85. On the Strong Summability of Fourier Series.

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Let $f(x)$ be a real function of period 2π, integrable L over $(0, 2\pi)$, and let

$$f(x)\sim\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x).$$

By $s_n(x)$ and $\sigma_n(x)$ we denote the n-partial sum and the n-th arithmetic mean of the above series, respectively.

Zygmund[1] has proved the following theorem.

If f is in L^p, where $p>1$, then

$$\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}(s_n-\sigma_n)^2/n\right\}^{\frac{1}{2}p}dx\leqq A_p\int_0^{2\pi}|f|^p dx,$$

where A_p depends on p.

In §1, the author proves that the exponent 2 in the left hand side series may be replaced by arbitrary index $m\geqq 2$. In §2, we give a theorem on the strong summability of double Fourier series. The case of index $m=2$ has been given by Marcinkiewicz.[2] Finally in §3, the strong summability theorem of lacunary sequence of partial sums is proved. The case of index $m=2$ has been investigated by Zalcwasser[3] and Zygmund.[4]

I. We begin with some preliminary lemmas.[5]

Lemma 1. If $\{n_k\}$ denotes any sequence of positive integers satisfying the condition $n_{k+1}/n_k>a>1$, then

$$\int_0^{2\pi}\left(\sum_{k=1}^{\infty}|s_{n_k}-\sigma_{n_k}|^2\right)^{\frac{1}{2}p}dx\leqq B_p\int_0^{2\pi}|f|^p dx.$$

This is known.[6]

Lemma 2. Let $f_1, f_2, \dots$ be a sequence of functions of period 2π, integrable L, and let $s_{n,\nu}$ denotes the ν-th partial sum of the Fourier series of f_n. Then

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|s_{n,k_n}|^m\right)^{\frac{p}{m}}dx\leqq C_{m,p}\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|f_n|^m\right)^{\frac{p}{m}}dx,$$

where $p>1$ and $m>1$.

This lemma is due to Boas and Bochner[7] when $k_n=\nu$. But the

1) A. Zygmund, Fund. Math., **30** (1938), 170–196.
2) J. Marcinkiewicz, Annali di Pisa, **8** (1939), 149–160.
3) Z. Zalcwasser, Studia Math., **6** (1936), 82–88.
4) A. Zygmund, loc. cit.
5) $A_{m,p}$, $B_{m,p}$, ... denote constants depending only on m and p.
6) A. Zygmund, loc. cit.
7) R. P. Boas, Jr. and S. Bochner, Journ. London Math. Soc., **14** (1939), 62–73.

above generalization is easily established by the ordinary method from their lemma.

Theorem 1. When $m \geqq 2$, $p > 1$,

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} | s_n - \sigma_n |^m / n \right)^{\frac{p}{m}} dx \leqq D_{m,p} \int_0^{2\pi} |f|^p dx .$$

In particular if f belongs to $L^p (p > 1)$, *then the series*

$$\sum_{n=1}^{\infty} | s_n - \sigma_n |^m / n \qquad (m \geqq 2)$$

converges almost everywhere, so that

$$(n+1)^{-1} \sum_{\nu=0}^{n} | s_\nu - f |^m \to 0 \qquad (m \geqq 1)$$

almost everywhere.

If dash denotes differentiation with respect to x, then

$$| s_n - \sigma_n | = | s'_n | / (n+1) .$$

Applying Lemma 2, Jensen's inequality and Lemma 1 successively, we obtain for $m \geqq 2$

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} | s_n - \sigma_n |^m / n \right)^{\frac{p}{m}} dx \leqq \int_0^{2\pi} \left(\sum_{n=1}^{\infty} | s'_n |^m / n^{m+1} \right)^{\frac{p}{m}} dx$$

$$\leqq \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} | s'_n |^m / n^{m+1} \right)^{\frac{p}{m}} dx$$

$$\leqq E_{m,p} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} | s'_{2^k} |^m \sum_{n=2^{k-1}}^{2^k-1} 1/n^{m+1} \right)^{\frac{p}{m}} dx$$

$$\leqq F_{m,p} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} | s'_{2^k} |^m / (2^k)^m \right)^{\frac{p}{m}} dx$$

$$\leqq G_{m,p} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} | s_{2^k} - \sigma_{2^k} |^m \right)^{\frac{p}{m}} dx$$

$$\leqq H_{m,p} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} | s_{2^k} - \sigma_{2^k} |^2 \right)^{\frac{1}{2} p} dx$$

$$\leqq I_{m,p} \int_0^{2\pi} | f |^p dx .$$

Hence if $f(x)$ belongs to $L^p (p > 1)$, the series $\sum_{n=1}^{\infty} | s_n - \sigma_n |^m / n (m \geqq 2)$ converges almost everywhere and by Kronecker's theorem $(n+1)^{-1} \sum_{\nu=0}^{n} | s_\nu - f |^m \to 0$ $(m \geqq 1)$ almost everywhere.

