44. A Screw Line in Hilbert Space and its Application to the Probability Theory*.

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(Comm. by S. KAKEYA, M.I.A., April 12, 1944.)

§ 1. A. Kolmogoroff has investigated the spectralization of the screw line in Hilbert space in his paper "*Kurven in Hilbertschen Raum, die gegenüber einer einparametrigen Gruppen von Bewegung invariant sind*"[1], where he has promised to give the complete proofs in another paper. In this note I will show his results, although the proofs may run in the same way as his own. And I will apply the results to the theory of two-dimensional brownian motions.

§ 2. Under a *congruent transformation in a Hilbert space* we understand an isometric mapping from $\mathfrak{H}$ to $\mathfrak{H}$ itself. On account of the Mazur-Ulam's theorem[2] any congruent transformation K is expressible in the form:

$$K\mathfrak{x} = \mathfrak{a} + U\mathfrak{x} \tag{2.1}$$

where $\mathfrak{a}$ is a fixed element in $\mathfrak{H}$ and U is a unitary operator.

Following after Kolmogoroff, we call a curve $\mathfrak{x}(t)$ in $\mathfrak{H}$ as *a screw line* (induced by a $\| \|$-continuous one-parameter group $\{K_t\}$ of congruent transformations), if we have $\mathfrak{x}(t) = K_t\mathfrak{x}(0)$ for any t. We have clearly $\mathfrak{x}(t+s) = K_{t+s}\mathfrak{x}(0) = K_sK_t\mathfrak{x}(0) = K_s\mathfrak{x}(t)$. We define the *moment function* $B_{\mathfrak{x}}(t, \tau, \sigma)$ of any curve $\mathfrak{x}(t)$ by

$$B_{\mathfrak{x}}(t, \tau, \sigma) = \big(\mathfrak{x}(t+\tau) - \mathfrak{x}(t), \mathfrak{x}(t+\sigma) - \mathfrak{x}(t)\big). \tag{2.2}$$

Theorem 1. A necessary and sufficient condition that $\mathfrak{x}(t)$ should be a screw line is that $B_{\mathfrak{x}}(t, \tau, \sigma)$ is independent of t and continuous in τ and σ.

Proof. The *necessity* is clear by the identity:

$$B_{\mathfrak{x}}(t, \tau, \sigma) = \big(K_\tau\mathfrak{x}(0) - \mathfrak{x}(0), K_\sigma\mathfrak{x}(0) - \mathfrak{x}(0)\big).$$

Sufficiency. The following proof is essentially due to Mr. K. Yosida. Suppose that $B_{\mathfrak{x}}(t, \tau, \sigma) = B(\tau, \sigma)$, where $B(\tau, \sigma)$ is continuous in τ and σ. Let $\mathfrak{H}_1$ be the linear manifold determined by the set $\mathfrak{x}(t) - \mathfrak{x}(s)$, $-\infty < s, t < \infty$, and $\mathfrak{H}_2$ be $\mathfrak{H} \ominus \bar{\mathfrak{H}}_1$. Since we have

$$\begin{aligned} &\Big\| \textstyle\sum_i a_i\big(\mathfrak{x}(t_i+\tau) - \mathfrak{x}(s_i+\tau)\big)\Big\|^2 \\ &\quad = \textstyle\sum_{ij} a_i\bar{a}_j B(t_i - s_i, t_j - s_j) = \Big\|\sum_i a_i\big(\mathfrak{x}(t_i) - \mathfrak{x}(s_i)\big)\Big\|^2, \end{aligned} \tag{2.3}$$

the following isometric mapping V_τ can be well defined in $\mathfrak{H}_1$:

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* The cost of this research has been defrayed from the Scientific Expenditure of the Department of Education.

1) C. R. (Doklady), 1940, vol. 26, 1. Cf. also Neumann and Schoenberg: Fourier integral and metric geometry, Trans. Amer. Math. Soc. vol. 50, 2, 1941.

2) Cf. S. Banach: Theorie des opérations linéaires, p. 166.

