\delta}x^{(1)\lambda}=S^{(0)(0)\lambda}_{\mu\sigma(1)}\underset{1}{dx^{\mu}}\underset{2}{dx^{\sigma}}+S^{(0)(2)\lambda}_{\mu\sigma(1)}[\underset{1}{dx^{\mu}}\underset{2}{\delta x^{(2)\sigma}}]$
$+S^{(0)(1)\lambda}_{\mu\sigma(1)}[\underset{1}{dx^{\mu}}\underset{2}{\delta x^{(1)\sigma}}]$,

(c) $\underset{21}{\delta\delta}x^{(2)\lambda}-\underset{12}{\delta\delta}x^{(2)\lambda}=S^{(0)(0)\lambda}_{\mu\sigma(2)}\underset{1}{dx^{\mu}}\underset{2}{dx^{\sigma}}+S^{(1)(1)\lambda}_{\mu\sigma(2)}\underset{1}{\delta x^{(1)\mu}}\underset{2}{\delta x^{(1)\sigma}}$
$+S^{(0)(1)\lambda}_{\mu\sigma(2)}[\underset{1}{dx^{\mu}}\underset{2}{\delta x^{(1)\sigma}}]$,

where

(0.11) (a) $S^{(0)(0)\lambda}_{\mu\sigma(0)}=\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}-\underset{(0)}{\overset{*}{\Gamma}}{}^{\lambda}_{\sigma\mu}$, (b) $S^{(0)(1)\lambda}_{\mu\sigma(0)}=\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}$,

(c) $S^{(0)(0)\lambda}_{\mu\sigma(1)}=\frac{1}{3}\bar{\nabla}^{(0)}_{\sigma}H^{\lambda}_{(2)\mu}-\frac{1}{3}\bar{\nabla}^{(0)}_{\mu}H^{\lambda}_{(2)\sigma}$, (d) $S^{(0)(2)\lambda}_{\mu\sigma(1)}=\frac{1}{2}\underset{(1)}{\overset{*}{\Gamma}}{}^{\lambda}_{\mu\sigma}$,

(e) $S^{(0)(0)\lambda}_{\mu\sigma(2)}=\frac{1}{3}\bar{\nabla}^{(0)}_{\sigma}H^{\lambda}_{(1)\mu}-\frac{2}{9}H^{\pi}_{(2)\sigma}\bar{\nabla}^{(0)}_{\mu}H^{\lambda}_{(2)\pi}-\frac{1}{3}\bar{\nabla}^{(0)}_{\mu}H^{\lambda}_{(1)\sigma}$
$+\frac{2}{9}H^{\pi}_{(2)\mu}\bar{\nabla}^{(2)}_{\sigma}H^{\lambda}_{(2)\pi}$,

(f) $S^{(1)(1)\lambda}_{\mu\sigma(2)} = S^{(0)(0)\lambda}_{\mu\sigma(0)}$,

(g) $S^{(0)(1)\lambda}_{\mu\sigma(2)} = \frac{1}{3}\bar{\nabla}^{(1)}_{\sigma} H^{\lambda}_{(1)\mu} - \frac{2}{3}\bar{\nabla}^{(0)}_{\mu} H^{\lambda}_{(2)\sigma} - \frac{2}{9} H^{\tau}_{(2)\mu} \bar{\nabla}^{(1)}_{\sigma} H^{\lambda}_{(2)\tau}$,

(h) $S^{(0)(1)\lambda}_{\mu\sigma(1)} = S^{(0)(0)\lambda}_{\mu\sigma(0)}$.

§ 1. Let us consider a hypersurface V_{n-1} immersed in the V_n whose parametric representation is

$$x^{\lambda} = x^{\lambda}(y^{\dot{1}}, y^{\dot{2}}, \dots, y^{\dot{n}-\dot{1}}).$$

In our V_{n-1}, we take a system of paths with projective parameter defined by

$$T^{i} = y^{(3)i} + H^{i}(y, y^{(1)}, y^{(2)}) = 0, \qquad (i = \dot{1}, \dot{2}, \dots, \dot{n}-\dot{1})$$

