On Bernstein-Chlodovsky operators preserving e^{-2x}

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Abstract

In this paper we introduce a generalization of Bernstein-Chlodovsky operators that preserves the exponential function e^{-2x} $(x \geq 0)$. We study its approximation properties in several function spaces, and we evaluate the rate of convergence by means of suitable moduli of continuity. Throughout some estimates of the rate of convergence, we prove better error estimation than the original operators on certain intervals.

1 Introduction

In 1912 S.N. Bernstein proposed the first constructive proof of the very well known Weierstrass theorem, furnishing an explicit example of a sequence of polynomials that approximates strongly every continuous functions f on a compact interval $[a,b]$. Without loss of generality we can think of $[a,b] = [0,1]$. For every bounded function f on $[0,1]$, $n \geq 1$ and $0 \leq x \leq 1$, such polynomials are defined as

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

and they are known as Bernstein polynomials on $[0,1]$ (see, e.g., [29, Chapter 1]). We point out that the B_n's fix the constants and the function x.

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In order to approximate functions defined on unbounded intervals, in [18] Chlodovsky introduced and studied the following Bernstein-type operators

$$B_{n,h_n}(f)(x) = \sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{x}{h_n}\right)^k\left(1-\frac{x}{h_n}\right)^{n-k}$$

($n \geq 1, x \geq 0$, f belonging to a suitable space), $(h_n)_{n\geq 1}$ being a sequence of strictly positive real numbers such that $\lim_{n\to\infty} h_n = +\infty$. Chlodovsky dealt with several questions related to the sequence $(B_{n,h_n})_{n\geq 1}$: pointwise convergence, uniform convergence, behavior on discontinuous functions, simultaneous approximation, and approximation properties for complex functions (see also [29, pp. 36–37]).

It is worth noticing that, while the B_{n,h_n}'s are not positive operators, many authors have dealt with a positive modification of theirs, that for an abuse of notation we continue to denote by B_{n,h_n} and to call Bernstein-Chlodovsky operators, defined as

$$B_{n,h_n}(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{x}{h_n}\right)^k\left(1-\frac{x}{h_n}\right)^{n-k} & \text{if } 0 \leq x \leq h_n\,, \\ f(x) & \text{if } x > h_n \end{cases} \tag{1.1}$$

(see, e.g., [9, 13]). Further results can be found in [19, 20]. For more recent developments on Bernstein-Chlodovsky operators and their variants, we refer the reader for instance to [1, 2, 26, 27]; moreover, in [17] a relation between Bernstein-Chlodovsky operators and Szász-Mirakyan operators can be found.

In this paper we are interested in a particular modification of the operators (1.1) that allows to reproduce constants and the exponential function e^{-2x} ($x \geq 0$). This kind of investigations fall into a research area which finds its roots in the pioneering work of King [28], where the classical Bernstein operators were modified in order to fix the function x^2, instead of the function x, getting in this way better error estimation than the B_n's on $[0, \frac{1}{3}]$.

Subsequently, King's idea has been successfully applied to several well known sequences of operators. In particular, a modification of the Bernstein-Chlodovsky operators preserving x^2 has been considered by Agratini in [7].

For a survey on the so-called King-type operators see [5]. Here we limit ourselves to recall what has been done in the case of King's type operators preserving exponential functions, which is a more recent development in this research area, as the considerable number of works on this topic appeared in the last couple of years shows.

In [6] the first case of positive linear operators fixing e^x, namely Bernstein-type operators, was treated. Another kind of exponential function is reproduced by the Bernstein-type operators studied by Birou in [14].

A systematic study on King-type operators preserving exponential functions has been done in [3] (see, also, [4]), where Acar, Aral, and Gonska defined Szász-Mirakyan operators preserving constants and e^{2ax}, $a > 0$. Later, other well known linear positive operators have been modified in order to fix constants and

$e^{ax}, a > 0$ ([15, 32]), constants and e^{-x} ([21, 22, 24]), constants and e^{-2x} ([21, 23]), constants and e^{Ax} with $A \in \mathbb{R}$ ([24]), e^{ax} and e^{2ax}, $a > 0$ ([11, 12, 30]).

In the present paper we introduce a sequence of Bernstein-Chlodovsky-type operators that fix constants and e^{-2x}. We deal with their approximation properties both in spaces of continuous functions and in some weighted functions spaces. Moreover, throughout some estimates of the rate of convergence, we are able to prove better error estimation than the original operators on certain intervals.

The paper is organized as follows. After some preliminaries, in Section 3 we define the Bernstein-Chlodovsky operators reproducing the function e^{-2x}. In Section 4 we study their approximation properties in several spaces of continuous functions, also providing some estimates of the rate of convergence by means of suitable moduli of smoothness. In the last section, we pass to investigate the behavior of our operators in polynomial weighted functions spaces, presenting also in this case estimates of the rate of convergence. The results proved are compared with previous ones present in the literature.

2 Preliminaries

In what follows, we shall denote by $C([0,+\infty[)$ the space of all continuous real valued functions on $[0,+\infty[$. $C_b([0,+\infty[)$ is the space consisting of all functions in $C([0,+\infty[)$ which are also bounded. $C_b([0,+\infty[)$, endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering, is a Banach lattice. Moreover, we shall use the symbols $C_*([0,+\infty[)$ and $C_0([0,+\infty[)$ for the Banach sublattices of $C_b([0,+\infty[)$ defined, respectively, as

$$C_*([0,+\infty[) = \{f \in C([0,+\infty[) \,:\, \exists \lim_{x\to+\infty} f(x) \in \mathbb{R}\},$$

and

$$C_0([0,+\infty[) = \{f \in C_*([0,+\infty[) \,:\, \lim_{x\to+\infty} f(x) = 0\}.$$

Further, given the weight function

$$w_m(x) = \frac{1}{1+x^m} \quad (m \geq 1, x \geq 0), \tag{2.1}$$

we consider the space

$$E_m := \{f \in C([0,+\infty[) \,:\, \sup_{x\geq 0} w_m(x)|f(x)| \in \mathbb{R}\},$$

endowed with the norm

$$\|f\|_m := \sup_{x\geq 0} w_m(x)|f(x)| \qquad (f \in E_m),$$

and its natural subspaces

$$E_m^* = \{f \in E_m \,:\, \exists \lim_{x\to+\infty} w_m(x)f(x) \in \mathbb{R}\}$$

and
$$E_m^0 = \{f \in E_m \,:\, \lim_{x\to+\infty} w_m(x)f(x) = 0\}\,.$$
Note that, by the Stone-Weierstrass theorem, $C_0([0,+\infty[)$ is dense in E_m^0.