(2.4) $$\sum_i a_i\big(\mathfrak{x}(t_i)-\mathfrak{x}(s_i)\big) \to \sum_i a_i\big(\mathfrak{x}(t_i+\tau)-\mathfrak{x}(s_i+\tau)\big)\,.$$

Since we have $V_\tau^{-1}=V_{-\tau}$ by the definition, we can extend V_τ and define a unitary operator in $\bar{\mathfrak{H}}_1$, say V_τ again.

$V_\tau V_\sigma = V_{\tau+\sigma}$ is clear. The continuity of V_τ (with respect to τ) follows from that of $B(\tau, \sigma)$.

Let I_2 be the identical operator in $\mathfrak{H}_2$. Then $V_\tau \oplus I_2$ will be a unitary operator in $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$, say U_τ. Thus we have a continuous one-parameter group $\{U_\tau\}$ of unitary operators.

We put $a_\tau=\mathfrak{x}(\tau)-U_\tau\mathfrak{x}(0)$ and define K_τ by $K_\tau\mathfrak{x}=a_\tau+U_\tau\mathfrak{x}$. By simple calculations we obtain $K_tK_s=K_{t+s}$. Therefore $\mathfrak{x}(t)$ is a screw line induced by the group $\{K_t\}$.

§ 3. *A canonical form of a screw line.*

Theorem 2 (*A. Kolmogoroff*). *A necessary and sufficient condition that $\mathfrak{x}(t)$ should be a screw line is that it is expressible in the form:*

(3.1) $$\mathfrak{x}(t)-\mathfrak{x}(0)=\int_{-\infty}^{\infty}\frac{e^{i\lambda t}-1}{i\lambda}\Phi(d\lambda)\,,$$

where Φ satisfies

(3.2) *orthogonality:* $E\frown E'=0$ *implies* $\big(\Phi(E),\Phi(E')\big)=0\,,$

and

(3.3) $$\int_{|\lambda|\le 1}\|\Phi(d\lambda)\|^2+\int_{|\lambda|>1}\frac{\|\Phi(d\lambda)\|^2}{\lambda^2}<\infty\,.$$

Proof. Sufficiency. We will calculate the moment function $B_{\mathfrak{x}}(t,\tau,\sigma)$.

$$\begin{aligned}B_{\mathfrak{x}}(t,\tau,\sigma)&=\big(\mathfrak{x}(t+\tau)-\mathfrak{x}(t),\,\mathfrak{x}(t+\sigma)-\mathfrak{x}(t)\big)\\&=\int_{-\infty}^{\infty}\frac{e^{i\lambda(t+\tau)}-e^{i\lambda t}}{i\lambda}\cdot\frac{e^{-i\lambda(t+\tau)}-e^{-i\lambda t}}{-i\lambda}\|\Phi(d\lambda)\|^2\\&=\int_{-\infty}^{\infty}\frac{e^{i\lambda\tau}-1}{i\lambda}\cdot\frac{e^{-i\lambda\sigma}-1}{-i\lambda}\|\Phi(d\lambda)\|^2\end{aligned}$$

By Theorem 1 we can see that $\mathfrak{x}(t)$ is a screw line.

Necessity. Let $\{K_t\}$ be the group that induces $\mathfrak{x}(t)$. Put $K_t\mathfrak{x}=a_t+U_t\mathfrak{x}$. Then $\{U_t\}$ is a continuous one-parameter group of unitary operators. By the Stone's theorem we have

(3.4) $$U_t=\int_{-\infty}^{\infty}e^{i\lambda t}E(d\lambda)\,.$$

Now we have

$$\begin{aligned}\mathfrak{x}\Big(\frac{k}{n}\Big)-\mathfrak{x}(0)&=\sum_{\nu=1}^{k}\Big(\mathfrak{x}\Big(\frac{\nu}{n}\Big)-\mathfrak{x}\Big(\frac{\nu-1}{n}\Big)\Big)\\&=\sum_{\nu=1}^{k}U_{\frac{\nu-1}{n}}\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big)\\&=\int_{-\infty}^{\infty}\Big(\sum_{\nu=1}^{k}e^{i\lambda\cdot\frac{\nu-1}{n}}\Big)E(d\lambda)\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big)\quad\text{(by (3.4))}\end{aligned}$$

