131. *Induced Measure Preserving Transformations.*

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§ **1.** The purpose of this paper is to investigate the properties of induced measure preserving transformations. We shall give only definitions and fundamental results, leaving the discussions of the details to another occasion.

§ **2.** A *measure space* $(\Omega, \mathfrak{B}, m)$ is a triple of a space $\Omega=\{\omega\}$, a Borel field $\mathfrak{B}=\{B\}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ satisfying $0 < m(\Omega) \leqq \infty$. In case $m(\Omega)=\infty$, it is assumed that there exists a sequence $\{B_n \mid n=1, 2, \ldots\}$ of subsets B_n of Ω such that $B_n \in \mathfrak{B}$, $m(B_n) < \infty$, $n=1, 2, \ldots$, and $\bigcup_{n=1}^{\infty} B_n = \Omega$. A subset B of Ω belonging to $\mathfrak{B}$ is called *$\mathfrak{B}$-measurable*, and $m(B)$ is its *m-measure*. A $\mathfrak{B}$-measurable subset N of Ω of m-measure zero is called a *null set* of $(\Omega, \mathfrak{B}, m)$, and the family of all null sets of $(\Omega, \mathfrak{B}, m)$ is denoted by $\mathfrak{N}(\Omega, \mathfrak{B}, m)$.

For any $\mathfrak{B}$-measurable subset Ω' of Ω with a positive m-measure, let us denote by $\mathfrak{B}_{\Omega'}$ the family of all $\mathfrak{B}$-measurable subsets B of Ω, and put $m_{\Omega'}(B)=m(B)$ on $\mathfrak{B}_{\Omega'}$. Then $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ is a measure space which we call *the measure space induced on Ω' by* $(\Omega, \mathfrak{B}, m)$, or simply an *induced measure space.*

A one-to-one mapping $\omega'=\varphi(\omega)$ of a measure space $(\Omega, \mathfrak{B}, m)$ onto another measure space $(\Omega', \mathfrak{B}', m')$ is a *measure preserving transformation* (m. p. t.) *in a strong sense* if $B \in \mathfrak{B}$ implies $\varphi(B) \in \mathfrak{B}'$, $m'\big(\varphi(B)\big)= m(B)$ and if conversely $B' \in \mathfrak{B}'$ implies $\varphi^{-1}(B') \in \mathfrak{B}$, $m\big(\varphi^{-1}(B')\big)=m'(B')$. If there exists two null sets $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ and $N' \in \mathfrak{N}(\Omega', \mathfrak{B}', m')$, and if $\omega'=\varphi(\omega)$ is a m. p. t. in a strong sense of $(\Omega-N, \mathfrak{B}_{\Omega-N}, m_{\Omega-N})$ onto $(\Omega'-N', \mathfrak{B}'_{\Omega'-N'}, m'_{\Omega'-N'})$, then $\omega'=\varphi(\omega)$ is called a *measure preserving transformation* (m. p. t.) *in a weak sense* of $(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$. $\mathfrak{D}(\varphi)=\Omega-N$ is the *domain* of φ and $\mathfrak{R}(\varphi)=\Omega'-N'$ is the *range* of φ. When we speak of a m. p. t. in a weak sense φ, the domain and the range of φ are usually not explicitly stated. Given two m. p. t. in a weak sense $\omega'=\varphi(\omega)$ and $\omega'=\psi(\omega)$ which map the same measure space $(\Omega', \mathfrak{B}', m')$ onto the same measure space $(\Omega', \mathfrak{B}', m')$, φ and ψ are called *almost equal* (notation: $\varphi \approxeq \psi$) if $\varphi(\omega)=\psi(\omega)$ almost everywhere on $(\Omega, \mathfrak{B}, m)$, or more precisely, if there exists a null set $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ such that $\Omega-N \subseteqq \mathfrak{D}(\varphi) \frown \mathfrak{D}(\psi)$ and $\varphi(\omega)=\psi(\omega)$ for all $\omega \in \Omega-N$. This notion of almost equality is clearly reflexive, symmetric and transitive. The class of all m. p. t. in a weak sense which are almost equal with φ is denoted by $[\varphi]$.

