72. *On the Torse-forming Directions in Riemannian Spaces.*

By Kentaro YANO.
Mathematical Institute, Tokyo Imperial University.

(Comm. by S. KAKEYA, M.I.A., June 12, 1944.)

§ 0. It is well known that a vector $v^\lambda(s)$ defined on each point of the curve $x^\lambda(s)$ in a Riemannian space V_n is said to be parallel along the curve, if it satisfies the differential equations of the form

$$(0.1)\qquad \frac{\delta v^\lambda}{ds}\equiv\frac{dv^\lambda}{ds}+\left\{{}^{\lambda}_{\mu\nu}\right\}v^\mu\frac{dx^\nu}{ds}=0\,,$$

$\left\{{}^{\lambda}_{\mu\nu}\right\}$ being the Christoffel symbols of the second kind. Following E. Cartan, the Euclidean connection without torsion of the Riemannian space being defined by

$$d\boldsymbol{M}=dx^\lambda\boldsymbol{e}_\lambda\,,\qquad d\boldsymbol{e}_\mu=\left\{{}^{\lambda}_{\mu\nu}\right\}dx^\nu\boldsymbol{e}_\lambda\,,$$

if we develop the curve on the tangent space at a point of the curve, the directions $v^\lambda(s)$ defined as above along the curve will be found to be parallel along the curve developed on the tangent space, for, the equations (0.1) just show that the geometrical variation of the vector $v^\lambda\boldsymbol{e}_\lambda$ along the curve vanishes. This will be the most natural interpretation of Levi-Civita's parallelism.

On the other hand, we have studied, in a previous paper[1], the concurrency of the directions defined along a curve in Riemannian spaces. A vector $v^\lambda(s)$ defined on each point of the curve $x^\lambda(s)$ in a Riemannian space is said to be concurrent along the curve, if it satisfies the differential equations of the form

$$(0.2)\qquad \frac{dx^\lambda}{ds}+\frac{\delta\alpha v^\lambda}{ds}=0\,,$$

where α is a suitable function of s. In fact, these equations show that the geometrical variation of the point $\boldsymbol{M}+\alpha v^\lambda\boldsymbol{e}_\lambda$ vanishes along the curve, and hence, if we develop the curve $x^\lambda(s)$ on the tangent space at a point of the curve, all the directions $v^\lambda(s)$ defined on each point of the curve pass through the fixed point $\boldsymbol{M}+\alpha v^\lambda\boldsymbol{e}_\lambda$.

Generalizing these concepts of parallelism and concurrency, we shall study in the present Note the torse-forming directions in Riemannian spaces. The torse-forming directions may be considered in affinely or projectively connected spaces. We have already indicated an application of torse-forming directions to the geometrical interpretation of the projective transformations of asymmetric affine connections[2].

---

1) K. Yano: Sur le parallélisme et la concourance dans l'espace de Riemann. Proc. **19** (1943), 189-197.

2) K. Yano: Über eine geometrische Deutung der projektiven Transformationen nicht-symmetrischer affiner Übertragungen. Proc. **20** (1944), 284-287.

§ 1. Consider a vector $v^\lambda(s)$ defined along a curve $x^\lambda(s)$ in a Riemannian space and develop the curve $x^\lambda(s)$ on the tangent space at a point of the curve. If, after the development, the directions defined by $v^\lambda(s)$ form a developable surface or torse, the directions defined by $v^\lambda(s)$ are said to be torse-forming along the curve in the Riemannian space.

In order that the directions $v^\lambda(s)$ defined along the curve $x^\lambda(s)$ be torse-forming, it is necessary and sufficient that the geometrical variation of the point $\boldsymbol{M}+\alpha v^\lambda \boldsymbol{e}_\lambda$ be in the direction $v^\lambda \boldsymbol{e}_\lambda$ for a suitable function α of s; say

$$\frac{d}{ds}(\boldsymbol{M}+\alpha v^\lambda \boldsymbol{e}_\lambda)=\beta v^\lambda \boldsymbol{e}_\lambda,$$

from which we obtain

$$\frac{dx^\lambda}{ds}+\frac{\delta \alpha v^\lambda}{ds}=\beta v^\lambda, \tag{1.1}$$

β being another suitable function of s.