(*) $$=\int_{-\infty}^{\infty}\frac{e^{i\lambda\frac{k}{n}}-1}{e^{i\frac{\lambda}{n}}-1}E(d\lambda)\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big)\,.$$

Put $$\Phi_n(\alpha,\beta)=\int_\alpha^\beta \frac{i\lambda}{e^{i\frac{\lambda}{n}}-1}E(d\lambda)\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big).$$

Then we have

$$\begin{aligned}\Phi_1(\alpha,\beta)&=\int_\alpha^\beta \frac{i\lambda}{e^{i\lambda}-1}E(d\lambda)\big(\mathfrak{x}(1)-\mathfrak{x}(0)\big)\\ &=\int_\alpha^\beta \frac{i\lambda}{e^{i\lambda}-1}E(d\lambda)\Big(\int_{-\infty}^{\infty}\frac{e^{i\sigma}-1}{e^{i\frac{\sigma}{n}}-1}E(d\sigma)\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big)\Big)\quad \text{(by (*))}\\ &=\int_\alpha^\beta \frac{i\lambda}{e^{i\frac{\lambda}{n}}-1}E(d\lambda)\Big(\mathfrak{x}\Big(\frac{1}{n}\Big)-\mathfrak{x}(0)\Big)\quad \Big(\text{by the orthogonality of } E(d\lambda)\Big)\\ &=\Phi_n(\alpha,\beta).\end{aligned}$$

Thus we see that $\Phi_n(\alpha,\beta)$ is independent of n, say $\Phi(\alpha,\beta)$. Then we obtain

$$\mathfrak{x}\Big(\frac{k}{n}\Big)-\mathfrak{x}(0)=\int_{-\infty}^{\infty}\frac{e^{i\lambda\frac{k}{n}}-1}{i\lambda}\Phi(d\lambda). \tag{3.5}$$

The orthogonality of Φ follows at once from that of $E(d\lambda)$. Next we have

$$\begin{aligned}\int_{-1}^{1}\|\Phi(d\lambda)\|^2&=\int_{-1}^{1}\Big|\frac{i\lambda}{e^{i\lambda}-1}\Big|^2\big\|E(d\lambda)\big(\mathfrak{x}(1)-\mathfrak{x}(0)\big)\big\|^2\\ &\leqq\Big(\frac{\pi}{2}\Big)^2\big\|E(-1,1)\big(\mathfrak{x}(1)-\mathfrak{x}(0)\big)\big\|^2<\infty\end{aligned}$$

We have only to prove $\int_{|\lambda|\geq 1}\frac{\|\Phi(d\lambda)\|^2}{\lambda^2}<\infty$. By the orthogonality of Φ we have,

$$\begin{aligned}B\Big(\frac{k}{n},\frac{k}{n}\Big)&=\Big(\mathfrak{x}\Big(\frac{k}{n}\Big)-\mathfrak{x}(0),\ \mathfrak{x}\Big(\frac{k}{n}\Big)-\mathfrak{x}(0)\Big)\\ &\geqq\Big\|\int_{1\leq|\lambda|<\Lambda}\frac{e^{i\lambda\frac{k}{n}}-1}{i\lambda}\Phi(d\lambda)\Big\|^2\qquad (1<\Lambda<\infty)\\ &=\int_{1\leq|\lambda|<\Lambda}\Big|\frac{e^{i\lambda\frac{k}{n}}-1}{i\lambda}\Big|^2\|\Phi(d\lambda)\|^2.\end{aligned}$$

On account of the continuity of $B(\tau,\tau)$ $\big(\equiv\big(\mathfrak{x}(\tau)-\mathfrak{x}(0),\mathfrak{x}(\tau)-\mathfrak{x}(0)\big)\big)$, we have $B(\tau,\tau)\geqq\int_{1\leq|\lambda|<\Lambda}\Big|\frac{e^{i\lambda\tau}-1}{i\lambda}\Big|^2\|\Phi(d\lambda)\|^2$, and so $\int_0^1 B(\tau,\tau)d\tau\geqq\frac{1}{4}\int_{1\leq|\lambda|<\Lambda}\frac{\|\Phi(d\lambda)\|^2}{\lambda^2}$. $\Big(\because\int_0^1|e^{i\lambda\tau}-1|^2d\tau=2\Big(1-\frac{\sin\lambda}{\lambda}\Big)\geqq\frac{1}{4}$ for $|\lambda|\geqq 1.\Big)$. Let λ tend to ∞. Then we have $\int_{1\leq|\lambda|}\frac{\|\Phi(d\lambda)\|^2}{\lambda^2}\leqq 4\int_0^1 B(\tau,\tau)d\tau<\infty$.