The quantities of V_{n-1} corresponding to that of V_n are derived from H^i in the same way as last paragraph. If we put

$$(1.1) \qquad \mathfrak{J}^{\lambda} = H^{\lambda} - \xi^{\lambda}_{i} H^{i} + 3\xi^{\lambda}_{jk} y^{(2)j} y^{(1)k} + \xi^{\lambda}_{jkh} y^{(1)j} y^{(1)k} y^{(1)h},$$

where $\xi^{\lambda}_{i} = \dfrac{\partial x^{\lambda}}{\partial y^{i}}$, $\xi^{\lambda}_{jk} = \dfrac{\partial^2 x^{\lambda}}{\partial y^{j} \partial y^{k}}$, $\xi^{\lambda}_{jkh} = \dfrac{\partial^3 x^{\lambda}}{\partial y^{j} \partial y^{k} \partial x^{h}}$,

then $\mathfrak{J}^{\lambda}$ is a vector of V_n and scalar of V_{n-1}.

According to the assumption on H^{λ} and H^{i}, $\mathfrak{J}^{\lambda}$ must satisfy the following relations:

$$(1.2) \quad \text{(a)} \quad \mathfrak{J}^{\lambda}_{(1)s} y^{(1)s} + 2\mathfrak{J}^{\lambda}_{(2)s} y^{(2)s} = 3\mathfrak{J}^{\lambda}, \qquad \text{(b)} \quad \mathfrak{J}^{\lambda}_{(2)s} y^{(1)s} = 0.$$

We have from (1.1)

$$(1.3) \quad \text{(a)} \quad \mathfrak{J}^{\lambda}_{(1)s} = H^{\lambda}_{(1)\nu} \xi^{\nu}_{s} + 2H^{\lambda}_{(2)\nu} \xi^{\nu}_{is} y^{(1)i} - \xi^{\lambda}_{i} H^{i}_{(1)s} + 3\xi^{\lambda}_{is} y^{(2)i} + \xi^{\lambda}_{sjk} y^{(1)j} y^{(1)k},$$

$$\text{(b)} \quad \mathfrak{J}^{\lambda}_{(2)s} = H^{\lambda}_{(2)\nu} \xi^{\nu}_{s} - \xi^{\lambda}_{i} H^{i}_{(2)s} + 3\xi^{\lambda}_{is} y^{(1)i}.$$

Now we get

$$(1.4) \quad \text{(a)} \quad \bar{\nabla}^{(0)}_{l} \Phi = \xi^{\lambda}_{l} \bar{\nabla}^{(1)}_{\lambda} \Phi + \frac{1}{3} \mathfrak{J}^{\lambda}_{(2)l} \bar{\nabla}^{(1)}_{\lambda} \Phi + \frac{1}{3} (\bar{\nabla}^{(1)}_{l} \mathfrak{J}^{\lambda}) \bar{\nabla}^{(2)}_{\lambda} \Phi,$$

$$\text{(b)} \quad \bar{\nabla}^{(1)}_{l} \Phi = \xi^{\lambda}_{l} \bar{\nabla}^{(1)}_{\lambda} \Phi + \frac{2}{3} \mathfrak{J}^{\lambda}_{(2)l} \bar{\nabla}^{(2)}_{\lambda} \Phi,$$

$$\text{(c)} \quad \bar{\nabla}^{(2)}_{l} \Phi = \xi^{\lambda}_{l} \bar{\nabla}^{(2)}_{\lambda} \Phi,$$

Φ being an arbitrary quantity of V_n.

On our hypersurface V_{n-1}, $x^{(1)\lambda}$ and $x^{(2)\lambda}$ being expressed as $x^{(1)\lambda} = \xi^{\lambda}_{i} y^{(1)i}$, $x^{(2)\lambda} = \xi^{\lambda}_{i} y^{(2)i} + \xi^{\lambda}_{ij} y^{(1)i} y^{(1)j}$, it can be readily proved that

$$(1.5) \quad \text{(a)} \quad \delta x^{(1)\lambda} = \xi^{\lambda}_{i} \delta y^{(1)i} + \frac{1}{3} \mathfrak{J}^{\lambda}_{(2)i} dy^{i},$$

$$\text{(b)} \quad \delta x^{(2)\lambda} = \xi^{\lambda}_{i} \delta y^{(2)i} + \frac{2}{3} \mathfrak{J}^{\lambda}_{(2)i} \delta y^{(1)i} + \frac{1}{3} (\bar{\nabla}^{(1)}_{j} \mathfrak{J}^{\lambda}) dy^{j}.$$