If $\alpha=0$, the vector v^λ is tangent to the curve. If $\alpha \neq 0$, we have from (1.1)

$$\frac{\delta v^\lambda}{ds}=p\frac{dx^\lambda}{ds}+qv^\lambda, \tag{1.2}$$

where

$$p=-\frac{1}{\alpha}, \qquad q=\frac{1}{\alpha}\left(\beta-\frac{d\alpha}{ds}\right). \tag{1.3}$$

Conversely, suppose that we have a vector $v^\lambda(s)$ defined along a curve $x^\lambda(s)$, and satisfying the differential equations of the form (1.2).

If $p=0$, the vector v^λ is parallel along the curve. If $p\neq 0$, putting

$$\alpha=-\frac{1}{p}, \qquad \beta=\frac{1}{p^2}\frac{dp}{ds}-\frac{q}{p},$$

we have the equations of the form (1.1). Hence, we have the

Theorem: In order that the directions $v^\lambda(s)$ defined along the curve $x^\lambda(s)$, and not tangent to the curve be torse-forming along the curve, it is necessary and sufficient that the covariant derivative of $v^\lambda(s)$ along the curve be a linear combination of the v^λ and the tangent vector $\frac{dx^\lambda}{ds}$.

§ 2. We shall consider, in this paragraph, a torse-forming vector field, that is, a vector field which is always torse-forming along any curve traced in the Riemannian space V_n. In this case, we have, from (1.2),

$$v^\lambda_{;\nu}\frac{dx^\nu}{ds}=p\frac{dx^\lambda}{ds}+qv^\lambda, \tag{2.1}$$

the semi-colon denoting the covariant derivatives with respect to $\left\{ {}^{\,\lambda}_{\mu\nu} \right\}$.

As these equations must be satisfied for any directions $\frac{dx^\lambda}{ds}$, we have

$$v^{\lambda}_{;\nu}=\rho\delta^{\lambda}_{\nu}+\sigma_{\nu}v^{\lambda} \tag{2.2}$$

for a suitable scalar ρ and a suitable vector σ_{ν}.

The vector v^{λ} satisfying the equations of the form (2.2), the unit vector v^{λ}/v, where v denotes the length of v^{λ}, satisfies also the equations of the same form, hence we can assume that the v^{λ} in (2.2) is a unit vector. Then multiplying (2.2) by $v_{\lambda}=g_{\lambda\mu}v^{\mu}$ and summing up for the index λ, we find

$$0=\rho v_{\nu}+\sigma_{\nu} \quad \text{from which} \quad \sigma_{\nu}=-\rho v_{\nu}\,.$$

Substituting this in (2.2) we have

$$v^{\lambda}_{;\nu}=\rho(\delta^{\lambda}_{\nu}-v_{\nu}v^{\lambda})\,, \tag{2.3}$$

or in covariant form

$$v_{\mu;\nu}=\rho(g_{\mu\nu}-v_{\mu}v_{\nu})\,. \tag{2.4}$$

These equations show that the covariant derivative $v_{\mu;\nu}$ of the vector v_{μ} is symmetric with respect to lower indices μ and ν, hence we have

$$\frac{\partial v_{\mu}}{\partial x^{\nu}}-\frac{\partial v_{\nu}}{\partial x^{\mu}}=0\,,$$

which shows that v_{μ} is a gradient vector of a scalar $F(x)$, that is to say,

$$v_{\mu}=\frac{\partial F}{\partial x^{\mu}}\,.$$

Thus, there exists, in our Riemannian space V_n, a family of hypersurfaces $F(x^1, x^2, \dots, x^n)=\text{constants}$ to which the vector field v^{λ} is normal.

On the other hand, we know that a hypersurface whose normals are always torse-forming along any curve traced on it is totally umbilical. Hence the hypersurfaces $F=\text{constants}$ are all totally umbilical. Moreover the equations (2.3) show that $v^{\lambda}_{;\nu}v^{\nu}$ vanishes, hence the curves generated by v^{λ} are all geodesics. Thus we have the

Theorem: If a Riemannian space V_n admits a torse-forming vector field v^{λ}, it contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics.

