# Some fixed point theorems for Meir-Keeler condensing operators and application to a system of integral equations

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**Abstract**

We introduce the concept of Meir-Keeler condensing operator in a Banach space via an arbitrary measure of weak noncompactness. We prove some generalizations of Darbo's fixed point theorem by considering a measure of weak noncompactness which not necessary has the maximum property. We prove some coupled fixed point theorems and we apply them in order to establish the existence of weak solutions for a system of functional integral equations of Volterra type.

In 1955, Darbo [10] proved a fixed point theorem which combines Schauder fixed point theorem and Banach contraction principle by considering the concept of measure of noncompactness, as introduced by Kuratowski [15]. Later, Sadovskii [18] proved a more general fixed point theorem by considering the concept of condensing mapping. On the other hand, in 1977, De Blasi [11] introduced the concept of measure of weak noncompactness, and in 1981 G. Emmanuele [12] stated a fixed point result for condensing mapping with respect to the measure of weak noncompactness. For more results concerning the weak topology of condensing operator which are weakly sequentially continuous, see [7, 8]. Observe that the different versions of Sadovskii's fixed point theorem that appeared in the literature are essentially based on the maximum property of the measure of noncompactness. Recently, Falset and Latrach [13] proved a Sadovskii's fixed point

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theorem without using the maximum property, in case the domain of the operator can, in some sense, be included in a weakly compact set.
In [1, 2, 3, 13] some generalizations of Darbo's theorem are given when the measure of noncompactness does not satisfy the maximum property. In particular, in [2], the authors introduce the notion of Meir-Keeler condensing operator and provide a few generalizations of Darbo's fixed point theorem. Also, they introduce the concept of a bivariate Meir-Keeler condensing operator and proved some coupled fixed point theorems.
In this paper, we introduce the notion of Meir-Keeler condensing operator with respect to a measure of weak noncompactness and we prove some generalization of Darbo's fixed point theorem by considering a measure of weak noncompactness which does not necessarily have the maximum property. Also, we prove some coupled fixed point theorems for weakly sequentially operators. These results are then used to investigate the existence of weak solutions to a system of functional integral equations of Volterra type

$$\begin{cases} x(t) = f(t, x(t), y(t)) + \int_0^t g(t, x(s), y(s))ds \\ \\ y(t) = f(t, y(t), x(t)) + \int_0^t g(t, y(s), x(s))ds \end{cases}$$

where $E$ is a Banach space, $T > 0$ and $f, g : [0, T] \times E \times E \to E$ are given functions.

## 1 Preliminaries

Let $E$ be a Banach space endowed with the norm $\| \, . \, \|$ and with the zero element $\theta$. We denote by $B(x, r)$ the closed ball centered at $x$ with radius $r$. In particular, we write $B_r(\theta) = B(\theta, r)$. For a subset $C$ of $E$, we write $\overline{C}$, $\overline{C}^{\omega}$, $convC$ and $\overline{convC}$, to denote the closure, the weak closure, the convex hull and the closed convex hull of the subset $C$, respectively. Moreover, we write $x_n \longrightarrow x$ and $x_n \rightharpoonup x$ to denote the strong convergence (with respect to the norm of $E$) and the weak convergence (with respect to the weak topology of $E$) of a sequence $\{x_n\}$ to $x$.
Further denote by $\mathfrak{B}_E$ the family of all nonempty and bounded subsets of a Banach space $E$, $\mathfrak{N}_E$ the family of all nonempty and relatively weakly compact subsets and let $\mathcal{K}^{\omega}$ be the family of all weakly compact subsets of $E$.
In the sequel we need the following definition of a measure of weak noncompactness [8].

**Definition 1.1.** *Let $E$ be a Banach space and $X, X_1, X_2 \in \mathfrak{B}_E$. A mapping $\omega : \mathfrak{B}_E \to [0, \infty)$ is said to be a measure of weak noncompactness if it satisfies the following conditions:*

*(1) The family $Ker(\omega) = \{X \in \mathfrak{B}_E : \omega(X) = 0\}$ is nonempty and $Ker(\omega) \subseteq \mathfrak{N}_E$.*

*(2) $X_1 \subset X_2$ implies $\omega(X_1) \leq \omega(X_2)$.*

*(3) $\omega(X_1) = \omega(\overline{X_1}^{\omega})$.*

*(4) $\omega(conv(X_1)) = \omega(X_1)$.*

(5) $\omega(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\omega(X_1) + (1-\lambda)\omega(X_2)$ *for* $\lambda \in [0,1]$.

(6) *If* $(X_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of E with* $\lim_{n\to+\infty} \omega(X_n) = 0$, *then* $\bigcap_{n=1}^{\infty} X_n$ *is nonempty and* $\omega(\bigcap_{n=1}^{\infty} X_n) = 0$.
*We say that a measure of weak noncompactness is regular if it satisfies additionally the following conditions :*

(7) $\omega(X_1 \cup X_2) = \max\{\omega(X_1), \omega(X_2)\}$ *(the maximum property).*

(8) $\omega(X_2 + X_1) \leq \omega(X_1) + \omega(X_2)$.

(9) $Ker(\omega) = \mathfrak{N}_E$.

The important example of a measure of weak noncompactness was defined by De Blasi [11]

$$\beta(X) = \inf\Big\{t > 0, \text{ there exists } Y \in \mathcal{K}^{\omega} \text{ such that } X \subset Y + B_t(\theta)\Big\},$$

here $X \in \mathfrak{B}_E$. Notice that the De Blasi measure of weak noncompactness $\beta$ is regular [11].