Throughout this paper, we use the symbol e_i for the power functions $e_i(t) = t^i$ ($t \geq 0$, $i \in \mathbb{N}$), and f_λ ($\lambda > 0$) for the exponential function
$$f_\lambda(x) = e^{-\lambda x} \quad (x \geq 0). \tag{2.2}$$

The purpose of this paper is to introduce a new sequence of positive linear operators on $C([0,+\infty[)$, which generalize Bernstein-Chlodovsky operators; we recall that, for any $n \geq 1$, the n-th Bernstein-Chlodovsky operator is defined by (1.1) and $(h_n)_{n\geq 1}$ is sequence of strictly positive real numbers satisfying $\lim_{n\to\infty} h_n = +\infty$.

Every B_{n,h_n} is linear, positive, and maps $C_*([0,+\infty[)$, $C_0([0,+\infty[)$ and E_m into themselves.

In particular, for every $n \geq 1$,
$$B_{n,h_n}(e_0) = e_0 \quad B_{n,h_n}(e_1) = e_1; \tag{2.3}$$
moreover, if $x \leq h_n$,
$$B_{n,h_n}(e_2)(x) = x^2 - \frac{1}{n}x^2 + \frac{h_n}{n}x\,. \tag{2.4}$$
More generally (see [18], pp. 391-392), for every $n \geq 1$ and $x \leq h_n$, if $m \leq n$,
$$\begin{aligned} B_{n,h_n}(e_m)(x) &= \sum_{k=0}^{m-1} a_{k,m}\left(\frac{h_n}{n}\right)^k \frac{[n]_{m-k}}{n^{m-k}}x^{m-k} \\ &= \frac{[n]_m}{n^m}x^m + \frac{h_n}{n}F_{m-1}(x) \end{aligned} \tag{2.5}$$
where, for $s \geq 1$, $[z]_s := z(z-1)\cdots(z-s+1)$, $[z]_0 := 1$ ($z \in \mathbb{R}$) is the falling difference polynomial, $a_{k,m}$ are suitable positive numbers and F_{m-1} is the polynomial of degree $m-1$
$$F_{m-1}(x) := \sum_{k=1}^{m-1} a_{k,m}\left(\frac{h_n}{n}\right)^{k-1} \frac{[n]_{m-k}}{n^{m-k}}x^{m-k}\,. \tag{2.6}$$

In particular, if we consider the function defined, for every $x \geq 0$, as
$$\psi_x = e_1 - xe_0, \tag{2.7}$$
then, for every $0 \leq x \leq h_n$,
$$B_{n,h_n}(\psi_x)(x) = 0 \quad \text{and} \quad B_{n,h_n}(\psi_x^2)(x) = \frac{h_n}{n}x - \frac{1}{n}x^2. \tag{2.8}$$

Finally, for the exponential functions f_λ (see (2.2)), if $n \geq 1$ and $x \leq h_n$, one has
$$B_{n,h_n}(f_\lambda)(x) = \left(1 - x\left(\frac{1-e^{-(\lambda h_n)/n}}{h_n}\right)\right)^n. \tag{2.9}$$

Under the additional hypothesis

$$\lim_{n\to\infty} \frac{h_n}{n} = 0, \tag{2.10}$$

it is well known that, for every $f \in C_*([0,+\infty[)$,

$$\lim_{n\to\infty} B_{n,h_n}(f) = f \quad \text{uniformly on } [0,+\infty[\,.$$

On the other hand (see [13]), under the assumption (2.10) it also follows that, for every $f \in E_m^0$ with $m > 2$,

$$\lim_{n\to\infty} B_{n,h_n}(f) = f \quad \text{in } E_m^0\,.$$

3 Bernstein-Chlodovsky operators preserving e^{-2x}

In this section we introduce a generalization of operators (1.1) that preserve the function f_2.

To this end, we determine a sequence $(r_n)_{n\geq 1}$ of real functions such that, for every $n \geq 1$, the operators

$$B_n^*(\cdot) = B_{n,h_n}(\cdot) \circ r_n(x) \quad \text{on } [0,h_n] \tag{3.1}$$

have the function f_2 as a fixed point. For that to happen, taking (2.9) into account, we need that, for $r_n(x) \leq h_n$,

$$e^{-2x} = \left(1 - r_n(x)\left(\frac{1-e^{-(2h_n)/n}}{h_n}\right)\right)^n,$$

that is

$$r_n(x) = h_n \frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}}. \tag{3.2}$$

Observe that, for every $n \geq 1$, thanks to the well known inequality $1 - e^{-x} \leq x$ ($x \geq 0$),

$$r_n(0) = 0, \quad 0 < r_n(x) \leq M_n x \quad \text{for every } x > 0\,, \tag{3.3}$$

where

$$M_n := \frac{2h_n}{n(1-e^{-(2h_n)/n})} \quad (n \geq 1)\,. \tag{3.4}$$

Moreover, $M_n \geq 1$ ($n \geq 1$) and, under assumption (2.10), $M_n \to 1$ as $n \to \infty$.