§ 3. Suppose that our Riemannian space V_n contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics. We shall then choose a coordinate system with respect to which the totally umbilical hypersurfaces are represented by the equations $x^n=\text{constants}$, and their orthogonal trajectories by $x^i=\text{constants}$. $(i, j, k, \dots=1, 2, \dots, n-1)$

Thus the fundamental quadratic differential form of the space may be written as

$$ds^2=g_{jk}(x^{\lambda})dx^j dx^k+g_{nn}(x^{\lambda})dx^n dx^n \tag{3.1}$$

The curves defined by x^i=constants being geodesics, we must have[1]

$$\{{}^{\,i}_{nn}\}=0 \quad \text{or} \quad \frac{1}{2}g^{ia}\left(\frac{\partial g_{an}}{\partial x^n}+\frac{\partial g_{an}}{\partial x^n}-\frac{\partial g_{nn}}{\partial x^a}\right)=0\,, \tag{3.2}$$

from which we have

$$g_{nn}=g_{nn}(x^n)\,.$$

The hypersurfaces defined by x^n=constants being totally umbilical, we must have

$$\{{}^{\,n}_{jk}\}=g_{jk}H \quad \text{or} \quad \frac{1}{2}g^{nn}\left(\frac{\partial g_{nj}}{\partial x^k}+\frac{\partial g_{nk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^n}\right)=g_{jk}H\,, \tag{3.3}$$

from which we have

$$g_{jk}=f(x^\lambda)g^*_{jk}(x^i)\,.$$

Thus, the (3.1) becomes

$$ds^2=f(x^\lambda)g^*_{jk}(x^i)dx^jdx^k+g_{nn}(x^n)dx^ndx^n\,, \tag{3.4}$$

or, writing x^n instead of $\int\sqrt{g_{nn}}\,dx^n$,

$$ds^2=f(x^\lambda)g^*_{jk}(x^i)dx^jdx^k+dx^ndx^n\,. \tag{3.5}$$

Conversely, if the fundamental quadratic differential form of a Riemannian space can be reduced to the form (3.5) by a suitable choice of the coordinate system, it is easily seen that the hypersurfaces defined by the equations x^n=constants are totally umbilical and the curves defined by x^i=constants are geodesics. Hence, we have the

Theorem: In order that a Riemannian space V_n contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics, it is necessary and sufficient that the fundamental quadratic differential form of the space V_n be reduced to the form (3.5) *by a suitable choice of the coordinate system.*

§ 4. Suppose that our Riemannian space V_n admits a torse-forming vector field $v^\lambda(x)$, then the space V_n contains a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics.

Conversely, if V_n contains such figures as above, there exists a coordinate system with respect to which the fundamental quadratic differential form is (3.5). In such a coordinate system, the Christoffel symbols are given by

$$\left\{\begin{array}{l}\{{}^{\,i}_{jk}\}=\{{}^{\,i}_{jk}\}^*+(\delta^i_jf_k+\delta^i_kf_j-f^ig^*_{jk})\,,\\ \{{}^{\,n}_{jk}\}=-ff_ng^*_{jk}\,,\qquad \{{}^{\,i}_{jn}\}=\{{}^{\,i}_{nj}\}=f_n\delta^i_j\,,\\ \{{}^{\,i}_{nn}\}=\{{}^{\,n}_{jn}\}=\{{}^{\,n}_{nj}\}=\{{}^{\,n}_{nn}\}=0\,,\end{array}\right. \tag{4.1}$$

where $\{{}^{\,i}_{jk}\}^*$ are Christoffel symbols formed with g^*_{jk} and

$$f_k=\frac{1}{2}\frac{\partial\log f}{\partial x^k}\,,\quad f^i=g^{*ia}f_a\,,\quad f_n=\frac{1}{2}\frac{\partial\log f}{\partial x^n}\,.$$

---

1) K. Yano: Concircular geometry II. Proc. **16** (1940), 354-360.

