48. Notes on Fourier Series (XIII). Remarks on the Strong Summability of Fourier Series.

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1. The purpose of this paper is to give some remarks to a paper of Mr. R. Salem[1] concerning the strong summability of Fourier series. He has given sufficient conditions, in terms of the mean modulus of continuity, for strong summability of the Fourier series of a function belonging to L_1. Especially he proved the theorem.

Theorem 1. Let $f(x)$ be an integrable function periodic with period 2π and let $\omega(\delta)$ be its mean modulus of continuity:

$$\omega(\delta)=\max_{0<h\leq\delta}\int_0^{2\pi}|f(x+h)-f(x)|\,dx\,. \tag{1}$$

If $\omega(\delta)=O(1/|\log\delta|^{1+\varepsilon})$, $\varepsilon>0$, then the series

$$\sum_{n=1}^{\infty}\frac{|S_n(x)-\theta_n(x)|}{n} \tag{2}$$

is convergent almost everywhere, where

$$\theta_n(x)=\frac{1}{2}\left[S_n\left(x+\frac{\pi}{2n}\right)+S_n\left(x-\frac{\pi}{2n}\right)\right] \tag{3}$$

and $S_n(x)$ is a partial sum of the Fourier series of $f(x)$.

Salem, in its proof, made use of the following well known theorem

$$\int_0^{2\pi}|S_n(x)|^p dx\leq A\sec\frac{p\pi}{2}\left[\int_0^{2\pi}|f(x)|\,dx\right]^p,\qquad 0<p<1. \tag{4}$$

2. We shall first remark that we can prove the theorem by using the following fact instead of (4):

$$\int_0^{2\pi}\frac{|S_n(x)|}{\log^{1+\eta}\left(2+|S_n(x)|\right)}dx\leq A\int_0^{2\pi}|f(x)|\,dx,\qquad \eta>0, \tag{5}$$

where A depends only on η and this is somewhat convenient for the proof in some point of view. (5) was proved in my previous paper[2] and is an easy consequence of a theorem due to E. T. Titchmarsh concerning the conjugate function. Since $S_n(x)-\theta_n(x)$ is the n-th partial sum of a function $f(x)-\frac{1}{2}f\left(x+\frac{\pi}{2n}\right)-\frac{1}{2}f\left(x-\frac{\pi}{2n}\right)$, we have

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1) R. Salem, Sur la convergence en moyenne des séries de Fourier. Comptes Rendus, Paris. t. **208** (1939), pp. 70–72.

2) T. Takahashi (=Kawata). On the conjugate function of an integrable function and Fourier series and Fourier transform, Sci. Rep. Tôhoku Univ. **25** (1936).

$$(6)\quad \int_0^{2\pi} \frac{|S_n(x)-\theta_n(x)|}{\log^{1+\eta}\left(2+|S_n(x)-\theta_n(x)|\right)}dx \leqq A\int_0^{2\pi}\left|f(x)-\frac{1}{2}f\left(x+\frac{\pi}{2n}\right)\right.$$

$$\left.-\frac{1}{2}f\left(x-\frac{\pi}{2n}\right)\right|dx \leqq A\omega\left(\frac{\pi}{2n}\right) \leqq A\omega\left(\frac{1}{n}\right)=\frac{A}{\log^{1+\varepsilon} n}.$$

If we divide both sides of (6) by λ_n and sum up, then we get

$$(7)\quad \int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|S_n(x)-\theta_n(x)|}{\lambda_n\log^{1+\eta}\left(2+S_n(x)-\theta_n(x)\right)}dx \leqq A\sum\frac{1}{\lambda_n(\log n)^{1+\varepsilon}},$$

provided that the right hand side is convergent. Thus the series in the integral sign of the left hand side is almost everywhere convergent. If we take $\lambda_n=n/(\log\log n)^{1+\eta}$, then since the series on the right of (7) converges, we get

$$(8)\quad \sum\frac{|S_n(x)-\theta_n(x)|}{n\log^{1+\eta}\left(2+|S_n(x)-\theta_n(x)|\right)}(\log\log n)^{1+\eta}<\infty$$

for almost all values of x. By the well known fact, $S_n(x)-\theta_n(x) = O(\log n)$ almost everywhere. Thus it follows that

$$(9)\quad \sum\frac{|S_n(x)-\theta_n(x)|}{n}<\infty$$

almost everywhere.

3. We now consider, in place of (2), the series

$$(10)\quad \sum_{n=1}^{\infty}\frac{|S_n(x)-\sigma_n(x)|}{n}$$

where $\sigma_n(x)$ is a Fejér's mean. We shall show that the similar results will be obtained.

Theorem 2. Under the same conditions as in Theorem 1, the series (10) *converges almost everywhere.*

To prove the theorem we need the following lemma which is known[1].