We also point out that $r_n(x) \leq h_n$ if and only if $x \leq h_n$, so that the sequence $(B_n^*)_{n\geq 1}$ turns into

$$B_n^*(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}}\right)^k \left(1 - \frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}}\right)^{n-k} & \text{if } 0 \leq x \leq h_n \\ f(x) & \text{if } x > h_n\,, \end{cases} \tag{3.5}$$

for every $n \geq 1$, $f \in C_*([0,+\infty[)$ and $x \geq 0$.

From (2.3), (2.4) and (2.5), we have that $B_n^*(e_0) = e_0$ and, for $x \leq h_n$,

$$B_n^*(e_1)(x) = r_n(x), \quad B_n^*(e_2)(x) = \frac{n-1}{n} r_n^2(x) + \frac{h_n}{n} r_n(x); \tag{3.6}$$

more generally, for every $n \geq 1$ and $x \leq h_n$, if $m \leq n$,

$$B_n^*(e_m)(x) = \frac{[n]_m}{n^m} r_n(x)^m + \frac{h_n}{n} F_{m-1}(r_n(x)) \tag{3.7}$$

where $F_{m-1}$is defined as in (2.6).

Also (see (2.7)), for any $n \geq 1$ and $x \in [0, h_n]$,

$$B_n^*(\psi_x)(x) = r_n(x) - x \tag{3.8}$$

and

$$B_n^*(\psi_x^2)(x) = \frac{n-1}{n} r_n^2(x) + \frac{h_n}{n} r_n(x) - 2x r_n(x) + x^2. \tag{3.9}$$

Finally, thanks to (2.9), for every $\lambda > 0$ and $x \leq h_n$,

$$B_n^*(f_\lambda)(x) = \left(1 - r_n(x)\left(\frac{1 - e^{-(\lambda h_n)/n}}{h_n}\right)\right)^n = \left(1 - \frac{(1 - e^{-(2x)/n})(1 - e^{-(\lambda h_n)/n})}{1 - e^{-(2h_n)/n}}\right)^n. \tag{3.10}$$

Before illustrating the approximation properties of the sequence $(B_n^*)_{n\geq 1}$, we list some further properties of the generating functions r_n.

Proposition 3.1. *For every $n \geq 1$,*

$$r_n(x) \geq x \quad \textit{for any } x \in [0, h_n]. \tag{3.11}$$

Moreover, suppose that the sequence $(h_n)_{n\geq 1}$ satisfies (2.10). *Then*

$$\lim_{n\to\infty} r_n = e_1 \textit{ uniformly on compact subintervals of } [0,+\infty[\,.$$

Proof. Fix $n \geq 1$. We first note that r_n is a concave increasing function in $[0, h_n]$, as it is the function $-f_{2/n}$. Since $r_n(0) = 0$ and $r_n(h_n) = h_n$, we have that, for every $x \in [0, h_n]$, $r_n(x) \geq x$, hence (3.11) holds true.

It is easy to check that $\lim_{n\to\infty} r_n = e_1$ pointwise on $[0,+\infty[$. Since each r_n is concave (and hence $-r_n$ is convex) the convergence is indeed uniform on every compact interval of $[0,+\infty[$. ∎

Remark 3.2. By applying (3.3) to (3.7), for every $n \geq 1$ and $x \leq h_n$,

$$B_n^*(e_m)(x) \leq M_n^m x^m + c_{n,m-1}(x) \quad \text{if } m \leq n\,,$$

with M_n as in (3.4) and $c_{n,m-1}(x) := \sum_{k=1}^{m-1} a_{k,m} M_n^{m-k} \left(\frac{h_n}{n}\right)^k x^{m-k}$; hence, if we now consider the weight w_m for $m \geq 2$ (see (2.1)),

$$w_m(x) B_n^*(e_m)(x) \leq w_m(x) M_n^m x^m + w_m(x) c_{n,m-1}(x) \leq w_m(x) M_n^m x^m + C_{n,m-1} \tag{3.12}$$

where $C_{n,0} := 0$ and, for $m > 1$,

$$C_{n,m-1} := \max_{x\geq 0} w_m(x) c_{n,m-1}(x) \leq \sum_{k=1}^{m-1} a_{k,m} M_n^{m-k} \left(\frac{h_n}{n}\right)^k . \tag{3.13}$$

Observe that, under the assumption (2.10), $C_{n,m-1} \to 0$ as $n \to \infty$; hence,

$$\lim_{n\to\infty} \|B_n^*(e_m) - e_m\|_m = 0 . \tag{3.14}$$

4 Uniform strongly approximation by the B_n^*'s

In this section we deal with some approximation properties of the sequence $(B_n^*)_{n\geq 1}$ in several spaces of continuous functions. We provide also estimates of the rate of convergence.

Theorem 4.1. *Consider the operators B_n^* ($n \geq 1$) defined by* (3.5). *Then, for a fixed $n \geq 1$,*

(i) B_n^* is a positive linear operator from $C_*([0, +\infty[)$ into itself; moreover, $\|B_n^*\|_{C_*([0,+\infty[)} = 1$.

(ii) $B_n^*(C_0([0, +\infty[)) \subset C_0([0, +\infty[)$.

Proof. (i) Fix $n \geq 1$. The positivity of the B_n^*'s is easily verified on account of (3.1) and (3.3). Moreover, if $f \in C_*([0, +\infty[)$, as quoted after (1.1), $B_{n,h_n}(f) \in C_*([0, +\infty[)$, hence in particular $B_{n,h_n}(f) \in C([0, +\infty[)$. Then, observing that the function r_n is continuous and $r_n(h_n) = h_n$, from (3.1) it follows that $B_n^*(f) \in C([0, +\infty[)$ and, from (3.5), $\lim_{x\to+\infty} B_n^*(f)(x) = \lim_{x\to+\infty} f(x) \in \mathbb{R}$. Finally, $\|B_n^*\|_{C_*([0,+\infty[)} = \|B_n^*(e_0)\|_\infty = 1$, thanks to the positivity of each B_n^*.

