3. *Positive Definite Integral Quadratic Forms and Generalized Potentials.*

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(Comm. by S. KAKEYA, M.I.A., Jan. 12, 1944.)

I. Positive definite integral quadratic forms.

Let Ω be a separable, metric space with metric $\rho(p, q)$ ($p \in \Omega$ and $q \in \Omega$). Suppose that the function $\Phi(p, q)$ defined for all points $p \in \Omega$ and $q \in \Omega$ satisfies the following conditions 1°)–5°):

1°) $\Phi(p, q) = \Phi(q, p) \geqq 0, \quad \Phi(p, p) = +\infty,$

2°) $\lim\limits_{\rho(p, q) \to 0} \Phi(p, q) = +\infty,$

3°) $\Phi(p, q)$ is a continuous function of (p, q) whenever $p \neq q$.

Before the condition 4°) is mentioned, it seems convenient to begin with some preliminary remarks.

Given a bounded Borel set E in Ω, let σ be any completely additive function of Borel sets on E. Then by Jordan's decomposition theorem[1], we may write $\sigma = \sigma^+ - \sigma^-$, where σ^+ and σ^- are the positive and negative variations of σ respectively, each of which is itself a non-negative, completely additive set-function defined for all Borel sets contained in E.

Now, consider the following integral:

$$\iint \Phi(p, q) d\sigma(p) d\tau(q)$$

$$= \lim_{N \to \infty} \iint \Phi_N(p, q) d\sigma^+(p) d\tau^+(q) + \lim_{N \to \infty} \iint \Phi_N(p, q) d\sigma^-(p) d\tau^-(q)$$
$$- \lim_{N \to \infty} \iint \Phi_N(p, q) d\sigma^-(p) d\tau^+(q) - \lim_{N \to \infty} \iint \Phi_N(p, q) d\sigma^+(p) d\tau^-(q),$$

where $\Phi_N(p, q) = \mathrm{Min}\,\{N, \Phi(p, q)\}$.

$\int$ is used for $\int_E$ throughout this Note, so that $\iint$ for $\int_E \int_E$.

If all the four terms involved are finite, then the integral is said to be *absolutely convergent*. Thus the 4th condition is:

4°) $$+\infty \geqq \iint \Phi(p, q) d\sigma(p) d\sigma(q) \geqq 0$$

except when the integral is meaningless,

5°) if $\iint \Phi(p, \sigma) d\sigma(p) d\sigma(q) = 0$, then we have $\sigma(e) = 0$ for any Borel set $e \subset E$.

1) S. Saks: Theory of the Integral, (1937), Chap. I.

In case when Ω is a three dimensional Euclidean space, the function $\Phi(p, q)=\frac{1}{[\rho(p, q)]^{\alpha}}$ $(2>\alpha>0)$ satisfies the required properties stated above[2)].

Let $\mathfrak{S}=\mathfrak{S}_E$ be the set of all the completely additive function σ of Borel sets on E such that the integral

$$\iint \Phi(p, q)d\sigma(p)d\sigma(q)$$

is absolutely convergent. Then $\mathfrak{S}$ is a linear space, namely, if $\sigma\in\mathfrak{S}$ and $\tau\in\mathfrak{S}$, then $\alpha\sigma+\beta\tau\in\mathfrak{S}$ for any real constants α and β.

It is evident from the definition that if $\sigma\in\mathfrak{S}$, then $\sigma^+\in\mathfrak{S}$ and $\sigma^-\in\mathfrak{S}$, so that if $\sigma\in\mathfrak{S}$ and $\tau\in\mathfrak{S}$, then

$$\iint \Phi(p, q)d\sigma(p)d\tau(q)$$

is also absolutely convergent, which we shall denote by (σ, τ) and say the *positive definite integral quadratic form.*

Evidently the symbol (σ, τ) defined above for any pair of $\sigma\in\mathfrak{S}$ and $\tau\in\mathfrak{S}$ possesses the following properties of *inner product*:

I_1) $(\sigma, \tau)=(\tau, \sigma)$ (Fubini-Tonelli's Theorem),

I_2) $(\alpha\sigma, \tau)=\alpha(\sigma, \tau)$ for any real constant α,

I_3) $(\sigma, \tau_1+\tau_2)=(\sigma, \tau_1)+(\sigma, \tau_2)$

I_4) $(\sigma, \sigma)\geqq 0$, where equal sign holds when and only when $\sigma\equiv 0$.