Let us now consider a case when two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ coincide. Then we obtain the notion of a m. p. t. in a strong sense or in a weak sense $\omega'=\varphi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto itself. The family $\Phi(\Omega, \mathfrak{B}, m)$ of all m. p. t. in a strong sense of a measure space $(\Omega, \mathfrak{B}, m)$ onto itself is clearly a group with respect to the usual way of taking product and inverse. Further it is clear that for every m. p. t. in a weak sense $\omega' = \varphi(\omega)$ which maps a measure space $(\Omega, \mathfrak{B}, m)$ onto itself, there exists a null set $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ such that $\Omega - N \subseteqq \mathfrak{D}(\varphi) \frown \mathfrak{R}(\varphi)$ and $\omega' = \varphi(\omega)$ is a m. p. t. in a strong sense of $(\Omega - N, \mathfrak{B}_{\Omega-N}, m_{\Omega-N})$ onto itself. Let now $\omega' = \varphi(\omega)$ and $\omega' = \psi(\omega)$ be two m. p. t. in a weak sense which map the same measure space $(\Omega, \mathfrak{B}, m)$ onto itself. Then $\chi = \varphi\psi$ is, by definition, a m. p. t. in a weak sense $\omega' = \chi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto itself for which $\mathfrak{D}(\chi) = \psi^{-1}\big(\mathfrak{D}(\varphi) \frown \mathfrak{R}(\psi)\big)$, $\mathfrak{R}(\chi) = \varphi\big(\mathfrak{D}(\varphi) \frown \mathfrak{R}(\psi)\big)$ and $\chi(\omega) = \varphi\big(\psi(\omega)\big)$ on $\mathfrak{D}(\chi)$. Further, if $\omega' = \varphi(\omega)$ is a m. p. t. in a weak sense which maps $(\Omega, \mathfrak{B}, m)$ onto itself, then $\omega' = \varphi^{-1}(\omega)$ is a m. p. t. in a weak sense which maps $(\Omega, \mathfrak{B}, m)$ onto itself for which $\mathfrak{D}(\varphi^{-1}) = \mathfrak{R}(\varphi)$, $\mathfrak{R}(\varphi^{-1}) = \mathfrak{D}(\varphi)$, and $\varphi^{-1}(\omega) = \omega'$ on $\mathfrak{D}(\varphi^{-1})$ if $\varphi(\omega') = \omega$. Since it is clear that $\varphi \approxeq \varphi'$ and $\psi \approxeq \psi'$ imply $\varphi\psi \approxeq \varphi'\psi'$ and $\varphi^{-1} \approxeq \varphi'^{-1}$, so we see that the family of all classes $[\varphi]$ of m. p. t. in a weak sense which map a measure space $(\Omega, \mathfrak{B}, m)$ onto itself may be considered as a group in which the product and the inverse are defined by $[\varphi][\psi] = [\varphi\psi]$ and $[\varphi]^{-1} = [\varphi^{-1}]$ respectively.

A m. p. t. in a strong sense $\omega' = \varphi(\omega)$ which maps a measure space $(\Omega, \mathfrak{B}, m)$ onto itself is *ergodic* if, for any two $\mathfrak{B}$-measurable subsets A and B of Ω with $m(A) > 0$ and $m(B) > 0$, there exists a positive integer n such that $m\big(\varphi^n(A) \frown B\big) > 0$. The ergodicity of a m. p. t. in a weak sense may be defined analogously.

Two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ are *isomorphic* with each other $\big($notation : $(\Omega, \mathfrak{B}, m) \approx (\Omega', \mathfrak{B}', m')\big)$ if there exists a m. p. t. in a weak sense $\omega' = \chi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$. Let further $\omega' = \varphi(\omega)$ and $\omega' = \psi(\omega)$ be two m. p. t. in a weak sense defined on the measure spaces $(\mathfrak{B}, \Omega, m)$ and $(\Omega', \mathfrak{B}', m')$ respectively. φ and ψ are *isomorphic* with each other (notation : $\varphi \approx \psi$) if the measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ are isomorphic with each other by means of a m. p. t. in a weak sense $\omega' = \chi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$ such that $\chi\big(\varphi(\omega)\big) = \psi\big(\chi(\omega)\big)$ almost everywhere on $(\Omega, \mathfrak{B}, m)$. It is obvious that $\varphi \approxeq \psi$ implies $\varphi \approx \psi$. The family of all m. p. t. in a weak sense which are isomorphic with φ is denoted by $\big[[\varphi]\big]$.