(ii) It is an easy consequence of (i) and the fact that $\lim_{x\to+\infty} B_n^*(f)(x) = \lim_{x\to+\infty} f(x) = 0$ whenever $f \in C_0([0, +\infty[)$. ■

Theorem 4.2. *Let $(B_n^*)_{n\geq 1}$ be the sequence of operators defined by* (3.5) *under assumption* (2.10). *The following statements hold true:*

(i) If $f \in C_*([0, +\infty[)$, then $\lim_{n\to+\infty} B_n^*(f) = f$ uniformly on $[0, +\infty[$.

(ii) If $f \in C_b([0, +\infty[)$, then $\lim_{n\to+\infty} B_n^*(f) = f$ uniformly on compacts subsets of $[0, +\infty[$.

Proof. In order to prove statement (i), we show that, for every $\lambda > 0$,

$$\lim_{n\to\infty} B_n^*(f_\lambda) = f_\lambda \quad \text{uniformly on } [0, +\infty[\tag{4.1}$$

(see (2.2)).

Indeed, we recall that (see [25, Lemma 3.1]), for every $t > 0$ and $n \geq 1$,

$$e^{-t\alpha_n} - e^{-t} < \frac{t_n}{2e}, \tag{4.2}$$

where $\alpha_n = \frac{1-e^{-t_n}}{t_n}$, and $(t_n)_{n\geq 1}$ is a sequence of strictly positive real numbers.

Following the same arguments in the proof of Corollary 3.4 in [25], we have that, for every $n \geq 1$ and $x \in [0, h_n]$,

$$|B_n^*(f_\lambda)(x) - e^{-\lambda x}| \leq e^{-\lambda r_n(x)\frac{1-e^{-\lambda h_n/n}}{\lambda h_n/n}} - e^{-\lambda x} \leq e^{-\lambda r_n(x)\frac{1-e^{-\lambda h_n/n}}{\lambda h_n/n}} - e^{-\lambda r_n(x)},$$

where the last inequality holds true by means of (3.11).

Hence, by applying (4.2) for $t = \lambda r_n(x)$ and $t_n = (\lambda h_n)/n$, we get that, for every $x \in [0, h_n]$,

$$|B_n^*(f_\lambda)(x) - e^{-\lambda x}| \leq \frac{\lambda h_n}{2ne}, \tag{4.3}$$

so that

$$\|B_n^*(f_\lambda) - f_\lambda\|_\infty \leq \frac{\lambda h_n}{2ne}, \tag{4.4}$$

and this completes the proof of (4.1). Statement (i) follows directly from (4.1) and the results in [16].

In order to prove statement (ii), we notice that (see (3.4)), for every $0 \leq x \leq h_n$,

$$|B_n^*(e_1)(x) - e_1(x)| \leq x\,(M_n - 1)$$

and

$$|B_n^*(e_2)(x) - e_2(x)| \leq x^2\left(M_n^2 - 1\right) + x\frac{h_n}{2n}M_n,$$

so that $\lim_{n\to+\infty} B_n^*(h) = h$ uniformly on compact subsets of $[0+\infty[$ for every $h \in \{e_0, e_1, e_2\}$ on account of (2.10) and the fact that $M_n \to 1$ as $n \to \infty$. Since $\{e_0, e_1, e_2\} \subset E_2^*$, the result follows from [8, Theorem 3.5]. ∎

A question about the usefulness, from an approximation theory point of view, of the sequence $(B_n^*)_{n\geq 1}$ naturally arises. To answer this question, we recall that, if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators acting on E_2^* such that $L_n(e_0) = e_0$ for all $n \geq 1$, then, for every $f \in C_b([0, +\infty[)$, $n \geq 1$ and $x \geq 0$,

$$|L_n(f)(x) - f(x)| \leq 2\omega(f, \sqrt{L_n(\psi_x^2)(x)}), \tag{4.5}$$

where $\omega(f, \delta)$ is the classical first modulus of continuity (see [9, Theorem 5.1.2]).

Hence, in approximating a function $f \in C_b([0, +\infty[)$, for a given $n \geq 1$, the operator $B_n^*(f)$ performs better than $B_{n,h_n}(f)$ in those points $x \in [0, h_n]$ such that

$$B_n^*(\psi_x^2)(x) \leq B_{n,h_n}(\psi_x^2)(x), \tag{4.6}$$

i.e, taking (2.8) and (3.9) into account, in all $x \in [0, h_n]$ verifying

$$(r_n(x) - x)\left[r_n(x) - x + \frac{h_n}{n} - \frac{1}{n}r_n(x) - \frac{1}{n}x\right] \leq 0.$$

Given (3.11), the latter leads to

$$g_n(x) := \frac{n-1}{n}r_n(x) - \frac{n+1}{n}x + \frac{h_n}{n} \leq 0$$

which is true for $x \in [x_{0,n}, h_n]$, $x_{0,n} \in (h_n/2, h_n)$ being unique. Indeed, $g_n(h_n) < 0 < g_n(0)$, so there exists $x_{0,n} \in (0, h_n)$ such that $g(x_{0,n}) = 0$. In addition, g_n is concave and $g_n(h_n/2) < 0$, hence the assertion.