Theorem 3. The measure Φ in the preceding theorem is expressible by $\mathfrak{x}(t)$ as follows:

$$(3.6)\qquad \Phi(\alpha,\beta)=\lim_{c\to\infty}\frac{1}{2\pi}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\int_{\alpha}^{\beta}e^{-i\mu\delta_k}d\mu\right)\left(\mathfrak{x}(\gamma_k)-\mathfrak{x}(\gamma_{k-1})\right),$$

$\gamma_k=c\left(-1+\frac{2k}{n}\right)$, $\delta_k=c\left(-1+\frac{2k-1}{n}\right)$, *for any continuity points* α, β *of* Φ.

For the proof we shall mention some preliminary lemmas concerning the Φ-integrability; we say that a complex-valued function $f(\lambda)$ is Φ-integrable, if we have

$$(3.7)\qquad \int_{-\infty}^{\infty}|f(\lambda)|^2\|\Phi(d\lambda)\|^2<\infty.$$

Lemma 1. Assume that $|f_n(\lambda)|$ *be bounded from above by a* Φ-*integrable function* $f(\lambda)$ *and that* $\lim_{n\to\infty}f_n(\lambda)=f_0(\lambda)$. *Then we have*

$$(3.8)\qquad \lim_{n\to\infty}\int_{-\infty}^{\infty}f_n(\lambda)\Phi(d\lambda)=\int_{-\infty}^{\infty}f_0(\lambda)\Phi(d\lambda).$$

By Lemma 1 and by (3.3) we obtain

Lemma 2. If there exist two positive numbers A, M *such that we have* $|f(\lambda)|<M$ *for* $|\lambda|<A$ *and that* $|f(\lambda)|<\frac{A}{|\lambda|}$ *otherwise, then* $f(\lambda)$ *is* Φ-*integrable.*

The proof of Theorem 3. We put

$$(3.9)\qquad S_n(c)=\sum_{k=1}^{n}\left(\int_{\alpha}^{\beta}e^{-i\mu\delta_k}d\mu\right)\left(\mathfrak{x}(\gamma_k)-\mathfrak{x}(\gamma_{k-1})\right),$$

and

$$(3.10)\qquad F_n(\lambda,\mu)=\sum_{k=1}^{n}e^{-i\mu\delta_k}\cdot\frac{e^{i\lambda\gamma_k}-e^{i\lambda\gamma_{k-1}}}{i\lambda}.$$

Then we have

$$(3.11)\qquad S_n(c)=\int_{-\infty}^{\infty}\int_{\alpha}^{\beta}F_n(d\mu)\Phi(d\lambda).$$

By simple calculations we obtain

$$(3.12)\qquad F_n(\lambda,\mu)=\sum_{k=1}^{n}e^{i(\lambda-\mu)\delta_k}\frac{e^{i\frac{c}{n}\lambda}-e^{-i\frac{c}{n}\lambda}}{i\lambda}=\frac{2\sin(\lambda-\mu)c\sin\frac{\lambda c}{n}}{\lambda\sin\frac{\lambda-\mu}{n}c},$$

and

$$(3.13)\qquad i\lambda F_n(\lambda,\mu)=\sum_{k=1}^{n}e^{i\lambda\gamma_k}(e^{-i\mu\delta_k}-e^{-i\mu\delta_{k+1}})+e^{i(\lambda\gamma_n-\mu\delta_{n+1})}$$
$$-e^{i(\lambda\gamma_0-\mu\delta_1)},\qquad \delta_{n+1}=c\left(1+\frac{1}{n}\right).$$