§ **3.** Let $\omega' = \varphi(\omega)$ be an ergodic m. p. t. in a strong sense defined on a measure space $(\Omega, \mathfrak{B}, m)$, and let Ω' be a $\mathfrak{B}$-measurable subset of Ω with a positive m-measure. Then

Lemma 1[1]. *There exists a null set $N \in \mathfrak{N}(\Omega, \mathfrak{B}, m)$ contained in Ω' such that $\omega \in \Omega' - N$ implies $\varphi^n(\omega) \in \Omega' - N$ for infinitely many positive integers n and for infinitely many negative integers n.*

For each $\omega \in \Omega' - N$, let us put $\varphi_{\Omega'}(\omega) = \varphi^n(\omega)$, where $n = n(\omega)$ is the smallest positive integer such that $\varphi^n(\omega) \in \Omega' - N$. Then

1) In case $m(\Omega) < \infty$, we have no need to assume that $\omega' = \varphi(\omega)$ is ergodic.

Lemma 2. $\omega'=\varphi_{\Omega'}(\omega)$ *is an ergodic m. p. t. in a strong sense which maps* $(\Omega'-N, \mathfrak{B}_{\Omega'-N}, m_{\Omega'-N})$ *onto itself.*

Thus, to every ergodic m. p. t. in a strong sense $\omega'=\varphi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto itself, there corresponds an ergodic m. p. t. in a weak sense $\omega'=\varphi_{\Omega'}(\omega)$ which maps the induced measure space $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ onto itself. $\varphi_{\Omega'}$ is called the *m. p. t. in a weak sense induced on* $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ *by* φ or simply an *induced m. p. t. in a weak sense.*

Further, it is easy to see that to every ergodic m. p. t. in a weak sense $\omega'=\varphi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto itself there corresponds, in a similar way, an ergodic m. p. t. in a weak sense $\omega'=\varphi_{\Omega'}(\omega)$ which maps the induced measure space $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ onto itself. Since it is clear that $\varphi \approxeq \psi$ on $(\Omega, \mathfrak{B}, m)$ implies $\varphi_{\Omega'} \approxeq \psi$ on $(\Omega', \mathfrak{B}', m')$, so we may consider that the operation of taking an induced m. p. t. is defined on the classes $[\varphi]$ by $[\varphi]_{\Omega'}=[\varphi_{\Omega'}]$.

§ **4.** Let φ and ψ be two ergodic m. p. t. in a weak sense which map the measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega' \mathfrak{B}' m')$ respectively onto themselves. ψ is called a *derivative* of φ or φ is called a *primitive* of ψ (notation: $\varphi \succ \psi$ or $\psi \prec \varphi$) if there exists a $\mathfrak{B}$-measurable subset Ω' of Ω with a positive m-measure such that the m. p. t. in a weak sense $\varphi_{\Omega'}$ which is induced on $(\Omega', \mathfrak{B}_{\Omega'}, m_{\Omega'})$ by φ is isomorphic with ψ. Since it is clear that $\varphi \approx \varphi'$, $\psi \approx \psi'$ and $\varphi \prec \psi$ imply $\varphi' \succ \psi'$, so we may say that a class $\left[[\psi]\right]$ is a *derivative* of a class $\left[[\varphi]\right]$ or that $\left[[\varphi]\right]$ is a *primitive* of $\left[[\psi]\right]$, if $\varphi' \succ \psi'$ for some (and hence for all) $\varphi' \in \left[[\varphi]\right]$ and $\psi' \in \left[[\psi]\right]$.

Lemma 3. $\varphi \succ \psi$ *and* $\psi \succ \chi$ *imply* $\varphi \succ \chi$.

