107. On Biorthogonal Systems in Banach Spaces.

By Sitiro HANAI.

Nagaoka Technical College.

(Comm. by S. KAKEYA, M.I.A., Oct. 12, 1944.)

1. Let $\{x_i\}$ be a sequence of elements of a Banach space E and $\{f_i\}$ a sequence of elements of its conjugate space $\bar{E}$, that is, the space of all bounded linear functionals $f(x)$ defined on E, with norm $\|f\| = \underset{\|x\|\leq 1}{\text{l. u. b.}} |f(x)|$.

The system $\{x_i\,;\,f_i\}$ $(i=1, 2, \dots)$ is called to be *biorthogonal* if

$$f_i(x_j)=\begin{cases}1 & \text{for } i=j\,,\\ 0 & \text{for } i\neq j\,.\end{cases}$$

We denote by X_k the closed linear subspace of E which consists of all linear combinations of terms of the subsequence of $\{x_i\}$ obtained by taking away only one term x_k from $\{x_i\}$ and of all limits of the combinations. The sequence $\{x_i\}$ is said to be *minimal* if $x_k \bar{\in} X_k$ for all k.

S. Kaczmarz and H. Steinhaus[1] have proved the following theorem:

Let $\{x_i\}$ be a sequence of elements of the space $L^{(p)}(p \geqq 1)$. The necessary and sufficient condition that there exists a sequence $\{f_i\}$ of bounded linear functionals defined on $L^{(p)}$ such that the system $\{x_i\,;\,f_i\}$ is biorthogonal is that the sequence $\{x_i\}$ is minimal.

The object of the present paper is to show that the above theorem is valid in the Banach space E and to get the conditions for the existence of $\{x_i\}$ of elements of E such that for a given sequence $\{f_i\}$ of elements of $\bar{E}$ the system $\{x_i\,;\,f_i\}$ is biorthogonal and finally to apply the obtained results to a trigonometrical system.

2. *Theorem 1. Let $\{x_i\}$ be a sequence of elements of E. The necessary and sufficient condition that there exists a sequence $\{f_i\}$ of elements of $\bar{E}$ such that $\{x_i, f_i\}$ is biorthogonal is that $\{x_i\}$ is minimal.*

Proof. Necessity. Suppose that there exists $\{f_i\}$ such that $\{x_i\,;\,f_i\}$ is biorthogonal and $x_1 \in X_1$. Then there are sequences of numbers $\{\gamma_k^{(n)}\}$ $(n=1, 2, \dots)$ such that $Z_n=\sum_{k=2}^{m_n}\gamma_k^{(n)}x_k$ and $\lim_{n\to\infty} Z_n=x_1$.

Therefore $$\lim_{n\to\infty} f_1(Z_n)=f_1(x_1)=1\,.$$

On the other hand $f_1(Z_n)=0$ for $n=1, 2, \dots$, thus we have arrived at a contradiction.

Sufficiency. If $\{x_i\}$ is minimal, then $x_1 \bar{\in} X_1$. Since X_1 is a closed linear subspace of E, there exists an $f_1 \in \bar{E}$ such that

1) S. Kaczmarz and H. Steinhaus: Theorie der Orthogonalreihen, Warszawa, 1935, p. 264.

$$f_1(x_1)=1\,,\qquad f_1(x)=0 \qquad \text{for all} \qquad x\in X_1{}^{2)}\,.$$

In the same manner, we have $f_i \in \bar{E}$ such that

$$f_i(x_i)=1\,,\qquad f_i(x)=0 \qquad \text{for all} \qquad x\in X_i$$

Thus the proof is completed.

Theorem 2. Let $\{f_i\}$ be a sequence of elements of the space $\bar{E}$. The necessary and sufficient condition that there exists a sequence $\{x_i\}$ of elements of E such that $\{x_i;\,f_i\}$ is biorthogonal is that $\{f_i\}$ is minimal.