We pass now to estimate the rate of convergence of $(B_n^*(f))_{n\geq 1}$ to f in Theorem 4.2, (i). To this end, recall the definition of a suitable modulus of continuity. More precisely, in [25] the author introduced the modulus of continuity $\omega^*(f,\delta)$ defined, for every $\delta \geq 0$ and $f \in C_*([0,+\infty[)$, by

$$\omega^*(f,\delta) = \sup_{\substack{x,t\geq 0\\ |e^{-x}-e^{-t}|\leq\delta}} |f(x) - f(t)| .$$

We remark that

$$\omega^*(f,\delta) = \omega(\Phi(f),\delta) ,$$

where $\omega(\cdot,\delta)$ stands for the usual modulus of continuity and $\Phi : C_*([0,+\infty[) \to C([0,1])$ is the isometric isomorphism defined by setting

$$\Phi(f)(t) = \begin{cases} f(-\ln t) & \text{if } 0 < t \leq 1 , \\ \lim\limits_{x\to+\infty} f(x) & \text{if } t = 0 , \end{cases} \quad \text{for every } f \in C_*([0,+\infty[) . \tag{4.7}$$

Let us mention the main result in [25]: for a sequence of positive linear operators $L_n : C_*([0,+\infty[) \to C_*([0,+\infty[)$ $(n \geq 1)$, if

$$a_n = \|L_n(e_0) - e_0\|_\infty , b_n = \|L_n(f_1) - f_1\|_\infty , c_n = \|L_n(f_2) - f_2\|_\infty \tag{4.8}$$

are null sequences, then for every $n \geq 1$ and $f \in C_*([0,+\infty[)$,

$$\|L_n(f) - f\|_\infty \leq \|f\|_\infty a_n + (2 + a_n)\,\omega^*(f, \sqrt{a_n + 2b_n + c_n}) . \tag{4.9}$$

Note that the modulus $\omega^*(f,\delta)$ is closely related to the particular Korovkin subset chosen for the space $C_*([0,+\infty[)$ (see [25, p. 135]).

The following result is indeed a direct consequence of (4.10).

Theorem 4.3. *Under the same assumptions of Theorem 4.2, for every $f \in C_*([0,+\infty[)$ and $n \geq 1$,*

$$\|B_n^*(f) - f\|_\infty \leq 2\,\omega^* \left(f, \sqrt{\frac{h_n}{ne}} \right) . \tag{4.10}$$

Proof. It is clear that, following the notation in (4.8), $a_n = c_n = 0$ and b_n is given by (4.4) with $\lambda = 1$ for every $n \geq 1$. ∎

Remark 4.4. Estimate (4.10) improves the one available in [25, Corollary 3.4], for the classical Bernstein-Chlodovsky operators (see (1.1)); in fact, for every $f \in C_*([0,+\infty[)$ and $n \geq 1$ (see (4.8)),

$$\|B_{n,h_n}(f) - f\|_\infty \leq 2\omega^* \left(f, \sqrt{\frac{2h_n}{ne}} \right)$$

In the same spirit of [25], we can consider on the space $C_*([0,+\infty[)$ a modulus of continuity of second order; namely, for every $\delta \geq 0$ and $f \in C_*([0,+\infty[)$, we set

$$\omega_2^*(f,\delta) = \omega_2(\Phi(f),\delta),$$

Φ being as in (4.7).

Note that the inverse operator Φ^{-1} is given by $\Phi^{-1}(g)(t) = g(e^{-t})$ for each $g \in C([0,1])$ and $t \geq 0$.

Starting from the operators B_n^* on $C_*([0,+\infty[)$, we obtain a new sequence of positive linear operators acting on $C([0,1])$, namely

$$Z_n^*(g) = \Phi(B_n^*(\Phi^{-1}(g))) \quad (g \in C([0,1]), n \geq 1). \tag{4.11}$$

Observe that the sequences $(B_n^*)_{n\geq 1}$ and $(Z_n^*)_{n\geq 1}$ are isomorphic, that is for every $f \in C_*([0,+\infty[)$ and $n \geq 1$,

$$\|B_n^*(f) - f\|_\infty = \|Z_n^*(\Phi(f)) - \Phi(f)\|_\infty, \tag{4.12}$$

hence the problem of estimating the rate of convergence of $(B_n^*)_{n\geq 1}$ in the space $C_*([0,+\infty[)$ might be transferred to evaluate the rate of convergence of the sequence $(Z_n^*)_{n\geq 1}$ in $C([0,1])$.

In this manner, rather that proving new estimates of the rate of convergence on $C_*([0,+\infty[)$, we can use some known results in $C([0,1])$ (see [10, Section 4] for a similar reasoning applied to other approximation processes). For example, we can apply [31, Theorem 2.2.1].

Theorem 4.5. *Under the same assumptions of Theorem 4.2, for every $f \in C_*([0,+\infty[)$ and $n \geq 1$,*

$$\|B_n^*(f) - f\|_\infty \leq \frac{1}{2e}\sqrt{\frac{h_n}{n}}\omega^*\left(\Phi(f), \sqrt{\frac{h_n}{n}}\right) + \left(1 + \frac{1}{e}\right)\omega_2^*\left(\Phi(f), \sqrt{\frac{h_n}{n}}\right).$$

Proof. As the equality (4.12) suggests, we search for a uniform estimate of $\|Z_n^*(\Phi(f)) - \Phi(f)\|_\infty$. For every $n \geq 1$, $f \in C_*([0,+\infty[)$, $0 \leq x \leq 1$ and $\delta > 0$, the following pointwise estimate holds (see [31, Theorem 2.2.1]):

$$\begin{aligned}|Z_n^*(\Phi(f))(x) - \Phi(f)(x)| &\leq |Z_n^*(e_0)(x) - 1||\Phi(f)(x)| + \frac{1}{\delta}|Z_n^*(\psi_x)(x)|\,\omega^*(f,\delta)\\ &+ \left(Z_n^*(e_0)(x) + \frac{1}{2\delta^2}Z_n^*(\psi_x^2)(x)\right)\omega_2^*(f,\delta).\end{aligned}$$