There now arises a natural question whether $\varphi \succ \psi$ and $\psi \succ \varphi$ imply $\varphi \approx \psi$ or not. As an answer to this question we first notice that it is possible to find two m. p. t. in a weak sense φ and ψ each defined on a measure space with an infinite total measure such that $\varphi \succ \psi$ and $\psi \succ \varphi$ without being $\varphi \approx \psi$. On the other hand, if one (and hence both) of the measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ on which φ and ψ are defined respectively has a finite total measure, then our problem is trivial: for, $\varphi \succ \psi$ and $\psi \succ \varphi$ imply $m(\Omega)=m'(\Omega')$, and $\varphi \succ \psi$ is compatible with $m(\Omega)=m'(\Omega') < \infty$ if and only if $\varphi \approx \psi$.

In order to investigate the situation in more detail, let us define a *weak isomorphism* as follows: two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ are *weakly isomorphic* with each other (notation: $(\Omega, \mathfrak{B}, m) \mathrel{\dot{\approx}} (\Omega', \mathfrak{B}', m')$) if there exist a one-to-one mapping $\omega'=\chi(\omega)$ of Ω onto Ω' and a positive number α such that $B \in \mathfrak{B}$ implies $\varphi(B) \in \mathfrak{B}$, $m'(\varphi(B))=\alpha m(B)$, and further that conversely $B' \in \mathfrak{B}'$ implies $\varphi^{-1}(B') \in B$, $m(\varphi^{-1}(B'))=\alpha^{-1}m'(B')$. (Such a mapping $\omega'=\chi(\omega)$ is called a *measure multiplying transformation*). Further, two m. p. t. in a weak sense φ and ψ defined on two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ respectively are *weakly isomorphic* with each other (notation: $\varphi \mathrel{\dot{\approx}} \psi$) if $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ are weakly isomorphic with each other by means of a measure multiplying transformation $\omega'=\chi(\omega)$ which maps

$(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$ such that $\chi(\varphi(\omega)) = \psi(\chi(\omega))$ almost everywhere on $(\Omega, \mathfrak{B}, m)$. The family of all ergodic m. p. t. in a weak sense which are weakly isomorphic with φ is denoted by $\left[[\varphi]\right]^{\cdot}$.

Let now φ and ψ be two ergodic m. p. t. in a weak sense. ψ is called a *weak derivative* of φ or φ is a *weak primitive* of ψ (notation $\varphi \dot{\succ} \psi$ or $\psi \dot{\prec} \varphi$) if there exists a derivative of φ which is weakly isomorphic with ψ. Since it is clear that $\varphi \dot{\approx} \varphi'$, $\psi \dot{\approx} \psi'$ and $\varphi \dot{\succ} \psi$ imply $\varphi' \dot{\succ} \psi'$, so we may say that a class $\left[[\psi]\right]^{\cdot}$ is a *weak derivative* of a class $\left[[\varphi]\right]^{\cdot}$ or that $\left[[\varphi]\right]^{\cdot}$ is a *weak primitive* of $\left[[\psi]\right]^{\cdot}$ if $\varphi' \dot{\succ} \psi'$ for some (and hence for all) $\varphi' \in \left[[\varphi]\right]^{\cdot}$ and $\psi' \in \left[[\psi]\right]^{\cdot}$

After these preliminaries we may say that it is possible to find two ergodic m. p. t. in a weak sense φ and ψ each defined on a measure space with a finite total measure such that $\varphi \dot{\succ} \psi$ and $\psi \dot{\succ} \varphi$ without being $\varphi \dot{\approx} \psi$.

§ **5.** We are now in a position to state

Lemma 4. For any two ergodic m. p. t. in a weak sense φ_1 and φ_2, the following two conditions are mutually equivalent: (i) *there exists an ergodic m. p. t. in a weak sense φ_3 such that $\varphi_3 \succ \varphi_1$ and $\varphi_3 \succ \varphi_2$.* (ii) *there exists an ergodic m. p. t. in a weak sense φ_4 such that $\varphi_1 \succ \varphi_4$ and $\varphi_2 \succ \varphi_4$.*

This result is due to J. von Neumann. The implication (i) → (ii) can be proved by appealing to Lemma 3 as well as to the following

Lemma 5. Let φ be an ergodic m. p. t. in a weak sense defined on a measure space $(\Omega, \mathfrak{B}, m)$, and let Ω_1 and Ω_2 be two $\mathfrak{B}$-measurable subsets of Ω with a positive m-measure. Then $\varphi_{\Omega_1} \approx \varphi_{\Omega_2}$ if there exists an integer n such that $\varphi^n(\Omega_1) = \Omega_2$.