Thus, writing $\|\sigma\|=\sqrt{(\sigma, \sigma)}$, we have introduced a "*norm*" into the linear space $\mathfrak{S}$ considered:

$$\|\sigma\|\geqq 0 \quad \text{for any} \quad \sigma\in\mathfrak{S}, \quad \|\sigma\|=0 \quad \text{implies} \quad \sigma\equiv 0,$$

$$\|\alpha\sigma\|=|\alpha|\cdot\|\sigma\|, \quad \alpha \text{ being any real constant, and} \quad \|\sigma+\tau\|\leqq\|\sigma\|+\|\tau\|.$$

Notice also for any $\sigma\in\mathfrak{S}$ and $\tau\in\mathfrak{S}$

$$|(\sigma, \tau)|\leqq\|\sigma\|\cdot\|\tau\| \quad \text{(Schwarz's inequality).}$$

The purpose of the present Note is to show how the general principle, introduced by O. Frostman[3)], of sweeping out process in the theory of Potential may be reduced to some simple considerations of the normed space $\mathfrak{S}$ just stated.

II. Positive mass-distributions on the compact set E.

Given a Borel set E in the space Ω, let μ be a completely additive, non-negative function of Borel sets on E. Then, μ is called a positive mass-distribution on E, of total mass $\mu(E)$. We have $\mu\equiv 0$, when and only when $\mu(E)=0$.

It is well known that *if $\{\mu_n\}$ is a sequence of positive mass-distributions on the compact set E, of total masses bounded*:

2) O. Frostman: Potentiel d'équilibre et Capacité des Ensembles, etc. Thèse, (1935), Chap. II.

3) O. Frostman: Loc. cit.

$$\mu_n(E) \leqq M(<+\infty) \qquad (n=1, 2, \dots),$$

then there exists a sub-sequence $\{\mu_{n_j}\}$ *such that, for every continuous function* $f(p)$ *defined on* E, *we have always*

$$\lim_{j\to\infty}\int f(p)d\mu_{n_j}(p)=\int f(p)d\mu(p), \tag{1}$$

which is often expressed as " μ_{n_j} *converging weakly to* μ *"*[4].

This fact is also true for the "product" distribution $\mu_n \times \mu_n$ in the product space $\Omega\times\Omega$, namely, for any continuous function $g(p, q)$ $\big((p, q)\in E\times E\big)$, we have always

$$\lim_{j\to\infty}\iint g(p, q)d\mu_{n_j}(p)d\mu_{n_j}(q)=\iint g(p, q)d\mu(p)d\mu(q). \tag{2}$$

If $g(p, q)$ is replaced by the discontinuous function $\Phi(p, q)$, then such relations as above will not hold in general.

But we have by (2) for the fixed $N(>0)$

$$\lim_{j\to\infty}\iint \Phi_N(p, q)d\mu_{n_j}(p)d\mu_{n_j}(q)=\iint \Phi_N(p, q)d\mu(p)d\mu(q)$$

and, from

$$\iint \Phi(p, q)d\mu_{n_j}(p)d\mu_{n_j}(q) \geqq \iint \Phi_N(p, q)d\mu_{n_j}(p)d\mu_{n_j}(q),$$

we have also

$$\varliminf_{j\to\infty}\iint \Phi(p, q)d\mu_{n_j}(p)d\mu_{n_j}(q) \geqq \iint \Phi_N(p, q)d\mu(p)d\mu(q),$$

whence, making $N\to\infty$,

$$\varliminf_{j\to\infty}\|\mu_{n_j}\|^2 \geqq \|\mu\|^2. \tag{3}$$

If $f(p)$ is lower semi-continuous on the compact set E, then there exists a monotone increasing sequence $\{f_m(p)\}$ of continuous functions on E such that $f_m(p)\uparrow f(p)$ $(m\to\infty)$. Similar argument as above shows also

$$\varliminf_{j\to\infty}\int f(p)d\mu_{n_j}(p) \geqq \int f(p)d\mu(p). \tag{4}$$

III. Generalized Potentials and Capacity.

Given a bounded Borel set E in Ω, let μ be a positive mass-distribution on E. We shall consider now the following function:

$$u(p)=\int \Phi(p, q)d\mu(q)=\lim_{N\to\infty}\int \Phi_N(p, q)d\mu(q), \tag{5}$$

which is well defined at every point of Ω. We call this function the *(generalized) potential of the distribution* μ *with respect to* $\Phi(p, q)$.

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4) N. Kryloff and N. Bogoliouboff: La théorie générale de la mesure dans son application à l'étude des systèmes de la mècanique non linéaire. Ann. of Math. Vol. **38** (1937).

