PAPERS COMMUNICATED

1. *On the Completion by Cuts of Distributive Lattices.*

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"Is the completion by cuts of modular lattices modular? Is that of distributive lattices necessarily distributive? This problem was presented by H. Macneille[1]. G. Birkhoff has listed this problem in his book among unsolved problems[2]. In § 2 we will solve this problem negatively by constructing an example of a distributive lattice, whose completion by cuts is not modular. In § 3 we will give a necessary and sufficient condition for the distributivity of the lattice completed by cuts of a distributive lattice.

1. *Explanation of the problem.*

Let S be a subset of a lattice L, S^+ the set of all upper bounds of S, and S^* the set of all lower bounds of S. We call $\bar{S}=(S^+)^*$ the "normal hull" of S, and S a "normal subset" if and only if it is its own normal hull. If S consists of an element x, then $\bar{x}$ is the set of $y \leqq x$, and $\bar{x}$ is called a "principal" normal subset. All the normal subsets of L, ordered with respect to set inclusion, form a complete lattice $\bar{L}$[3]. All the principal normal subsets form a sublattice isomorphic to L. Our problem is to discuss the distributivity of $\bar{L}$ assuming that L is distributive.

In the discussion of distributivity, the notion of "neutral element" is very important. We define an element a to be neutral if and only if every triple $\{a, x, y\}$ generates a distributive sublattice. The neutral elements of a lattice L constitute a distributive sublattice of L. Thus $\bar{L}$ is distributive if and only if all the elements of $\bar{L}$ are neutral[4].

2. *Example.*

Let L_1, L_2 and L_3 be three simply ordered lattice (i. e. chain) such that

$$L_1;\ a_1 > a_2 > \cdots > a_i > \cdots b_j > \cdots > b_2 > b_1$$

$$L_2;\ p > q$$

$$L_3;\ c_1 > c_2 > \cdots > c_k > \cdots > d_1 > \cdots d_2 > d_1$$

Let L be a sublattice of the direct product $L_1 \times L_2 \times L_3$, consisting of the following elements,

1) H. Macneille, Partially ordered sets, Trans. Amer. Math. Soc., **42** (1937).
2) G. Birkhoff, Lattice theory, 146.
3) loc. cit. 1) or 2).
4) G. Birkhoff, Neutral elements in general lattice, Bull. Amer. Math. Soc., **46** (1940).

$$A_{ik}=(a_i, p, c_k) \qquad (i, k=1, 2, \dots)$$
$$B_{jk}=(b_j, p, c_k) \qquad (j, k=1, 2, \dots)$$
$$C_{jk}=(b_j, q, c_k) \qquad (j, k=1, 2, \dots)$$
$$D_{jl}=(b_j, q, d_l) \qquad (j, l=1, 2, \dots)$$

We can verify that L is distributive as a sublattice of the distributive lattice $L_1 \times L_2 \times L_3$.

Let S be the normal subset consisting of $D_{j,l}(j, l=1, 2, \dots)$, $\bar{B}_{11}$ and $\bar{C}_{11}$ be two principal normal subsets. Then clearly

$$S \frown \bar{B}_{11}=S \frown \bar{C}_{11}=(D_{1l}\,;\ l=1, 2, \dots).$$
$$S \smile \bar{B}_{11}=S \smile \bar{C}_{11}=(B_{j,k}, C_{jk}, D_{j,l}\,;\ j, k, l=1, 2, \dots),$$
$$\bar{B}_{11} > \bar{C}_{11}.$$

Thus five normal subsets S, $\bar{B}_{11}, \bar{C}_{11}, S \smile \bar{B}_{11}=S \smile \bar{C}_{11}$ and $S \frown \bar{B}_{11}= S \frown \bar{C}_{11}$ form a non-modular sublattice of $\bar{L}$. By this example Macneille's problem is solved negatively.

3. *A necessary and sufficient condition for the distributivity of* $\bar{L}$.

We will seek a necessary and sufficient condition for the neutrality of all the elements of $\bar{L}$. Let S be a fixed normal subset of L, then $x \to \bar{x} \frown S$ and $x \to \bar{x} \smile S$ are lattice-homomorphisms from L into $\bar{L}$. Thus $x \to \{\bar{x} \frown S, \bar{x} \smile S\}$ is a lattice-homomorphism.

Lemma. An element of a lattice L is neutral if and only if it is carried into $\{I, 0\}$ under isomorphism of L onto a sublattice of a direct product[1].

Theorem. A necessary and sufficient condition for the distributivity of $\bar{L}$ is that the lattice-homomorphism $x \to \{\bar{x} \smile S, \bar{x} \frown S\}$ be isomorphic for any normal subset S.

Proof. Necessity. If the lattice-homomorphic mapping $x \to \{\bar{x} \smile S, \bar{x} \frown S\}$ is not isomorphic, there exist two distinct elements x and y such as $\{\bar{x} \smile S, \bar{x} \frown S\}=\{\bar{y} \smile S, \bar{y} \frown S\}$. We have $\{\overline{x \smile y} \smile S, \overline{x \smile y} \frown S\}=\{\bar{x} \smile S, \bar{x} \frown S\}$ and $\bar{x} \neq \overline{x \smile y}$. Thus five elements $S, \bar{x}, \overline{x \smile y}$, $S \smile \bar{x}=S \smile \overline{x \smile y}$ and $S \frown \bar{x}=S \frown \overline{x \smile y}$ form a non-distributive sublattice of $\bar{L}$.

Sufficiency. If L is isomorphic with a sublattice $\{\bar{x} \smile S, \bar{x} \frown S\}$ of the direct product $\{\bar{x} \smile S\} \times \{\bar{x} \frown S\}$, $\bar{L}$ is isomorphic to $\{\overline{\bar{x} \smile S}, \overline{\bar{x} \frown S}\}$, which is a sublattice of the direct product $\overline{\{\bar{x} \smile S\}} \times \overline{\{\bar{x} \frown S\}}$. Under this isomorphism S is carried into $\{I, O\}$. Thus S is a neutral element in $\bar{L}$, and then $\bar{L}$ is distributive.

1) loc. cit. 4).