The inverse implication (ii) → (i) may be proved by constructing a measure space $(\Omega, \mathfrak{B}, m)$, an ergodic m. p. t. in a weak sense $\omega' = \varphi(\omega)$ defined on it, and two $\mathfrak{B}$-measurable subsets Ω_1 and Ω_2 of Ω with a positive m-measure such that the m. p. t. in a weak sense φ_{Ω_1} and φ_{Ω_2} induced on the induced measure spaces $(\Omega_1, \mathfrak{B}_{\Omega_1}, m_{\Omega_1})$ and $(\Omega_2, \mathfrak{B}_{\Omega_2}, m_{\Omega_2})$ respectively are isomorphic with φ_1 and φ_2 respectively.

In case one (and hence both) of the conditions (i) and (ii) of Lemma 4 is satisfied, φ_1 and φ_2 are called *equivalent* with each other (notation: $\varphi_1 \sim \varphi_2$). It is clear that this relation is reflexive and symmetric. But the transitivity is not obvious:

Lemma 6. $\varphi_1 \sim \varphi_2$ and $\varphi_2 \sim \varphi_3$ imply $\varphi_1 \sim \varphi_3$.

We shall denote by $\left[\left[[\varphi]\right]\right]$ the family of all ergodic m. p. t. in a weak sense which are equivalent with φ.

Thus we have obtained a notion of equivalence among ergodic m. p. t. in a weak sense. Our next problem is to find out what it means in ergodic theory that two ergodic m. p. t. in a weak sense are equivalent with each other, or in other words, to obtain an ergodic-theoretical interpretation of this equivalence relation.

First it is interesting to observe that our equivalence relation seems to have nothing to do with the spectral properties of the ergodic

m. p. t. in a weak sense in question. In fact, as was shown by J. von Neumann, there exists a pair of ergodic m. p. t. in a weak sense φ and ψ equivalent with each other such that φ has a pure point spectrum and yet ψ has a continuous spectrum. Further, for the most of the well known concrete examples of ergodic m. p. t. φ, it is possible to find an ergodic m. p. t. in a weak sense with a continuous spectrum which is equivalent with φ. It is conjectured that every equivalence class contains an ergodic m. p. t. with a continuous spectrum. Finally we may even state a rather bold conjecture that any two ergodic m. p. t. in a weak sense are equivalent with each other, or in other words that there exists only one equivalence class $\left[\left[[\varphi]\right]\right]$. The meaning of this equivalence relation and of these conjectures will be explained, to a certain extent, in the following section.

§ **6.** In order to state the main result of this paper, we need the notion of a *flow* and of a *flow built under a function*[1]. *A flow in a strong sense* is a one-parameter family $\Phi=\{\varphi_t(\omega)\,|-\infty<t<\infty\}$ of m. p. t. in a strong sense of a measure space $(\Omega, \mathfrak{B}, m)$ onto itself with the group property: $\varphi_t\big(\varphi_s(\omega)\big)=\varphi_{t+s}(\omega)$ for all $\omega\in\Omega$ and for all t, s with $-\infty<t, s<\infty$. If there exists a null set $N\in\mathfrak{N}(\Omega, \mathfrak{B}, m)$ and if Φ is a flow in a strong sense defined on $(\Omega-N, \mathfrak{B}_{\Omega-N}, m_{\Omega-N})$, then Φ is called a *flow in a weak sense.*