If there exists a neighbourhood U of $p \in \Omega$ such that $\mu(U \cdot E)=0$, then the potential $u(p)$ is continuous at that point.

Being a limit of a monotone increasing sequence of continuous functions as seen by the right-hand side of (5), *$u(p)$ is in general lower semi-continuous at $p \in \Omega$, if, for every neighbourhood U of p, $\mu(U \cdot E) > 0$.*

The set E_0 of points $p \in \Omega$ such that, for every neighbourhood U of p, $\mu(U \cdot E) \neq 0$ is called the *kernel* of μ. It is evident that E_0 is closed and contained in E^b, the closure of E.

It is also evident that $u(p)$ is continuous outside E_0.

Let μ be a positive mass-distribution on E, of total mass 1, and consider the upper bound of the potential:

$$V_\mu^\Phi(E) = \sup_{p \in \Omega} \int \Phi(p, q) d\mu(q) .$$

Writing $V^\Phi(E) = \inf_\mu V_\mu^\Phi(E)$, we shall define

$$C^\Phi(E) = [V^\Phi(E)]^{-1} ,$$

which is called the Φ-capacity of E. We notice here that $C^\Phi(E)=0$ if and only if $V^\Phi(E) = +\infty$.

A little consideration shows that the necessary and sufficient condition for the existence of a positive mass-distribution μ on E, of total mass positive, with *bounded* potential is that the Φ-capacity of E should be positive[5].

Let E be a compact set of positive capacity. Then there exists a positive mass-distribution μ on E, of total mass 1, such that for every $p \in \Omega$

$$u(p) = \int \Phi(p, q) d\mu(q) < M (< +\infty) ,$$

from which we have, integrating both sides with μ,

$$\iint \Phi(p, q) d\mu(p) d\mu(q) < M ,$$

whence $\mu \in \mathfrak{S}_E$, so that we have

$$\inf_\mu \| \mu \|^2 = W(E) < \infty ,$$

the lower bound being taken over all $\mu \in \mathfrak{S}_E$ with $\mu(E)=1$.

Then, there exists a sequence of positive distributions $\{\mu_n\}$, of total mass 1, such that $\lim_{n\to\infty} \| \mu_n \|^2 = W(E)$.

From the sequence, we can select a sub-sequence $\{\mu_{n_j}\}$ converging weakly to the positive distribution μ_0 of total mass 1. By (3), we have

$$W(E) = \lim_{j\to\infty} \| \mu_{n_j} \|^2 \geqq \| \mu_0 \|^2 \geqq W(E) ,$$

5) S. Kametani: On some Properties of Hausdorff's Measure and the Concept of Capacity in Generalized Potentials, Proc. **18** (1942), 617.

from which we find $W(E)=\|\mu_0\|^2>0$, since otherwise, we would find $\mu_0\equiv 0$. Hence we have at last

$$0<W(E)<+\infty\,. \tag{6}$$

IV. Gauss' Variation.

Let E be a compact set of *positive* capacity. Given a *upper semi-continuous* function $f(p)\geqq 0$ on E, let us write

$$G(\mu)=\|\mu\|^2-2\int f(p)d\mu(p)\,,$$

where $\mu\in\mathfrak{S}$ is any positive mass-distribution on E.

$C^{\Phi}(E)$ being positive, there exists a positive distribution $\mu\in\mathfrak{S}$, of total mass *positive*. Let $\nu=m^{-1}\cdot\mu$, where $m=\mu(E)\,(>0)$, then $\nu\in\mathfrak{S}$ and $\nu(E)=1$. Hence

$$G(\mu)=m^2\|\nu\|^2-2m\int f(p)d\nu(p)$$

$$\geqq m^2W(E)-2mM=m^2\big(W(E)-2M\cdot m^{-1}\big) \tag{7}$$

where $M=\sup_{p\in E}f(p)<+\infty$ by the upper semi-continuity of $f(p)$, and $W(E)>0$ by (6). Since the right-hand side of (7) is positive for $m>2M/W(E)=K$, $G(\mu)>0$ holds for any positive distribution μ on E, of total mass $\mu(E)>K$.