II. Let $f(x, y)$ be a function of period 2π, integrable L. By $s_{m,n}(f;\ x, y)$ and $\sigma_{m,n}(f;\ x, y)$, we denote the n-th partial sum and the n-th arithmetic mean of Fourier series of $f(x, y)$. Then we have

Lemma 3. If $m>1$, $p>1$, we have

$$\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|\sigma_{k_n}(f_n,x)|^m\Big)^{\frac{p}{m}}dx \leqq J_{m,p}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|f_n|^m\Big)^{\frac{p}{m}}dx .$$

For,

$$|\sigma_{k_n}(f_n,x)|^m \leqq (k_n+1)^{-1}\sum_{\nu=0}|s_\nu(f_n,x)|^m$$

and Lemma 2 give us the lemma.

Theorem 2. If $f(x,y)\in L^p(p>1)$, then

$$\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|s_{n,n}-\sigma_{n,n}|^m/n\Big)^{\frac{p}{m}}dxdy \leqq K_{m,p}\int_0^{2\pi}\int_0^{2\pi}|f|^p dxdy$$

for $m\geqq 2$. Especially

$$(n+1)^{-1}\sum_{\nu=0}^{n}|s_{\nu,\nu}-f|^m \to 0 \qquad (m\geqq 1)$$

almost everywhere.

For every function $h(x,y)$, we describe it by $h^{(1)}(x,y)$ when we consider $h(x,y)$ as a function of x only, and by $h^{(2)}(x,y)$ when we consider as a function of y only. Then

$$|s_{n,n}-\sigma_{n,n}| = s_n\{s_n(f^{(2)})-\sigma_n(f^{(2)})\}^{(1)}+\sigma_n\{s_n(f^{(1)})-\sigma_n(f^{(1)})\}^{(2)} = P_n+Q_n ,$$

say. From Lemma 2.

$$\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|P_n|^m/n\Big)^{\frac{p}{m}}dx \leqq L_{m,p}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|s^n(f^{(2)})-\sigma_n(f^{(2)})|^m/n\Big)^{\frac{p}{m}}dx .$$

Integrating with respect to y and applying Theorem 1, we obtain

$$\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|P_n|^m/n\Big)^{\frac{p}{m}}dxdy \leqq M_{m,p}\int_0^{2\pi}\int_0^{2\pi}|f|^p dxdy .$$

Applying Lemma 3, we get by the analogous calculation

$$\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|s_{n,n}-\sigma_{n,n}|^m/n\Big)^{\frac{p}{m}}dxdy \leqq P_{m,p}\int_0^{2\pi}\int_0^{2\pi}|f|^p dxdy .$$

Thus we get the first part of the theorem. The remaining part is obvious.

III. *Theorem 3. If $\{p_n\}$ is an increasing sequence of natural numbers, which satisfies*

$$\sum_{n_{k-1}}^{n_k-1} 1/np_n^m = O(1/p_{n_k}^m)$$

for some $\{n_k\}$ such as $\beta > n_k/n_{k-1} > \alpha > 1$, then for $f(x)\in L^p(p>1)$,

$$\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|s_{p_n}-\sigma_{p_n}|^m/n\Big)^{\frac{p}{m}}dx \leqq Q_{m,p}\int_0^{2\pi}|f|^p dx$$

where $m \geqq 2$. *In particular*

$$(n+1)^{-1}\sum_{\nu=0}^{n}|s_{p_\nu}-f|^m \to 0 \qquad (m\geqq 1)$$

almost everywhere.

The above condition is satisfied by $p_n=[n^l]$, $l\geqq 1$.

For,

$$\sum_{n=1}^{\infty}|s_{p_n}-\sigma_{p_n}|^m/n=\sum_{n=1}^{\infty}|s'_{p_n}|^m/n(p_n+1)^m.$$

From Lemma 2 and 1, we have

$$\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|s'_{p_n}|^m/np_n^m\Big)^{\frac{p}{m}}dx=\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}\sum_{n=n_{k-1}}^{n_k-1}|s'_{p_n}|^m/np_n^m\Big)^{\frac{p}{m}}dx$$

$$\leqq R_{m,p}\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|s'_{p_{n_k}}|^m\sum_{n=n_{k-1}}^{n_k-1}1/np_n^m\Big)^{\frac{p}{m}}dx$$

$$\leqq S_{m,p}\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|s'_{p_{n_k}}|^m/p_{n_k}\Big)^{\frac{p}{m}}dx$$

$$\leqq T_{m,p}\int_0^{2\pi}|f|^p dx.$$

Thus we obtain the theorem.