A flow in a strong sense $\Phi=\{\varphi_t(\omega)\,|-\infty<t<\infty\}$ defined on a measure space $(\Omega, \mathfrak{B}, m)$ is *ergodic* if, for any two $\mathfrak{B}$-measurable subsets A and B of Ω with $m(A)>0$ and $m(B)>0$, there exists a real number t such that $m\big(\varphi_t(A)\frown B\big)>0$. The ergodicity of a flow in a weak sense is defined analogously. Two flows in a weak sense $\Phi=\{\varphi_t(\omega)\,|-\infty<t<\infty\}$ and $\Psi=\{\psi_t(\omega)\,|-\infty<t<\infty\}$ defined on two measure spaces $(\Omega, \mathfrak{B}, m)$ and $(\Omega', \mathfrak{B}', m')$ respectively are *isomorphic* with each other if there exists a m. p. t. in a weak sense $\omega'=\chi(\omega)$ which maps $(\Omega, \mathfrak{B}, m)$ onto $(\Omega', \mathfrak{B}', m')$ such that $\chi\big(\varphi_t(\omega)\big)=\psi_t\big(\chi(\omega)\big)$ for all $\omega\in\mathfrak{D}(\chi)$ and for all real numbers t.

A measure space $(\Omega, \mathfrak{B}, m)$ is *completed* if every subset of a null set of $(\Omega, \mathfrak{B}, m)$ is again a null set of $(\Omega, \mathfrak{B}, m)$. A flow $\Phi=\{\varphi_t(\omega)\,|-\infty<t<\infty\}$ defined on a completed measure space $(\Omega, \mathfrak{B}, m)$ is *measurable* if the mapping $(\omega, t)\rightarrow\varphi_t(\omega)$ is measurable as a mapping of the cartesian product $\widetilde{\Omega}=\Omega\times R^1$ onto Ω, where $R^1=\{t\,|-\infty<t<\infty\}$, i. e. if, for any $B\in\mathfrak{B}$, the set $\widetilde{B}=\{\widetilde{\omega}=(\omega, t)\,|\,\varphi_t(\omega)\in B\}\subseteqq\widetilde{\Omega}$ is $\mathfrak{B}$-measurable with respect to the completed direct product measure space $(\widetilde{\Omega}, \widetilde{\mathfrak{B}}, \widetilde{m})=(\Omega, \mathfrak{B}, m)\otimes(R^1, \mathfrak{L}, l)$, where we denote by $(R^1, \mathfrak{L}, l)$ the ordinary Lebesgue measure space defined on R^1.

Let $\omega'=\varphi(\omega)$ be a m. p. t. in a strong sense defined on a completed measure space $(\Omega, \mathfrak{B}, m)$, and let $f(\omega)\in L^+(\Omega, \mathfrak{B}, m)$, where we denote by $L^+(\Omega, \mathfrak{B}, m)$ the family of all positive valued $\mathfrak{B}$-measurable functions $f(\omega)$ which are defined and integrable on $(\Omega, \mathfrak{B}, m)$. Consider

1) Cf. W. Ambrose, Representation of ergodic flows, Annals of Math., **42** (1941).

the subset $\bar{\Omega}=\{\bar{\omega}\}$ of all pairs $\bar{\omega}=\{\omega, x\}$, $\omega\in\Omega$, $0\leqq x<f(\omega)$ as a subset of $\tilde{\Omega}=\Omega\times R^1$, and let $(\bar{\Omega}, \bar{\mathfrak{B}}, \bar{m})=(\bar{\Omega}, \tilde{\mathfrak{B}}_{\bar{\Omega}}, \tilde{m}_{\bar{\Omega}})$ be the measure space induced on $\bar{\Omega}$ by $(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{m})$. Further, define a flow $\bar{\Phi}=\{\bar{\varphi}_t(\bar{\omega})\mid -\infty<t<\infty\}$ on $(\bar{\Omega}, \bar{\mathfrak{B}}, \bar{m})$ by

$$\begin{aligned}\bar{\varphi}_t(\omega, x)&=\Big(\varphi^n(\omega), x+t-\textstyle\sum_{k=0}^{n-1}f\big((\varphi^k(\omega))\big)\Big),\\ &\qquad\text{if}\quad \textstyle\sum_{k=0}^{n-1}f\big(\varphi^k(\omega)\big)-x\leqq t<\sum_{k=0}^{n}f\big(\varphi^k(\omega)\big)-x,\ n=1,2,\dots,\\ &=(\omega, x+t),\quad\text{if}\quad -x\leqq t<f(\omega)-x,\\ &=\Big(\varphi^{-n}(\omega),\ x+t+\textstyle\sum_{k=1}^{n}f\big(\varphi^{-k}(\omega)\big)\Big),\\ &\qquad\text{if}\quad -\textstyle\sum_{k=1}^{n}f\big(\varphi^{-k}(\omega)\big)-x\leqq t<-\sum_{k=1}^{n-1}f\big(\varphi^{-k}(\omega)\big),\ n=1,2,\dots\end{aligned}$$