As $G(\mu)=0$ for $\mu\equiv 0$, we have

$$g=\inf_{\mu}G(\mu)\leqq 0\,,$$

from which and also from the consideration made above we have

$$\inf_{\mu}G(\mu)=\inf_{\mu(E)\leq K}G(\mu)\,.$$

Let $\{\mu_n\}$ be a sequence of positive mass-distributions such that

$$\lim_{n\to\infty}G(\mu_n)=g \quad\text{and}\quad \mu_n(E)\leqq K\,.$$

Then there exists a sub-sequence $\{\mu_{n_j}\}$ which converges weakly to a positive mass-distribution μ_0 with the following properties:

$$\varliminf_{j\to\infty}\|\mu_{n_j}\|\geqq\|\mu_0\| \qquad \big(\text{by }(3)\big)$$

and $$\varliminf_{j\to\infty}\int\big(-f(p)\big)d\mu_{n_j}(p)\geqq\int\big(-f(p)\big)d\mu_0(p) \qquad \big(\text{by }(4)\big).$$

Hence we have

$$g=\lim_{j\to\infty}G(\mu_{n_j})\geqq\varliminf_{j\to\infty}\|\mu_{n_j}\|^2+2\varliminf_{j\to\infty}\int\big(-f(p)\big)d\mu_{n_j}(p)$$

$$\geqq\|\mu_0\|^2+2\int\big(-f(p)\big)d\mu_0(p)$$

$$=G(\mu_0)\geqq g\,,$$

from which follows

$$G(\mu_0)=g\,.$$

Thus we have proved the following:

Theorem 1. *Let E be a compact set of positive capacity. Then there exists a positive mass-distribution $\mu_0 \in \mathfrak{S}$ which minimizes the value of $G(\mu)$ for all the positive distributions μ on E. (Gauss' Variation).*

Let $\nu \in \mathfrak{S}$ be any positive distribution on E. Then $\mu_0+\varepsilon\nu \in \mathfrak{S}$ for $\varepsilon \geqq 0$. By Theorem 1, we have for any $\varepsilon > 0$

$$\begin{aligned} 0 &\leqq G(\mu_0+\varepsilon\nu)-G(\mu_0) \\ &=\left\{\|\mu_0+\varepsilon\nu\|^2-2\int f(p)d\mu_0(p)-2\varepsilon\int f(p)d\nu(p)\right\} \\ &\qquad -\left\{\|\mu_0\|^2-2\int f(p)d\mu_0(p)\right\} \\ &=2\varepsilon\left\{(\mu_0,\nu)-\int f(p)d\nu(p)\right\}+\varepsilon^2\|\nu\|^2\,. \end{aligned}$$

Noticing $\|\nu\|^2 < \infty$ and $\varepsilon(>0)$ arbitrary, we must have the following inequality

$$(\mu_0,\nu)-\int f(p)d\nu(p) \geqq 0\,,$$

which is valid for any positive mass-distribution $\nu \in \mathfrak{S}$.

If $\|\mu_0\| \neq 0$ on the one-hand, $G(t\mu_0)=t^2\|\mu_0\|^2-2t\int f(p)d\mu_0(p)$ attains its minimum at $t=\int f(p)d\mu_0(p)/\|\mu_0\|^2$, while $G(t\mu_0)$ must be minimized at $t=1$, from which we have

$$\|\mu_0\|^2-\int f(p)d\mu_0(p)=0\,. \tag{8}$$

If $\|\mu_0\|=0$ on the other hand, then we have $\mu_0\equiv 0$, from which we have also (8).

Thus we have proved the following fundamental:

Theorem 2. *Let E be a compact set with $C^{\Phi}(E)>0$.*

Then, for any positive mass-distribution ν on E such that $\nu \in \mathfrak{S}$, we have

$$(\mu_0,\nu)-\int f(p)d\nu(p) \geqq 0\,.$$

In particular, if $\nu=\mu_0$, then we have

$$\|\mu_0\|^2-\int f(p)d\mu_0(p)=0\,.$$

Theorem 3. *The positive mass-distribution μ on E which minimizes $G(\mu)$ is uniquely determined.*

Proof. Let μ_0 and ν_0 be two distributions minimizing $G(\mu)$. Then by the preceding theorem, we have

$$(\mu_0,\nu_0) \geqq \int f(p)d\nu_0(p)=\|\nu_0\|^2\,,$$

$$(\nu_0, \mu_0) \geqq \int f(p) d\mu_0(p) = \|\mu_0\|^2 .$$

Hence $$\|\mu_0 - \nu_0\|^2 = \|\mu_0\|^2 - 2(\mu_0, \nu_0) + \|\nu_0\|^2 \leqq 0 ,$$

which shows $$\|\mu_0 - \nu_0\| = 0 , \quad \text{that is,} \quad \mu_0 = \nu_0 .$$

IV. Properties of the potential of the distribution μ_0.

Throughout this section, we shall denote by $u(p)$ the potential $\int \Phi(p, q) d\mu_0(q)$ of the minimizing distribution μ_0 on the compact set E of positive capacity.

