134. *On Baire Functions on Infinite Product Spaces.*

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A set will be called a Baire set if its characteristic function is a Baire function. In euclidean space, it is well known that the set of all Baire sets coincides with the set of all borel sets. But in general this is not true for other spaces. Of course in any space, it is evident that the set of Baire sets is contained in the set of borel sets. But the converse is not true, namely the characteristic function of a Borel set is not always a Baire function. Such an example is easily derived in an infinite product space, by using some property of Baire functions on this space.

Theorem 1. The value of real valued continuous function on the product space of closed intervals [0 1] is determined by at most countable coordinates. Namely let $\Omega=\prod_{\alpha\in\Theta}\Omega_\alpha$, where for any α $\Omega_\alpha=[0\ 1]$ and Θ is a set of indexes. For any continuous function $f(p)$ on Ω, there exist at most countable coordinates $\alpha_1, \alpha_2, \dots$ depending on $f(p)$, such that for any two points $p=\prod_{\alpha\in\Theta}p_\alpha$, $q=\prod_{\alpha\in\Theta}q_\alpha$ of Ω $f(p)=f(q)$ when $p_{\alpha_i}=q_{\alpha_i}$ $(i=1, 2, \dots)$

Proof. We define the continuous function $f_\alpha(p)$ by $f_\alpha(p)=p_\alpha$. Let R be the smallest Ring of real-valued continuous functions which contains all $f_\alpha(p)$. Then for any two different points q and r there exists $f_\alpha(p)$ such that $f_\alpha(q)\neq f_\alpha(r)$. By a theorem of Gelfand-Silov[1] we see that any continuous function $f(p)$ on Ω may uniformly be approximated by a sequence of elements of R. On the other hand the element of R is the function which depends only on finite coordinates. So $f(p)$ is a function which depends on at most countable coordinates.

Theorem 2. The value of continuous function on the product space of bicompact spaces is determined by at most countable coordinates.

Proof. Let $\Omega=\prod_{\alpha\in\Theta}\Omega_\alpha$, where for any α Ω_α is a bicompact space and Θ is a set of indexes. By the well known theorem every bicompact space may be embedded homeomorphically in an infinite product of intervals [0 1]. So every Ω_α can be embedded homeomorphically in $\bar{\Omega}_\alpha=\prod_{\beta\in\Theta_\alpha}\Omega_\beta$, where for any $\beta\Omega_\beta=[0\ 1]$ and Θ_α is a set of indexes. We put $\bar{\Theta}=\bigcup_\alpha\Theta_\alpha$ and let $\bar{\Omega}=\prod_{\alpha\in\Theta}\bar{\Omega}_\alpha=\prod_{\beta\in\bar{\Theta}}\Omega_\beta$. Then Ω can be embedded homeomorphically in $\bar{\Omega}$. Since Ω is bicompact, the homeomorphic image Ω' of Ω is closed in $\bar{\Omega}$. So every continuous function on Ω' can always be extended to a continuous function on $\bar{\Omega}$. In virtue of Theorem 1 any continuous function $f(p)$ on $\bar{\Omega}$ is determined by at most countable

1) Rec. Math, **9.7** (1941), 25.

coordinates $\beta_1, \beta_2, \cdots$ of $\bar{\Omega} = \prod_{\beta \in \bar{\Theta}} \Omega_\beta$ and so is determined by at most countable coordinates $\alpha_1, \alpha_2, \cdots$ of $\bar{\Omega} = \prod_{\alpha \in \Theta} \bar{\Omega}_\alpha$. Thus any continuous function $g(p)$ on Ω' is determined by at most countable coordinates $\alpha_1, \alpha_2, \cdots$.

As every continuous function on Ω' may be considered as a continuous function on Ω', our theorem is completely proved.

Theorem 3. In general, the value of continuous function on an infinite product of completely regular spaces is determined only by at most countable coordinates.

Proof. Let $\Omega = \prod_{\alpha \in \Theta} \Omega_\alpha$, where for any α Ω_α is a completely regular space and Θ is a set of indexes. By the theorem of compactification of completely regular space, there exists for every Ω_α a bicompact space $\bar{\Omega}_\alpha$ such that Ω_α is dense in $\bar{\Omega}_\alpha$. Let $\bar{\Omega} = \prod_{\alpha \in \Theta} \bar{\Omega}_\alpha$, then it is easily seen that Ω is dense in $\bar{\Omega}$. Moreover any continuous function $f(p)$ on Ω may be extended to a continuous function $\bar{f}(p)$ on $\bar{\Omega}$. In virtue of Theorem 2, $\bar{f}(p)$ is determined by at most countable coordinates and so $f(p)$ is determined by at most countable coordinates. Q. E. D.

Theorem 4. The value of any Baire function on the product space of completely regular spaces is determined by at most countable coordinates.

Proof. This is evident from the definition of Baire function and Theorem 3.

Application. Let $\Omega = \prod_{\alpha \in \Theta} \Omega_\alpha$, where for any α $\Omega_\alpha = [0\ 1]$ and $\Theta = [0\ 1]$. Let $\mathfrak{K}$ be the smallest countably additive class which contains all elementary open sets. It is easily seen that every elementary open set is a Baire set, so the element of $\mathfrak{K}$ is also a Baire set. Conversely it may be proved, by theorem 4, that every Baire set in Ω is contained in $\mathfrak{K}$.

Theorem 5. In Ω, $\mathfrak{K}$ coincides with the set of all Baire sets.

Proof. It will be sufficient to show that for any continuous function $f(p)$ and for any real number λ the set $E\{p : f(p) > \lambda\}$ is an element of $\mathfrak{K}$. For, then the sets $E\{p : f(p) \geqq \lambda\}$, $E\{p : f(p) < \lambda\}$, and $E\{p : f(p) \leqq \lambda\}$ also belong to $\mathfrak{K}$ and hence by transfinite induction we see that for any Baire function $\varphi(p)$ and for any real number λ the set $E\{p : \varphi(p) > \lambda\}$ belongs to $\mathfrak{K}$.

Let $G = E\{p : f(p) > \lambda\}$, then clearly G is open, so G is expressible as the sum of elementary open sets. By theorem 1 we may suppose the value of $f(p)$ is determined by at most countable coordinates $\alpha_1, \alpha_2, \cdots$. So we may assume that every elementary open set which is contained in G is an open set whose coordinates are restricted to a finite number of indexes from $\alpha_1, \alpha_2, \cdots$. The sum of such elementary open sets can be considered as the sum of at most countable elementary open sets. This completes the proof. Q. E. D.

Of course a point p of Ω is a closed set, but its characteristic function is a function which depends on all coordinates.

Therefore p is not a Baire set. Thus we obtain an example of a borel set which is not a Baire set. Next we will show examples of semi-continuous function which is not a Baire function. We put $\varphi(p)=\underset{\alpha\in\Theta}{\mathrm{Min}}\, p_\alpha$ where $p=\prod_{\alpha\in\Theta} p_\alpha$. We easily see that $\varphi(p)$ is an upper semi-continuous function, but as the value of this function is not determined by countable coordinates, so $\varphi(p)$ is not a Baire function.

Similarly if we put $\psi(p)=\underset{\alpha\in\Theta}{\mathrm{Max}}\, P_\alpha$, then we have an example of lower semi-continous function which is not a Baire function.

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