Then we have from (1.5)

$$\delta\xi^\lambda_i = F^\lambda_{jk}dy^k + \frac{2}{3}\mathfrak{J}^\lambda_{(2)j(2)k}\delta y^{(1)k}, \tag{1.6}$$

where

$$F^\lambda_{jk} = \xi^\lambda_{jk} + \overset{*}{\underset{(0)}{\Gamma}}{}^\lambda_{\mu\nu}\xi^\mu_j\xi^\nu_k - \xi^\lambda_i\overset{*}{\underset{(0)}{\Gamma}}{}^i_{jk} + \frac{1}{3}\overset{*}{\underset{(1)}{\Gamma}}{}^\lambda_{\mu\nu}\xi^\mu_j\mathfrak{J}^\nu_{(2)k}, \tag{1.7}$$

$$\frac{2}{3}\mathfrak{J}^\lambda_{(2)j(2)k} = \overset{*}{\underset{(1)}{\Gamma}}{}^\lambda_{\mu\nu}\xi^\mu_j\xi^\nu_k - \xi^\nu_i\overset{*}{\underset{(1)}{\Gamma}}{}^i_{jk}. \tag{1.8}$$

Furthermore we get from (1.3) and (1.4)

$$F^\lambda_{jk} = \frac{1}{3}\bar{V}^{(1)}_k\mathfrak{J}^\lambda_{(2)j}. \tag{1.9}$$

If we have a vector v^λ tangent to the hypersurface V_{n-1}, then v^λ being expressed as $v^\lambda = \xi^\lambda_i v^i$, we have along V_{n-1}

$$\delta v^\lambda = \xi^\lambda_i\delta v^i + v^j\left(F^\lambda_{jk}dy^k + \frac{2}{3}\mathfrak{J}^\lambda_{(2)j(2)k}\delta y^{(1)k}\right). \tag{1.10}$$

Definition 1. If we have $\mathfrak{J}^\lambda = 0$ *at every point of* V_{n-1}, *such a* V_{n-1} *is called totally geodesic* V_{n-1}.

Then it is evident from (1.1), that any path of a totally geodesic V_{n-1} immersed in V_n is also a path of V_n.

Definition 2. If we have $\mathfrak{J}^\lambda_{(2)k} = 0$ *at every point on* V_{n-1}, *such a* V_{n-1} *is called semi-geodesic* V_{n-1}.

From (1.9) and (1.2), it can be proved that semi-geodesic V_{n-1} has following properties:

(a) $\delta\xi^\lambda_j = 0$,

(b) With respect to $x^{(1)}_s$, $\mathfrak{J}^\lambda$ are homogenous of degree 3.

That is to say: *When a vector of semi-geodesic* V_{n-1} *is displaced parallelly to itself in* V_n *along* V_{n-1}, *it moves also parallelly to itself in* V_{n-1}.

We have from (1.7) and (1.8) the next fundamental relations,

$$\xi^\mu_j w^\lambda_\mu = \xi^\lambda_i w^i_j - \xi^\lambda_{jk}dy^k + F^\lambda_{jk}dy^k + \frac{2}{3}\mathfrak{J}^\lambda_{(2)j(2)k}\delta y^{(1)k}. \tag{1.11}$$

§ 2. With use of (1.6) and (1.10), we obtain the integrability conditions of (1.11) as follows:

$$\begin{aligned}
(2.1)\quad (a)\quad & \overset{*}{R}{}^{\lambda(0)(0)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_k\xi^\pi_h + \frac{1}{9}R^{\lambda(1)(1)}_{\mu\sigma\pi}\mathfrak{J}^\sigma_{(2)k}\mathfrak{J}^\pi_{(2)h}\xi^\mu_j \\
& + \frac{1}{3}R^{\lambda(0)(1)}_{\mu\sigma\pi}\xi^\mu_j(\xi^\sigma_k\mathfrak{J}^\pi_{(2)h} - \xi^\sigma_h\mathfrak{J}^\pi_{(2)k}) \\
& + \frac{1}{3}R^{\lambda(0)(2)}_{\mu\sigma\pi}\xi^\mu_j(\xi^\sigma_k\bar{V}^{(1)}_h\mathfrak{J}^\pi - \xi^\sigma_h\bar{V}^{(1)}_k\mathfrak{J}^\pi) \\
& + \frac{1}{9}R^{\lambda(1)(2)}_{\mu\sigma\pi}\xi^\mu_j(\mathfrak{J}^\sigma_{(2)k}\bar{V}^{(1)}_h\mathfrak{J}^\pi - \mathfrak{J}^\sigma_{(2)h}\bar{V}^{(1)}_k\mathfrak{J}^\pi)
\end{aligned}$$

$$=\xi^\lambda_i R^{i(0)(0)}_{jkh}+\nabla^{(0)}_h F^\lambda_{jk}-\nabla^{(0)}_k F^\lambda_{jh}+F^\lambda_{ja}S^{(0)(0)a}_{kh(0)}$$

$$+\frac{2}{3}\mathfrak{S}^\lambda_{(2)j(2)a}S^{(0)(0)a}_{kh(1)},$$

(b) $$R^{\lambda(1)(1)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_k\xi^\pi_h+\frac{2}{3}R^{\lambda(1)(2)}_{\mu\sigma\pi}\xi^\mu_j(\xi^\sigma_k\mathfrak{S}^\pi_{(2)h}-\xi^\sigma_h\mathfrak{S}^\pi_{(2)k})$$

$$=\xi^\lambda_i R^{i(1)(1)}_{jkh}+\frac{2}{3}(\nabla^{(1)}_h\mathfrak{S}^\lambda_{(2)j(2)k}-\nabla^{(1)}_k\mathfrak{S}^\lambda_{(2)j(2)h}),$$

(c) $$R^{\lambda(0)(1)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_k\xi^\pi_h+\frac{1}{3}R^{\lambda(1)(1)}_{\mu\sigma\pi}\xi^\mu_j\mathfrak{S}^\sigma_{(2)k}\xi^\pi_h+\frac{2}{3}R^{\lambda(0)(2)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_k\mathfrak{S}^\pi_{(2)h}$$

$$-\frac{1}{3}R^{\lambda(1)(2)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_h\bar{\nabla}^{(1)}_k\mathfrak{S}^\pi+\frac{2}{9}R^{\lambda(1)(2)}_{\mu\sigma\pi}\xi^\mu_j\mathfrak{S}^\sigma_{(2)k}\mathfrak{S}^\pi_{(2)\pi}$$

$$=\xi^\lambda_i R^{i(0)(1)}_{jkh}+\nabla^{(1)}_k F^\lambda_{jh}-\frac{2}{3}\nabla^{(0)}_k\mathfrak{S}^\lambda_{(2)j(2)h}+F^\lambda_{ja}S^{(0)(1)a}_{kh(0)}$$

$$+\frac{2}{3}\mathfrak{S}^\lambda_{(2)j(2)a}S^{(0)(1)a}_{kh(1)},$$

(d) $$R^{((0)(2)}_{\mu\sigma\pi}\xi^\mu_j\xi^\sigma_k\xi^\pi_h+\frac{1}{3}R^{\lambda(1)(2)}_{\mu\sigma\pi}\xi^\mu_i\mathfrak{S}^\sigma_{(2)k}\xi^\pi_h$$

$$=\xi^\lambda_i R^{i(0)(2)}_{jkh}+\nabla^{(2)}_h F^\lambda_{jk}+\frac{2}{3}\mathfrak{S}^\lambda_{(2)j(2)a}S^{(0)(2)a}_{kh(1)},$$

where

$$\overset{*}{R}{}^{\lambda(0)(0)}_{\mu\sigma\pi}=R^{\lambda(0)(0)}_{\mu\sigma\pi}+\underset{(1)}{\overset{*}{\Gamma}}{}^\lambda_{\mu\tau}S^{(0)(0)\tau}_{\sigma\pi(1)}.$$