Theorem 4. *The potential $u(p)$ is not less than $f(p)$ everywhere on E with the possible exception of Φ-capacity 0, namely*

$$C^{\Phi}\{p ;\ p \in E,\ u(p) < f(p)\} = 0 .$$

Proof. Denoting the set $\{p ;\ p \in E,\ u(p) < f(p)\}$ by e_0, we find this Borelian, since $u(p) - f(p)$ is lower semi-continuous. Now if $C^{\Phi}(e_0) > 0$, then there would be a positive distribution ν on e_0, of total mass 1, such that

$$\int_{e_0}\int_{e_0} \Phi(p, q) d\nu(p) d\nu(q) < +\infty .$$

Hence, regarding as the distribution on E, ν belongs to $\mathfrak{S}$.

Then, by Theorem 2, we must have

$$0 \leqq (\mu_0, \nu) - \int f(p) d\nu(p) = \int \big(u(p) - f(p)\big) d\nu(p)$$
$$= \int_{e_0} \big(u(p) - f(p)\big) d\nu(p) < 0 ,$$

which is absurd.

Lemma. *The set $e_0 = \{p ;\ p \in E,\ u(p) < f(p)\}$ contains no μ_0-mass, namely :*

$$\mu_0\{p ;\ p \in E,\ u(p) < f(p)\} = 0 .$$

Proof. Supposing the contrary, let $\mu_0(e_0) > 0$.

Then we can define a positive mass-distribution ν on E, of total mass $\mu_0(e_0) (> 0)$, writing for every Borel set e, $\nu(e) = \mu_0(e_0 \cdot e)$.

Since $$\iint \Phi(p, q) d\nu(p) d\nu(q) = \int_{e_0}\int_{e_0} \Phi(p, q) d\mu_0(p) d\mu_0(q)$$
$$\leqq \|\mu_0\|^2 ,$$

we have $\nu \in \mathfrak{S}$. By Theorem 2, it would follow

$$0 \leqq (\mu_0, \nu) - \int f(p) d\nu(p)$$
$$= \int \big(u(p) - f(p)\big) d\nu(p) = \int_{e_0} \big(u(p) - f(p)\big) d\mu_0(p) < 0 ,$$

whence a contradiction.

Theorem 5. *Let E_0 be the kernel of μ_0. Then, at every point $p \in E_0$, we have*

$$u(p) \leqq f(p) .$$

Proof. Supposing the contrary, let p_0 be a point of E_0 such that

$$u(p_0)-f(p_0)>0 .$$

The function $u(p)-f(p)$ being lower semi-continuous on E, there would exist a neighbourhood U of p_0 such that

$$u(p)-f(p)>0 \tag{9}$$

for every $p\in U\cdot E$. Since p_0 is a point of the kernel of μ_0, we have

$$\mu_0(U\cdot E)>0 .$$

By the preceding lemma, $e_0=\{p;\ p\in E,\ u(p)<f(p)\}$ contains no μ_0-mass, whence we must have by Theorem 2

$$\begin{aligned} 0=\int \big(u(p)-f(p)\big)d\mu_0(p)&=\int_{E-e_0}\big(u(p)-f(p)\big)d\mu_0(p)\\ &\geqq\int_{E\cdot U-e_0}\big(u(p)-f(p)\big)d\mu_0(p)\\ =\int_{E\cdot U}\big(u(p)-f(p)\big)d\mu_0(p)>0 &\qquad \big(\text{by (9)}\big), \end{aligned}$$

which is a contradiction.

Remark 1. If $f(p)$ is a positive constant, then the distribution μ_0 gives the potential which corresponds to the equilibrium potential.

If $f(p)=\Phi(p,s)$, where $s\notin E$, or more generally, if $f(p)=\int_F\Phi(p,s)d\nu(s)$, where $\rho(E,F)>0$, then the potential obtained corresponds to the one given by sweeping out the mass charging s or F.

Remark 2. Having not assumed on $\Phi(p,q)$ none of subharmonicity, etc., the results obtained seem general enough, though there might be rooms for more precise and complete results under additional conditions.

The author wishes to express his gratitude to all the members of the Mathematical Institute of Nagoya Imperial University for their close attention and kind remarks given concerning his lecture during his stay with them.