For any $\lambda\neq 0$ and for any n we have, by (3.13),

$$\left|\int_{\alpha}^{\beta}F_n(\lambda,\mu)d\mu\right|\leqq\frac{1}{\lambda}\int_{\alpha}^{\beta}\left(\sum_{k=1}^{n}|\mu|(\delta_{k+1}-\delta_k)+2\right)d\mu=\frac{1}{\lambda}\int_{\alpha}^{\beta}2(|\mu|c+1)d\mu.$$

If $|\lambda|>\max\{|\alpha|,|\beta|\}$ and $n>\frac{4Ac}{\pi}$, then we have $\left|\frac{\lambda-\mu}{n}c\right|\leqq\frac{2Ac}{n}<\frac{\pi}{2}$ for $\alpha\leqq\mu\leqq\beta$, and so

$$\left|\int_\alpha^\beta F_n(\lambda,\mu)d\mu\right|\leqq\int_\alpha^\beta 2\frac{|\sin(\lambda-\mu)c|}{\frac{2}{\pi}\left|\frac{\lambda-\mu}{n}c\right|}\cdot\frac{\left|\frac{\lambda}{n}c\right|}{|\lambda|}d\mu=\pi\int_\alpha^\beta\frac{|\sin(\lambda-\mu)c|}{|\lambda-\mu|}d\mu$$

$$\leqq\pi c(\beta-\alpha).$$

Making use of Lemma 1 and 2 we obtain, by (3.11),

$$S_\infty=\int_{-\infty}^\infty\left(\lim_{n\to\infty}\int_\alpha^\beta F(\lambda,\mu)d\mu\right)\Phi(d\lambda)$$

$$=\int_{-\infty}^\infty\int_\alpha^\beta\lim_{n\to\infty}F_n(\lambda,\mu)d\mu\Phi(d\lambda)\qquad\left(\because\ |F_n(0,\mu)|\leqq 2c,\ |F_n(\lambda,\mu)|\leqq 2(|\mu|c+1)\ \text{for}\ \lambda\neq 0\right)$$

$$=2\int_{-\infty}^\infty\int_\alpha^\beta\frac{\sin(\lambda-\mu)c}{\lambda-\mu}d\mu\Phi(d\lambda)\qquad\text{(by (3.12))}$$

$$(\dagger)\quad =2\int_\alpha^\beta\int_{c(\alpha-\lambda)}^{c(\beta-\lambda)}\frac{\sin\theta}{\theta}d\theta\Phi(d\lambda).$$

In order to obtain $S\equiv\lim_{c\to\infty}S_\infty(c)$ we shall first estimate $\int_{c(\alpha-\lambda)}^{c(\beta-\lambda)}\frac{\sin\theta}{\theta}d\theta$ in two ways:

$$\left|\int_{c(\alpha-\lambda)}^{c(\beta-\lambda)}\frac{\sin\theta}{\theta}d\theta\right|\leqq\int_{-\pi}^{\pi}\frac{\sin\theta}{\theta}d\theta,$$

and

$$\left|\int_{c(\alpha-\lambda)}^{c(\beta-\lambda)}\frac{\sin\theta}{\theta}d\theta\right|\leqq\frac{c(\beta-\alpha)}{c\min(|\alpha-\lambda|,|\beta-\lambda|)}\leqq\frac{2(\beta-\alpha)}{|\lambda|}$$

for $|\lambda|>2\max\{|\alpha|,|\beta|\}$.

Therefore, making use of Lemma 1 and 2 again, we obtain, from (†), $S=2\int_{-\infty}^\infty\lim_{c\to\infty}\int_{c(\alpha-\lambda)}^{c(\beta-\lambda)}\frac{\sin\theta}{\theta}d\theta\Phi(d\lambda)$, if α and β are both continuity points of Φ.

§ 4. *A canonical form of a screw function.* A complex-valued function $B(\tau,\sigma)$, $-\infty<\tau,\ \sigma<\infty$ is called a *screw function*, if there exists a screw line $\mathfrak{x}$ such that $B_{\mathfrak{x}}(t,\tau,\sigma)=B(\tau,\sigma)$.