It is easy to see that $Z_n^*(e_0) = e_0$. Moreover, for $n \geq 1$ and $x \in [0,1]$,

$$Z_n^*(\psi_x)(t) = \begin{cases} B_n^*(f_1 - xe_0)(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

$$Z_n^*(\psi_x^2)(t) = \begin{cases} B_n^*(f_2 - 2xf_1 + x^2e_0)(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

where f_λ, $\lambda = 1, 2$, is defined by (2.2). Hence,

$$|Z_n^*(\psi_x)(x)| = \begin{cases} |B_n^*(f_1)(-\log x) - x| & \text{if } e^{-h_n} \le x \le 1\,, \\ 0 & \text{if } 0 \le x < e^{-h_n}\,, \end{cases}$$

and from (4.3) it follows that $|Z_n^*(\psi_x)(x)| \le h_n/(2en)$. Moreover,

$$Z_n^*(\psi_x^2)(x) = \begin{cases} B_n^*(f_2)(-\log x) - x^2 - 2x(B_n^*(f_1)(-\log x) - x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } e^{-h_n} \le x \le 1\,, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } 0 \le x < e^{-h_n}\,, \end{cases}$$

and again from (4.3) we have

$$Z_n^*(\psi_x^2)(x) \le \frac{h_n}{en} + \frac{h_n}{en} = \frac{2h_n}{en}\,.$$

Setting $\delta = \sqrt{h_n/n}$ we get the desired estimate. ∎

5 Uniform weighted approximation by the B_n^*'s

The approximation properties of the operators B_n^* on the weighted function spaces E_m^0, E_m^* and E_m are shown below. Moreover, some estimates of the rate of convergence are presented. For (a generalization of) Bernstein-Chlodovsky operators (1.1) similar matters were tackled in [13], where the authors dealt with the space E_m^0, $m > 2$.

Theorem 5.1. *Consider the operators B_n^* ($n \ge 1$) defined by* (3.5). *Then, for a fixed $n \ge 1$, the following propositions hold:*

(i) For $m \le n$, B_n^* is a positive linear operator from E_m into itself and $\|B_n^*\|_{E_m} \le 1 + M_n^m + C_{n,m}$, where M_n and $C_{n,m}$ are given by (3.4) *and* (3.13), *respectively. In particular, if* (2.10) *holds true,* $\sup_{n\ge1} \|B_n^*\|_{E_m} < +\infty$. *Finally, $B_n^*(E_m^*) \subset E_m^*$.*

(ii) For $m \le n$, $B_n^*(E_m^0) \subset E_m^0$.

Proof. (i) Fix $n \ge 1$. First of all, note that, from (3.5) and (3.12), for $x \le h_n$,

$$w_m(x) B_n^*(e_m)(x) \le w_m(x) M_n^m x^m + C_{n,m-1} \le M_n^m + C_{n,m-1}\,,$$

where M_n and $C_{n,m-1}$ are give by (3.4) and (3.13).

Moreover, for $x > h_n$,

$$w_m(x) B_n^*(e_m)(x) = x^m w_m(x) \le 1 \le M_n \le M_n^m + C_{n,m-1}.$$

Therefore, for every $x \ge 0$, $w_m(x) B_n^*(e_m)(x) \le M_n^m + C_{n,m-1}$, where M_n is defined by (3.4). We remark that, under assumption (2.10), $(M_n)_{n\ge1}$ is indeed a bounded sequence.

Summing up, if $f \in E_m$ and $x \geq 0$,

$$\begin{aligned} w_m(x)|B_n^*(f)(x)| &\leq \|f\|_m w_m(x) B_n^*(e_0+e_m)(x) \\ &= \|f\|_m w_m(x)(B_n^*(e_0)(x) + B_n^*(e_m)(x)) \leq \|f\|_m (1 + M_n^m + C_{n,m-1}), \end{aligned}$$

hence (i) holds true.

(ii) The statement can be achieved by noting that the subspace D generated by the family $\{f_\lambda : \lambda > 0\}$ is dense in $C_0([0,+\infty[)$, and consequently in E_m^0; moreover, $B_n^*(D) \subset C_0([0,+\infty[) \subset E_m^0$. ■

Theorem 5.2. *Let $(B_n^*)_{n\geq 1}$ be the sequence of operators defined by* (3.5) *and assume that* (2.10) *holds true. If $f \in E_m^*$ (and, in particular, if $f \in E_m^0$), then*

$$\lim_{n\to+\infty} B_n^*(f) = f \quad \text{with respect to } \|\cdot\|_m. \tag{5.1}$$

Moreover, if $f \in E_m$, then

$$\lim_{n\to+\infty} w_m(x)(B_n^*(f)(x) - f(x)) = 0 \tag{5.2}$$

uniformly on compact subsets of $[0,+\infty[$.

Proof. We begin the proof by showing (5.1) for functions in E_m^0 with a density argument. Indeed, under assumption (2.10), the sequence $(B_n^*)_{n\geq 1}$ is equibounded on E_m^0 by means of Theorem 5.1 and the linear subspace generated by $(f_\lambda)_{\lambda>0}$ is dense in E_m^0. As we have shown in Theorem 4.2, for every $\lambda > 0$, $\lim_{n\to+\infty} B_n^*(f_\lambda) = f_\lambda$ with respect to $\|\cdot\|_\infty$ and hence with respect to $\|\cdot\|_m$, and this completes the proof.

On the other hand, if $f \in E_m^*$, then $f = g + \alpha_m(e_0 + e_m)$, where $\alpha_m := \lim_{x\to+\infty} w_m(x) f(x) \in \mathbb{R}$ and $g = f - \alpha_m(e_0+e_m) \in E_m^0$. Therefore (5.1) follows for f too, in virtue of (3.14).