Proof. Necessity. Suppose that $\{f_i\}$ is not minimal. Let F_k denote the closed linear subspace of $\bar{E}$ obtained from the subsequence of $\{f_i\}$ by taking away only one term f_k from $\{f_i\}$. Without loss of generality we may assume that $f_1 \in F_1$. Then, there are sequences $\{\gamma_k^{(n)}\}$ $(n=1, 2, \ldots)$ such that $f^{(n)}=\sum_{k=2}^{m_n}\gamma_k^{(n)} f_k$ and $\lim_{n\to\infty}\|f_1-f^{(n)}\|=0$.

On the other hand we have

$$f_1(x_1)=1\,,\qquad \text{and} \qquad f^{(n)}(x_1)=\sum_{k=2}^{m_n}\gamma_k^{(n)} f_k(x_1)=0\,.$$

Since $\lim_{n\to\infty}\|f_1-f^{(n)}\|=0$, we get $f_1(x_1)=0$, which contradicts to $f_1(x_1)=1$.

Sufficiency. Since $\{f_i\}$ is minimal and the space $\bar{E}$ is a Banach space, by Theorem 1 there exists a $g_1 \in \bar{\bar{E}}$ such that

$$g_1(f_i)=\begin{cases}1 & \text{for } i=1\,,\\ 0 & \text{for } i\neq 1\,,\end{cases}$$

where $\bar{\bar{E}}$ denotes the conjugate space of $\bar{E}$.

By S. Kakutani's theorem[3] for every $f_1, f_2, \cdots, f_n$ where n denotes an arbitrary integer there exists at least one element $x_1^{(n)} \in E$ such that

$$f_i(x_1^{(n)})=g_1(f_i) \qquad \text{for} \qquad i=1, 2, \cdots, n\,.$$

Let $E_1^{(n)}$ denote the set of all such elements $x_1^{(n)}$. Then it is easy to see that the set $E_1^{(n)}$ is a closed non-null set for each n. And we have clearly $E_1^{(1)} \geqq E_1^{(2)} \geqq \cdots$.

Since the space E is a complete metric space, by the common part theorem of Cantor, the set $E_1=\prod_{n=1}^{\infty} E_1^{(n)}$ is a non-null set. Therefore there exists an element $x_1 \in E_1$.

Then $\qquad f_1(x_1)=1\,,\qquad f_i(x_1)=0 \qquad \text{for} \qquad i\neq 1\,.$

In the same way as the above argument, we have x_j such that

$$f_i(x_j)=\begin{cases}1 & \text{for } i=j\,,\\ 0 & \text{for } i\neq j\,.\end{cases}$$

---

2) S. Banach: Théorie des opérations linéaires, Warszawa, 1932, p. 57.

3) S. Kakutani: Weak topology and regularity of Banach spaces, Proc. **15** (1939), 169–173.

The theorem is thus proved.

Remark. From the proof of Theorem 1, we see that Theorem 1 remains true in normed linear spaces which are more general than Banach spaces.

3. Now we will apply Theorem 2 to a trigonometrical system. Let $\{x_i\}$ be a sequence whose terms are as follows: $x_{2k-1}=\frac{\sin kt}{k}$, $x_{2k}=\frac{\cos kt}{k}$ $(k=1, 2, \dots)$ defined on the closed interval $[-\pi, \pi]$. Then each x_i is a function of bounded variation. Let $V(x_i)$ denote the total variation of x_i on $[-\pi, \pi]$. Then we have the following theorem.

Theorem 3. For each x_i of the system $\{x_i\}$, there exists an $\varepsilon_i > 0$ such that $V(x_i - x) \geqq \varepsilon_i$ for all $x \in X_i$.

Proof. We define a sequence $\{f_i\}$ of bounded linear functionals defined on the space (C) of all real-valued continuous functions on $[-\pi, \pi]$ as follows:

$$f_i(y)=(-1)^{i+1}\frac{1}{\pi}\int_{-\pi}^{\pi} y(t)dx_i \qquad (i \geqq 1).$$

Next we define a sequence $\{y_i\}$ of elements of the space (C) as follows: $y_{2k-1}=\cos kt$, $y_{2k}=\sin kt (k \geqq 1)$. Then it is evident that $\{y_i ; f_i\}$ is a biorthogonal system.

Since the space (C) is a Banach space and the norm $\|f_i\|=\frac{1}{\pi}V(x_i)$, by Theorem 2 we have the theorem.