Lemma 1. If $\alpha(t)=O(1/|\log t|^k)$ *for small* t, $k>0$, then the Dirichlet integral of $\alpha(t)$ is $O(1/|\log t|^k)$, or

$$(11)\quad I_n=\frac{1}{n\pi}\int_0^{\pi}\alpha(t)\frac{\sin^2\left(n+\frac{1}{2}\right)t}{\sin^2\frac{t}{2}}dt=O\left(\frac{1}{\log^k n}\right)$$

We prove the theorem. Since the n-th partial sum of the function $f(x)-\sigma_n(x)$ is $S_n(x)-\sigma_n(x)$, (5) shows that

1) For ex. see, Izumi-Kawata, Notes on Fourier Series (V), On the Absolute Riesz' Summability, Tôhoku Math. Journ. **45** (1938).

$$(12)\quad \int_0^{2\pi}\frac{|S_n(x)-\sigma_n(x)|}{\log^{1+\eta}\left(2+|S_n(x)-\sigma_n(x)|\right)}\leqq A\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx$$

$$\leqq A\int_0^{2\pi}dx\int_0^{2\pi}|f(x+t)+f(x-t)-2f(x)|\,\frac{\sin^2\left(n+\frac{1}{2}\right)t}{n\sin^2\frac{t}{2}}dt$$

$$\leqq A\int_0^{2\pi}dx\int_0^{\pi}|f(x+t)-f(x)|\,\frac{\sin^2\left(n+\frac{1}{2}\right)t}{n\sin^2\frac{t}{2}}dt$$

$$=A\int_0^{\pi}\frac{\sin^2\left(n+\frac{1}{2}\right)t}{n\sin^2\frac{t}{2}}dt\int_0^{2\pi}|f(x+t)-f(x)|\,dx$$

which is, by the lemma, of order $O(1/\log^{1+\varepsilon}n)$, where A is some constant. Thus we obtain

$$(13)\quad \int_0^{2\pi}\frac{|S_n(x)-\sigma_n(x)|}{\log^{1+\eta}\left(2+|S_n(x)-\sigma_n(x)|\right)}dx\leqq\frac{A}{\log^{1+\varepsilon}n}.$$

From this inequality, the same arguments show the validity of the conclusion of our theorem.

4. *Theorem 4. Let $f(x)$ be integrable and be such that*

$$\omega(\delta)=O(1/|\log\log\delta|^{2+\varepsilon}),\qquad \varepsilon>0.$$

Then at almost all points x, the series

$$(14)\quad \sum\frac{|S_n(x)-\sigma_n(x)|}{n\log n}$$

converges.

This theorem will be proved by the similar arguments as in § 3. But we need the following lemma in place of Lemma 1.

Lemma 2. If $\alpha(t)=O(1/|\log\log t|^{2+\varepsilon})$, then the Dirichlet integral is $O\left(1/\log\log n)^{2+\varepsilon}\right)$.

Let the Dirichlet integral of $\alpha(t)$ be J_n. Then

$$J_n=\frac{1}{n\pi}\int_0^{\frac{1}{n}}+\frac{1}{n\pi}\int_{\frac{1}{n}}^{\pi}\equiv T_1+T_2.$$

$$T_1=n\int_0^{\frac{1}{n}}\frac{dt}{|\log\log t|^{2+\varepsilon}}=O\left(\frac{1}{(\log\log n)^{2+\varepsilon}}\right).$$

$$T_2=O\left(\frac{1}{n\pi}\int_{\frac{1}{n}}^{\pi}\frac{dt}{t^2\left(\log\log\frac{1}{t}\right)^{2+\varepsilon}}\right)$$

$$=O\left\{\frac{1}{n\pi}\int_{\frac{1}{n}}^{\frac{1}{\sqrt{n}}}\frac{dt}{t^2\left(\log\log\frac{1}{t}\right)^{2+\varepsilon}}+\frac{1}{n\pi}\int_{\frac{1}{\sqrt{n}}}^{\pi}\frac{dt}{t^2\left(\log\log\frac{1}{t}\right)^{2+\varepsilon}}\right\}$$

$$=O\left(\frac{1}{n}\frac{1}{(\log\log n)^{2+\varepsilon}}\int_{\frac{1}{n}}^{\frac{1}{\sqrt{n}}}\frac{dt}{t^2}\right)+O\left(\frac{1}{n}\cdot\frac{n}{(\log\log n)^{2+\varepsilon}}\int_{\frac{1}{\sqrt{n}}}^{\pi}dt\right)$$

$$=O\left(\log\log n)^{-2-\varepsilon}\right).$$

We prove Theorem 4. Making use of Lemma 2 instead of Lemma 1, by similar methods we can obtain

$$\int_0^{2\pi}\frac{|S_n(x)-\sigma_n(x)|}{\log^{1+\eta}\left(2+|S_n(x)-\sigma_n(x)|\right)}dx\leqq A\frac{1}{(\log\log n)^{2+\varepsilon}}. \tag{15}$$

Let now $0<\eta<\varepsilon$. Dividing both sides by $n\log n/(\log\log n)^{1+\eta}$ and summing up we have

$$\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{|S_n(x)-\sigma_n(x)|\,(\log\log n)^{1+\eta}}{n\log n\log^{1+\eta}\left(2+|S_n(x)-\sigma_n(x)|\right)}dx \tag{16}$$

$$\leqq A\sum_{n=1}^{\infty}\frac{1}{n\log n\,(\log\log \mathrm{n})^{1+\varepsilon-\eta}}.$$

The right hand side is obviously convergent. From this by the former arguments we can conclude the almost everywhere convergence of the series (14).