The integrability conditions of (1.5) are

(2.2) (a) $$S^{(0)(0)\lambda}_{\mu\sigma(0)}\xi^\mu_j\xi^\sigma_k+\frac{1}{3}(\xi^\mu_j\mathfrak{S}^\sigma_{(2)k}-\xi^\mu_k\mathfrak{S}^\sigma_{(2)j})S^{(0)(1)\lambda}_{\mu\sigma(0)}=\xi^\lambda_i S^{(0)(0)i}_{jk(0)}+F^\lambda_{jk}-F^\nu_{kj},$$

(b) $$S^{(0)(1)\lambda}_{\mu\sigma(0)}\xi^\mu_j\xi^\sigma_k=\xi^\lambda_i S^{(0)(1)i}_{jk(0)}+\frac{2}{3}\mathfrak{S}^\lambda_{(2)j(2)k},$$

(c) $$S^{(0)(1)\lambda}_{\mu\sigma(1)}\xi^\mu_j\xi^\sigma_k+\frac{1}{3}S^{(0)(1)\lambda}_{\mu\sigma(1)}\xi^\mu_j\mathfrak{S}^\sigma_{(2)k}-\frac{1}{3}S^{(0)(1)\lambda}_{\mu\sigma(1)}\xi^\mu_k\mathfrak{S}^\sigma_{(2)j}$$

$$-\frac{1}{3}S^{(0)(2)\lambda}_{\mu\sigma(1)}\xi^\mu_k\bar{\nabla}^{(1)}_j\mathfrak{S}^\sigma+\frac{1}{3}S^{(0)(2)\lambda}_{\mu\sigma(1)}\xi^\mu_j\bar{\nabla}^{(1)}_k\mathfrak{S}^\sigma$$

$$=\xi^\lambda_i S^{(0)(0)i}_{jk(1)}+\frac{1}{3}\mathfrak{S}^\lambda_{(2)i}S^{(0)(0)i}_{jk(0)}+\frac{1}{3}\nabla^{(0)}_k\mathfrak{S}^\lambda_{(2)j}-\frac{1}{3}\nabla^{(0)}_j\mathfrak{S}^\lambda_{(2)k},$$

(d) $$S^{(0)(0)\lambda}_{\mu\sigma(2)}\xi^\mu_j\xi^\sigma_k+\frac{1}{3}S^{(0)(1)\lambda}_{\mu\sigma(2)}\xi^\mu_j\mathfrak{S}^\sigma_{(2)k}-\frac{1}{3}S^{(0)(1)\lambda}_{\mu\sigma(2)}\xi^\mu_k\mathfrak{S}^\sigma_{(2)j}$$

$$+\frac{1}{9}S^{(1)(1)\lambda}_{\mu\sigma(2)}\mathfrak{S}^\mu_{(2)j}\mathfrak{S}^\sigma_{(2)k}=\xi^\lambda_i S^{(0)(0)i}_{jk(2)}+\frac{2}{3}\mathfrak{S}^\lambda_{(2)i}S^{(0)(0)i}_{jk(1)}$$

$$+\frac{1}{3}(\bar{\nabla}_i^{(1)}\mathfrak{J}^\lambda)S_{jk(0)}^{(0)(0)i}+\frac{1}{3}\nabla_k^{(0)}\bar{\nabla}_j^{(1)}\mathfrak{J}^\lambda-\frac{1}{3}\nabla_j^{(0)}\bar{\nabla}_k^{(0)}\mathfrak{J}^\lambda\,,$$

$$\text{(e)}\quad S_{\mu\sigma(2)}^{(0)(1)\lambda}\xi_j^\mu\xi_k^\sigma+\frac{1}{3}S_{\mu\sigma(2)}^{(1)(1)\lambda}\mathfrak{J}_{(2)j}^\mu\xi_k^\sigma-\frac{1}{3}S_{\mu\sigma(2)}^{(1)(1)\lambda}\mathfrak{J}_{(2)k}^\mu\xi_j^\sigma$$

$$=\xi_i^\lambda S_{jk(2)}^{(0)(1)i}-\frac{2}{3}\nabla_j^{(0)}\mathfrak{J}_{(2)k}^\lambda+\frac{2}{3}\mathfrak{J}_{(2)i}^\lambda S_{jk(2)}^{(0)(1)i}$$

$$+\frac{1}{3}\nabla_k^{(1)}\bar{\nabla}_j^{(1)}\mathfrak{J}^\lambda+\frac{1}{3}(\bar{\nabla}_i^{(1)}\mathfrak{J}^\lambda)S_{jk(0)}^{(0)(1)i}\,.$$

§ 3. The equations of (2.2) or (2.1) correspond to the equations of Gauss and Codazzi in Riemannian geometry. Let us consider the geometric meaning of them. At first, we must go back to § 0 and consider the classification of spaces.