Theorem 4 (A. Kolmogoroff). A necessary and sufficient condition that $B(\tau,\sigma)$ should be a screw function is that $B(\tau,\sigma)$ is expressible in the form:

$$(4.1)\qquad B(\tau,\sigma)=\int_{-\infty}^\infty\frac{e^{i\lambda\tau}-1}{i\lambda}\,\frac{e^{-i\lambda\sigma}-1}{-i\lambda}F(d\lambda),$$

where

$$\int_{|\lambda|<1}F(d\lambda)+\int_{|\lambda|\geqq 1}\frac{F(d\lambda)}{\lambda^2}<\infty$$

Proof. The *necessity* is evident by Theorem 3.

Sufficiency. We define the functions ξ_τ depending on a real parameter τ as follows: $\xi_\tau(t)=0$ $(t \neq \tau)$, $\xi_\tau(\tau)=1$. Let $\mathfrak{H}^*$ be the system of all linear forms of ξ_τ's with complex coefficients. We introduce an inner product (f, g) into $\mathfrak{H}^*$ by

$$\Big(\sum_\mu c_\mu(\xi_{\tau_\mu}-\xi_0),\ \sum_\nu d_\nu(\xi_{\sigma_\mu}-\xi_0)\Big)=\sum_{\mu,\nu} c_\mu \bar{d}_\nu B(\tau_\mu, \sigma_\nu);$$

(f, g) is evidently linear (conjugate linear) with respect to $f(g)$, and we have further

$$\Big(\sum_\mu c_\mu(\xi_{\tau_\mu}-\xi_0),\ \sum_\mu c_\mu(\xi_{\tau_\mu}-\xi_0)\Big)=\int_{-\infty}^{\infty}\Big|\sum_\mu \frac{e^{i\lambda\tau_\mu}-1}{i\lambda}c_\mu\Big|^2 F(d\lambda) \geqq 0 .$$

Let $\mathfrak{N}^*$ denote the set of all f's such that $(f, f)=0$. Then $\mathfrak{H}=\overline{\mathfrak{H}^*/\mathfrak{N}^*}$ may be considered as a Hilbert space. Let $\mathfrak{x}(t)$ denote the element of $\mathfrak{H}$ corresponding to ξ_t. Then we have

$$\begin{aligned} B_{\mathfrak{x}}(t, \tau, \sigma) &= \big(\mathfrak{x}(t+\tau)-\mathfrak{x}(t),\ \mathfrak{x}(t+\sigma)-\mathfrak{x}(t)\big) \\ &= (\xi_{t+\tau}-\xi_t,\ \xi_{t+\sigma}-\xi_t) \\ &= B(t+\tau, t+\tau)-B(t+\tau, t)-B(t, t+\sigma)+B(t, t) \\ &= \int_{-\infty}^{\infty} \frac{e^{i\lambda\tau}-1}{i\lambda}\cdot\frac{e^{-i\lambda\sigma}-1}{-i\lambda}F(d\lambda)=B(\tau, \sigma) . \end{aligned}$$

Therefore $\mathfrak{x}(t)$ is a screw line by Theorem 1, and so $B(\tau, \sigma)$ is a screw function.

Theorem 5. The measure F in the preceding theorem is expressible in the form:

$$(4.2)\quad F(\alpha, \beta)=\lim_{n\to\infty}\Big(\frac{1}{2\pi}\Big)^2\lim_{n\to\infty}\sum_{k,h=1}^{n}\Big(\int_\alpha^\beta e^{-i\mu(\delta_k-\delta_h)}d\mu\Big)\Big(B(\gamma_k, \gamma_h) -B(\gamma_k, \gamma_{h-1})-B(\gamma_{k-1}, \gamma_h)+B(\gamma_{k-1}, \gamma_{h-1})\Big),$$

$\gamma_k=c\Big(-1+\dfrac{2k}{n}\Big)$, $\delta_k=c\Big(-1+\dfrac{2k-1}{n}\Big)$ *for any continuity points* α, β of F.

The proof can be achieved in the same way as in Theorem 4 and so will be omitted.