Formula (5.2) is a consequence of the preceding result and the inclusion $E_m \subset E_{m+1}^0$ since, if J is a compact subset of $[0,+\infty[$, then

$$w_m(x)|B_n^*(f)(x) - f(x)| \leq K \|B_n^*(f) - f\|_{m+1}$$

for every $x \in J$, where $K := \sup_{x\in J} \frac{w_m(x)}{w_{m+1}(x)}$. ■

For the convergence in E_m^0 we have the following result.

Theorem 5.3. *For sufficiently large $n \geq 1$ and $f \in E_m^0$ with $m > 2$,*

$$\|B_n^*(f) - f\|_m \leq 2\omega\left(f, \alpha_m\sqrt{M_n^2 - 1} + \beta_m\sqrt{\frac{h_n}{n} M_n}\right), \tag{5.3}$$

where M_n is given by (3.4) and α_m, β_m are suitable constants depending only on m.

Proof. It is known that (see [9, Theorem 5.1.2]), for any $n \geq 1$, $x \in [0, h_n]$, $f \in E_m^0$ with $m > 2$ and $\delta > 0$,

$$|B_n^*(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta}\sqrt{B_n^*(\psi_x^2)(x)}\right)\omega(f, \delta), \tag{5.4}$$

where $\omega(f, \delta)$ is the classical first modulus of continuity. Hence,

$$\|B_n^*(f) - f\|_m \leq \left(1 + \frac{1}{\delta}\sup_{x \in [0, h_n]} \frac{\sqrt{B_n^*(\psi_x^2)(x)}}{1 + x^m}\right)\omega(f, \delta),$$

so that from the pointwise estimate (5.4) it is possible to get a uniform weighted estimate. To this end, we evaluate

$$\sup_{0 \leq x \leq h_n} \frac{\sqrt{B_n^*(\psi_x^2)(x)}}{1 + x^m}.$$

First, by a straightforward computation, for largely enough $n \geq 1$, we have

$$\sup_{0 \leq x \leq h_n} \frac{x}{1 + x^m} = \max\left\{\frac{h_n}{1 + h_n^m}, \frac{\frac{1}{\sqrt[m]{m-1}}}{1 + \frac{1}{m-1}}\right\} = \frac{\frac{1}{\sqrt[m]{m-1}}}{1 + \frac{1}{m-1}} =: \alpha_m,$$

and

$$\sup_{0 \leq x \leq h_n} \frac{\sqrt{x}}{1 + x^m} = \max\left\{\frac{\sqrt{h_n}}{1 + h_n^m}, \frac{\frac{1}{\sqrt[2m]{2m-1}}}{1 + \frac{1}{2m-1}}\right\} = \frac{\frac{1}{\sqrt[2m]{2m-1}}}{1 + \frac{1}{2m-1}} =: \beta_m,$$

since

$$\lim_{n \to \infty} \frac{\sqrt{h_n}}{1 + h_n^m} = 0 = \lim_{n \to \infty} \frac{h_n}{1 + h_n^m}.$$

Moreover, on account of (3.3), (3.9) and (3.11),

$$B_n^*(\psi_x^2)(x) \leq M_n^2 x^2 + \frac{h_n}{n} M_n x - x^2 = \left(M_n^2 - 1\right) x^2 + \frac{h_n}{n} M_n x.$$

Hence,

$$\sup_{0 \leq x \leq h_n} \frac{\sqrt{B_n^*(\psi_x^2)(x)}}{1 + x^m} \leq \alpha_m \sqrt{M_n^2 - 1} + \beta_m \sqrt{\frac{h_n}{n} M_n},$$

and from this, taking $\delta = \alpha_m \sqrt{M_n^2 - 1} + \beta_m \sqrt{\frac{h_n}{n} M_n}$, (5.3) follows. ∎

In order to estimate the rate of the convergence in E_m^*, we may use a similarity technique as made at page 690. In this case, we define the isometric isomorphism Φ_m between E_m^* and $C([0,1])$ defined by

$$\Phi_m(f)(t) = \begin{cases} (w_m f)(-\log t) & 0 < t \leq 1 \\ \lim_{x \to +\infty} (w_m f)(x) & t = 0 \end{cases} \quad \text{for any } f \in E_m^*.$$

Then, on the space E_m^* we can define two new moduli of continuity; namely, for every $\delta \geq 0$ and $f \in E_m^*$, we set

$$\omega_m^*(f,\delta) = \omega(\Phi_m(f),\delta)\,,$$

and

$$\omega_{2,m}^*(f,\delta) = \omega_2(\Phi_m(f),\delta)\,.$$

If we now consider the operators B_n^* acting on E_m^*, we can define

$$W_n^*(g) = \Phi_m(B_n^*(\Phi_m^{-1}(g))) \quad (g \in C([0,1]), n \geq 1)\,, \tag{5.5}$$

where the inverse operator Φ_m^{-1} is given by $\Phi_m^{-1}(g) = w_m^{-1}(t)g(e^{-t})$ for each $g \in C([0,1])$ and $t \geq 0$; $(W_n^*)_{n\geq 1}$ is a sequence of positive linear operators on $C([0,1])$.

Further, for every $f \in E_m^*$ and $n \geq 1$,

$$\|B_n^*(f) - f\|_m = \|W_n^*(\Phi_m(f)) - \Phi_m(f)\|_\infty\,,$$

hence again we may estimate the rate of convergence of $(B_n^*)_{n\geq 1}$ in E_m^* by means of estimates of the sequence $(W_n^*)_{n\geq 1}$ in $C([0,1])$, as the following result shows.