Definition 3. If $\overset{*}{R}{}_{\mu\sigma\pi}^{\lambda(0)(0)}=0$, *at every point of* V_n, *such a* V_n *is called* 0*-flat.*

Definition 4. If $R_{\mu\sigma\pi}^{\lambda(1)(1)}=0$, *at every point of* V_n, *such a* V_n *is called* 1*-flat.*

Then, we have from (0.8) the following theorems:

Theorem 1. In order that $\underset{21}{\delta}\underset{12}{\delta}v^\lambda-\underset{12}{\delta}\underset{21}{\delta}v^\lambda=0$ *for an arbitrary* v^λ *and vanishing* $\delta x^{(1)\lambda}$, $\delta x^{(2)\lambda}$, *it is necessary and sufficient that* V_n *is* 0*-flat.*

Definition 5. If $S_{\mu\sigma(0)}^{(0)(0)\lambda}=0$, *at every point of* V_n, *such a* V_n *is called* 0*-symmetric.*

Definition 6. If $S_{\mu\sigma(1)}^{(0)(0)\lambda}=0$, *at every point of* V_n, *such a* V_n *is called* 1*-symmetric.*

Definition 7. If $S_{\mu\sigma(2)}^{(0)(0)\lambda}=0$, *at every point of* V_n, *such a* V_n *is called* 2*-symmetric.*

Then we have from (0.10) the following theorems:

Theorem 3. In order that $\underset{21}{\delta}dx^\lambda-\underset{12}{\delta}dx^\lambda=0$, *for vanishing* $\delta x^{(1)\lambda}$, *it is necessary and sufficient that* V_n *is* 0*-symmetric.*

Theorem 4. In order that $\underset{21}{\delta}\delta x^{(1)\lambda}-\underset{12}{\delta}\delta x^{(1)\lambda}=0$, *for vanishing* $\delta x^{(2)\lambda}$, *it is necessary and sufficient that* V_n *is* 1*-symmetric.*

Theorem 5. In order that $\underset{21}{\delta}\delta x^{(2)\lambda}-\underset{12}{\delta}\delta x^{(2)\lambda}=0$, *for vanishing* dx^λ, *it is necessary and sufficient that* V_n *is* 2*-symmetric.*

Remark. If V_n is 0-flat, it is also 1-symmetric. If V_n is 0-symmetric, it is also 1-flat, because we have after some calculation

$$R_{\tau\mu\sigma}^{\lambda(0)(0)}x^{(1)\tau}=S_{\mu\sigma(1)}^{(0)(0)\lambda}\,,\qquad (S_{\mu\sigma(0)}^{(0)(0)\lambda})_{(2)\tau}=\frac{1}{2}R_{\tau\mu\sigma}^{\lambda(1)(1)}\,.$$

By means of these theorems and (2.1) or (2.2), we have the following results.

Theorem 6. (i) *The totally geodesic* V_{n-1} *immersed in the* 0*-flat* V_n *is also* 0*-flat.*

(ii) *The totally geodesic* V_{n-1} *immersed in the* 1*-symmetric* V_n *is also* 1*-symmetric.*

(iii) *The totally geodesic* V_{n-1} *immersed in the* 2-*symmetric* V_n *is also* 2-*symmetric.*

(iv) *The semi-geodesic* V_{n-1} *immersed in the* 1-*flat* V_n *is also* 1-*flat.*

(v) *The semi-geodesic* V_{n-1} *immersed in the* 0-*symmetric* V_n *is also* 0-*symmetric.*

In conclusion I wish to express my cordial thanks to Prof. K. Yano for his kind guidance and many benefical suggestions.