§ 5. *The two-dimensional brownian motion as a screw line on $L^2(\Omega, P)$.* A system of complex-valued random variables $x_\alpha(\omega)$, $\omega \in (\Omega, P)$, $\alpha \in A$, is called *normal*, if any random variable of the form: $\sum c_i x_{\alpha_i}(\omega)$, c_i's being any complex numbers, is subjected to the normal law in the complex plane, i.e. to the law of the form: $\dfrac{1}{\pi}e^{-\frac{\xi^2+\eta^2}{a^2}}d\xi d\eta$. *In any normal system the orthogonality in $L^2(\Omega, P)$ implies the (stochastic) independence.* A stochastic process $x(t, \omega)$, $S \leqq t \leqq T$, is called *normal*, if the system $\{x(t, \omega)\}$ is normal. Under *a (two-dimensional) brownian motion* in the time-interval (S, T), $-\infty \leqq S$, $T \leqq \infty$, we understand a normal, differential[3], and temporally homogeneous (complex-valued) process.

A screw line $x(t,\omega)$, $-\infty<t<\infty$, in the Hilbert space $L^2(\Omega,P)$ is called *normal*, if it is a normal process. *The moment function* $B(\tau,\sigma)$ *of any normal screw line is real-valued* and so is determined by $B(\tau,\tau)$ as follows: $B(\tau,\sigma)=\frac{1}{2}\big(B(\tau+\sigma,\tau+\sigma)-B(\tau,\tau)-B(\sigma,\sigma)\big)$.

Theorem 6. Let $x(t,\omega)$ be a normal screw line in $L^2(\Omega,P)$. A necessary and sufficient condition that it should be a brownian motion is that $B(\tau,\tau)=a^2|\tau|$, a being a positive constant, i.e. that $B(\tau,\sigma)$ $=\frac{a^2}{2}(|\tau+\sigma|-|\tau|-|\sigma|)$.

The proof is brief and so will be omitted.

Usually we obtain a mathematical scheme of brownian motions by introducing a convenient probability distribution into a functional space[4]. But the above theorem gives another method of constructing the scheme.

Let C be the complex plane, and G be the probability distribution on C such that $G(E)=\iint_E \frac{1}{\pi}e^{-(\xi^2+\eta^2)}d\xi d\eta$. We consider the product measure space $(C,G)^{\aleph_0}$, say (Ω,P). We denote by $a_n(\omega)$ the n-th coordinate of ω, $n=0,1,2,\ldots$. Now we define $x(t,\omega)$, $0\leqq t\leqq 2\pi$, by

$$x(t,\omega)=ta_0(\omega)+\sum_{n\neq 0}\frac{a_n(\omega)e^{int}}{in}.$$

Since a_n, $n=0,1,2,\ldots$, are independent and normally distributed, the system $\{a_n\}$ is normal and so $x(t,\omega)$ is a normal process. On account of the identities:

$$x(t,\omega)-\lambda(0,\omega)=\sum_n a_n\frac{e^{int}}{in}, \qquad \|a_0\|^2+\sum_{n\neq 0}\frac{\|a_n\|^2}{n^2}=1+\frac{\pi^2}{3}<\infty,$$

we see by Theorem 2 that $x(t,\omega)$ is a screw line, whose moment function $B(t,\sigma)$ is determined by $B(\tau,\tau)=\sum_n\left|\frac{e^{int}-1}{in}\right|^2\|a_n\|^2=2\pi t$ for $0\leqq t\leqq 2\pi$. By Theorem 6 $x(t,\omega)$, $0\leqq t\leqq 2\pi$, is a brownian motion. This is *the scheme obtained by N. Wiener*[5].

By this method we cannot construct a brownian motion on an infinite time-interval; in this point the former is more advantageous, while the latter may be more convenient on account of its concreteness.

I wish to express my gratitude for the kindness with which Mr. K. Yosida has encouraged and directed me through the course of this investigation.

3) Cf. J. L. Doob: Stochastic processes depending on a continuous parameter, Trans. Amer. Math. Soc. vol. 42.

4) Cf. J. L. Doob. loc. cit. (3).

5) R. E. A. C. Paley and N. Wiener: Fourier transforms in the complex domain, Chap. 9. Random functions, Amer. Math. Soc. Coll. Publ. 19, 1934.