Thus, in such a coordinate system, the vector field $v^\lambda=\delta^\lambda_n$ is torse-forming, for

$$v^\lambda_{;\nu}=\frac{\partial\delta^\lambda_n}{\partial x^\nu}+\{^{\lambda}_{\mu\nu}\}\delta^\mu_n=f_n\delta^\lambda_\nu-f_n\delta^n_\nu\delta^\lambda_n$$
$$=f_n(\delta^\lambda_\nu-v_\mu v^\lambda)\,.$$

Thus, we have the

Theorem: In order that a Riemannian space V_n admit a torse-forming vector field, it is necessary and sufficient that the space V_n contain a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesics.

Theorem: In order that a Riemannian space V_n admit a torse-forming vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form (3.5).

§ 5. It is interesting to observe that the form of the function $f(x^\lambda)$ in (3.5) gives us various special cases which have already been studied by the present author.

The Riemannian space V_n admitting always a torse-forming vector field v^λ, we have seen that the space V_n contains a family of totally umbilical hypersurfaces whose orthogonal trajectories are geodesics. Suppose especially that the normals to these hypersurfaces are concurrent along these hypersurfaces. Then the mean curvatures of these hypersurfaces must be constant and the space must admit concircular transformations[1]. Thus $f(x^\lambda)$ in (3.5) becomes a function of x^n only and the ds^2 of our space takes the form

$$ds^2=f(x^n)g^*_{jk}(x^i)dx^jdx^k+dx^ndx^n\,. \tag{5.1}$$

Conversely, if the fundamental quadratic differential form of the space may be reduced to the form (5.1) by a suitable choice of the coordinate system, the Riemannian space admits concircular transformations and hence it contains a family of ∞^1 totally umbilical hypersurfaces with constant mean curvatures whose orthogonal trajectories are geodesics, and we can conclude that the vector field defined as the normals to these hypersurfaces is torse-forming and concurrent especially along the hypersurfaces.

If the torse-forming vector field is especially a concurrent one[2], the space contains a family of ∞^1 totally umbilical hypersurfaces with constant mean curvature whose orthogonal trajectories are geodesics, and the length of the geodesics between two of these totally umbilical hypersurfaces must be constants[3]. Then the function $f(x^n)$ in (5.1) takes the special form $(x^n)^2$, and the ds^2 of our space becomes

$$ds^2=(x^n)^2g^*_{jk}(x^i)dx^jdx^k+dx^ndx^n\,. \tag{5.2}$$

---

1) K. Yano: Concircular geometry I. Proc. **16** (1940), 195-200; II, 354-360; III, 442-448; IV, 505-511; V, Proc. **18** (1942), 446-451.

2) K. Yano: Sur le parallélisme et la concourance dans l'espace de Riemann. loc. cit.

3) K. Yano and T. Adati: Parallel tangent deformation, concircular transformation and concurrent vector field. Proc. **20** (1944), 123-127.

Conversely, if the fundamental quadratic differential form of the space may be reduced to the form (5.2), the space contains a family of ∞^1 totally umbilical hypersurfaces with constant mean curvatures whose orthogonal trajectories are geodesics and the length of the geodesics between two of these totally umbilical hypersurfaces are constant, thus the vector field defined as the normals to these hypersurfaces is concurrent not only along the hypersurfaces but also along their orthogonal trajectories, and the vector field is concurrent in all the space.

Finally if the torse-forming vector field is especially a parallel one, the space contains a family of ∞^1 totally geodesic hypersurfaces whose orthogonal trajectories are geodesics, thus the function $f(x^\lambda)$ in (3.5) must be equal to a constant, and the ds^2 of our space becomes

$$ds^2 = g^*_{jk}(x^i)dx^j dx^k + dx^n dx^n \,. \tag{5.3}$$

Conversely, if the fundamental quadratic differential form of our space may be reduced to the form (5.3), the space contains a family of ∞^1 totally geodesic hypersurfaces whose orthogonal trajectories are geodesics. Thus the vector field defined as the normals to these hypersurfaces is parallel not only along these totally geodesic hypersurfaces but also along their orthogonal trajectories and the vector field is parallel in all the space.