Theorem 5.4. *For every $n, m \geq 1$, $m \leq n$, and $f \in E_m^*$,*

$$\begin{aligned}&\|B_n^*(f) - f\|_m \leq (M_n^m - 1 + C_{n,m-1})||\Phi_m(f)||_\infty \\ &+ H_{1,m}\sqrt[4]{\frac{h_n}{n}}\omega_m^*\left(f, \sqrt[4]{\frac{h_n}{n}}\right) + \left(M_n^m + C_{n,m-1} + \frac{1}{2}H_{2,m}\sqrt{\frac{h_n}{n}}\right)\omega_{2,m}^*\left(f, \sqrt[4]{\frac{h_n}{n}}\right),\end{aligned}$$

where $H_{1,m}, H_{2,m}$ are suitable positive constants depending only on m, and M_n and $C_{n,m-1}$ are given in (3.4) *and* (3.13).

Proof. Thanks to (5.5) we evaluate $\|W_n^*(\Phi_m(f)) - \Phi_m(f)\|_\infty$ by applying [31, Theorem 2.2.1].

Easy calculations show that, for $n \geq 1$ and $x \in [0,1]$,

$$W_n^*(e_0)(t) = \begin{cases} (w_m B_n^*(e_0 + e_m))(-\log t) & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t = 0, \end{cases}$$

$$W_n^*(\psi_x)(t) = \begin{cases} (w_m B_n^*((1 + e_m)(f_1 - xe_0)))(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

and

$$W_n^*(\psi_x^2)(t) = \begin{cases} (w_m B_n^*((1 + e_m)(f_2 - 2xf_1 + x^2 e_0)))(-\log x) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

where f_λ, $\lambda = 1, 2$, is defined by (2.2).

Now, fix $0 < x \leq 1$; recalling the definition of the operators B_n^*, and thanks to (3.12), we have

$$|w_m(-\log x)B_n^*(e_0 + e_m)(-\log x) - 1| \begin{cases} \leq M_n^m - 1 + C_{n,m-1} & \text{if } e^{-h_n} \leq x \leq 1, \\ = 0 & \text{if } 0 \leq x < e^{-h_n}; \end{cases}$$

hence,

$$||W_n^*(e_0) - e_0||_\infty \le M_n^m - 1 + C_{n,m-1}\,.$$

Moreover, for every $e^{-h_n} \le x \le 1$, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|W_n^*(\psi_x)(x)| &\le w_m(-\log x) B_n^*(|(1+e_m)(f_1 - xe_0)|)(-\log x)\\ &\le w_m(-\log x)\sqrt{B_n^*((e_0+e_m)^2)(-\log x)}\sqrt{B_n^*((f_1 - xe_0)^2)(-\log x)}\\ &= w_m(-\log x)\sqrt{B_n^*((e_0+e_m)^2)(-\log x)}\sqrt{Z_n^*(\psi_x^2)(x)}.\end{aligned}$$

We point out that, by (3.12),

$$L_m := \sup_{0<x\le 1} w_m(-\log x)\sqrt{B_n^*((e_0+e_m)^2)(-\log x)} \in \mathbb{R};$$

therefore, on account of what has been made in the proof of Theorem 4.5, there exists $H_{1,m} > 0$ such that

$$|W_n^*(\psi_x)(x)| \le H_{1,m}\sqrt{\frac{h_n}{n}}.$$

Note that, for $0 \le x < e^{-h_n}$, $W_n^*(\psi_x)(x) = 0$, therefore the previous inequality holds for every $x \in [0,1]$.

Finally, for every $0 < x \le 1$, we get

$$\begin{aligned}W_n^*(\psi_x^2)(x) &= w_m(-\log x) B_n^*((1+e_m)(f_2 - 2xf_1 + x^2))(-\log x)\\ &\le w_m(-\log x)\sqrt{B_n^*((1+e_m)^2)(-\log x)}\sqrt{B_n^*((f_2 - 2xf_1 + x^2)^2)(-\log x)}\\ &\le L_m\sqrt{Z_n^*(\psi_x^4)(x)}.\end{aligned}$$

Hence, keeping (4.3) in mind, there exists $K_3 > 0$ such that

$$\begin{aligned}W_n^*(\psi_x^4)(x) &= B_n^*(f_4)(-\log x) - x^4 - 4x(B_n^*(f_3)(-\log x) - x^3)\\ &+ 6x^2(B_n^*(f_2)(-\log x) - x^2) - 4x^3(B_n^*(f_1)(-\log x) - x) \le K_3\frac{h_n}{n},\end{aligned}$$

so that

$$W_n^*(\psi_x^2)(x) \le H_{2,m}\sqrt{\frac{h_n}{n}},$$

for a suitable constant $H_{2,m} > 0$ depending on m, only.

From the above considerations, applying [31, Theorem 2.2.1], for every $n \ge 1$, $f \in E_m^*$, $0 \le x \le 1$ and $\delta > 0$, we get

$$\begin{aligned}&|W_n^*(\Phi_m(f))(x) - \Phi_m(f)(x)| \le |W_n^*(e_0)(x) - 1||\Phi_m(f)(x)|\\ &+ \frac{1}{\delta}|W_n^*(\psi_x)(x)|\omega_m^*(f,\delta) + \left(W_n^*(e_0)(x) + \frac{1}{2\delta^2}W_n^*(\psi_x^2)(x)\right)\omega_{2,m}^*(f,\delta)\\ &\le (M_n^m - 1 + C_{n,m-1})|\Phi_m(f)(x)|\\ &+ \frac{1}{\delta}H_{1,m}\sqrt{\frac{h_n}{n}}\omega_m^*(f,\delta) + \left(M_n^m + C_{n,m-1} + \frac{1}{2\delta^2}H_{2,m}\sqrt{\frac{h_n}{n}}\right)\omega_{2,m}^*(f,\delta),\end{aligned}$$

(see (3.13)) and, for $\delta := \sqrt[4]{h_n/n}$ we get the required assertion. ■

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