$\bar{\Phi}$ is called the *flow in a strong sense built under the ceiling function* $f(\omega)$ *on the basis space* $(\Omega, \mathfrak{B}, m)$ *with respect to the m. p. t. in a strong sense* $\omega'=\varphi(\omega)$, or simply a *flow built under a function.* Similarly, a *flow in a weak sense built under* $f(\omega)$ *on* $(\Omega, \mathfrak{B}, m)$ *with respect to the basis m. p. t. in a weak sense* $\omega'=\varphi(\omega)$ is defined. We shall denote this flow by $\Phi(\varphi, f)$. It is clear that $\Phi(\varphi, f)$ is ergodic if and only if φ is so. Further, it is easy to see that a flow in a strong or a weak sense is always measurable, and it was shown by W. Ambrose[1] that conversely every measurable ergodic flow defined on a completed measure space is isomorphic with a flow in a strong sense built under a function.

Let now $\omega'=\varphi(\omega)$ be an ergodic m. p. t. in a weak sense defined on a completed measure space $(\Omega, \mathfrak{B}, m)$, and let $\boldsymbol{\Phi}(\varphi)=\{\Phi(\varphi, f)\mid f(\omega)\in L^+(\Omega, \mathfrak{B}, m)\}$ be the family of all flows in a weak sense built under a function $f(\omega)$ on the basis space $(\Omega, \mathfrak{B}, m)$ with respect to $\omega'=\varphi(\omega)$, where $f(\omega)$ runs through $L^+(\Omega, \mathfrak{B}, m)$. Then

Theorem. In order that two ergodic m. p. t. in a weak sense φ *and* ψ *defined on two completed measure spaces* $(\Omega, \mathfrak{B}, m)$ *and* $(\Omega', \mathfrak{B}', m')$ *respectively be equivalent with each other, it is necessary and sufficient that the corresponding families of flows* $\boldsymbol{\Phi}(\varphi)$ *and* $\boldsymbol{\Phi}(\psi)$ *are identical up to an isomorphism,* i. e. *that, for any* $f(\omega)\in L^+(\Omega, \mathfrak{B}, m)$, *there exists an* $f'(\omega')\in L^+(\Omega', \mathfrak{B}', m')$ *such that the flow built under a function* $\Phi(\varphi, f)$ *is isomorphic with* $\Phi(\psi, f')$, *and further that conversely, for any* $f'(\omega')\in L^+(\Omega', \mathfrak{B}', m')$, *there exists an* $f(\omega)\in L^+(\Omega, \mathfrak{B}, m)$ *such that* $\Phi(\psi, f')$ *is isomorphic with* $\Phi(\varphi, f)$.

According to this result, the conjecture stated above that any two ergodic m. p. t. in a weak sense are equivalent with each other means that, for any ergodic m. p. t. in a weak sense $\omega'=\varphi(\omega)$ defined on a completed measure space $(\Omega, \mathfrak{B}, m)$ and for any measurable ergodic flow Φ defined on another completed measure space, there exists an $f(\omega)\in L^+(\Omega, \mathfrak{B}, m)$ such that the flow $\Phi(\varphi, f)$ is isomorphic with Φ.

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1) W. Ambrose, loc. cit. 2).

In concluding this note it is not without interest to observe that the well known conjecture of E. Hopf[1] to the effect that, by changing the velocity of a flow on each trajectory, we might obtain a flow with a continuous spectrum may be stated as follows: every class $\Phi(\varphi)$ contains at least one flow with a continuous spectrum. This conjecture is clearly weaker than the preceding one.

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1) E. Hopf, Ergodentheorie, 1937.