**Definition 1.2.** *Let E be a Banach space. A mapping* $T : E \to E$ *is called D−set-Lipschitzian, with respect to the measure of weak noncompactness* $\omega$, *if there exists a continuous nondecreasing function* $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\Theta(0) = 0$ *such that*

$$\omega(T(C)) \leq \Theta(\omega(C)),$$

*for all* $C \in \mathfrak{B}_E$. *The function* $\Theta$ *is called D−function of T. If* $\Theta$ *satisfies* $\Theta(r) < r$ *for* $r > 0$, *then T is called a nonlinear D−set-contraction.*

**Definition 1.3.** *[7] Let C be a nonempty subset of Banach space E. We say that* $T : C \to E$ *is condensing with respect to the measure of weak noncompactness* $\omega$ *if* $T(X)$ *is bounded, and*

$$\omega(T(X)) < \omega(X),$$

*for all bounded subset X of C with* $\omega(X) > 0$.

**Definition 1.4.** *[7] Let E be a Banach space. An operator* $T : E \to E$ *is said to be weakly compact if* $T(C)$ *is relatively weakly compact for every bounded subset* $C \subset E$.

**Definition 1.5.** *Let E be a Banach space. An operator* $T : E \to E$ *is said to be weakly sequentially continuous on E, if for every* $(x_n)_n$ *with* $x_n \rightharpoonup x$, *we have* $Tx_n \rightharpoonup Tx$.

We recall the weak version of the Schauder-Tikhonov fixed point principle which was obtained by Arino, Gautier and Penot [4].

**Theorem 1.1.** *Let C be a nonempty, convex and weakly compact subset of a Banach space E and* $T : C \to C$ *a weakly sequentially continuous operator. Then T has at least one fixed point in the set C.*

## 2 Fixed Point Results for Meir-Keeler condensing operators

In 1969, Meir and Keeler [17] introduced the notion of Meir-Keeler contraction and proved an interesting fixed-point theorem which is a generalization of the Banach contraction principle. In this section, we introduce the notion of Meir-Keeler condensing operator via an arbitrary measure of weak noncompactness on a Banach space E and we present some fixed point results.

**Definition 2.1.** *Let $C$ be a nonempty subset of a Banach space $E$ and $\omega$ an arbitrary measure of weak noncompactness on $E$. We say that an operator $T : C \to C$ is Meir-Keeler condensing (with respect to $\omega$) if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq \omega(X) < \epsilon + \delta \Rightarrow \omega(T(X)) < \epsilon, \tag{1}$$

*for all $X \in \mathfrak{B}_C$.*

**Theorem 2.1.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and $\omega$ be an arbitrary measure of weak noncompactness on $E$. If $T : C \to C$ is a weakly sequentially continuous and Meir-keeler condensing mapping, then $T$ has at least one fixed point and the set of all fixed points of $T$ in $C$ is weakly compact.*

*Proof.* Define the sequence $(C_n)$ of subsets of $C$ by

$$C_0 = C \text{ and } C_n = \overline{conv(TC_{n-1})}^{\omega}, n \geq 1.$$

If there exists an integer $N \geq 0$ such that $\omega(C_N) = 0$, then $C_N$ is relatively weakly compact. Thus, it suffices to apply Theorem 2.1.
Assume that $\omega(C_n) \neq 0$ for $n \geq 0$. Define $\epsilon_n = \omega(C_n)$ and let $\delta_n = \delta(\epsilon_n) > 0$ be chosen according to (1). By the definition of $\epsilon_n$, we have

$$\epsilon_{n+1} = \omega(C_{n+1}) = \omega(\overline{conv}(TC_n)) = \omega(TC_n) < \omega(C_n) = \epsilon_n.$$

Since $(\epsilon_n)_{n\geq 0}$ is a positive decreasing sequence of real numbers, there exists $r \geq 0$ such that $\epsilon_n \to r$ as $n \to \infty$. We show that $r = 0$.
Suppose the contrary, then there exists $N_0$ such that

$$n > N_0 \Longrightarrow r \leq \epsilon_n < r + \delta(r),$$

then, by the definition of Meir-Keeler condensing operator, we get $\epsilon_{n+1} < r$. This is absurd, so $r = 0$. Hence $(C_n)$ is a decreasing sequence of nonempty, bounded and weakly closed subsets. Consequently, by condition (6), we deduce that the set $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty, weakly closed convex and $C_\infty \in ker\omega$. It is clear that $T(C_\infty) \subset C_\infty$ and, so, $T : C_\infty \to C_\infty$ is well defined. Thus, applying Theorem 2.1, $T$ has at least one fixed point.
We put

$$F_T = \{x \in C : T(x) = x\} \textit{ and } \epsilon_0 = \omega(F_T).$$

If $\epsilon_0 \neq 0$, then by (1), we have

$$\omega(F_T) = \omega(T(F_T)) < \epsilon_0 = \omega(F_T),$$

which is absurd, then $\omega(F_T) = 0$. On the other hand, it is clear that $F_T$ is weakly closed, so $F_T$ is weakly compact. ∎

If we consider in Theorem 3.1 the measure of weak noncompactness $diam(.)$, we obtain the following result which is a partial answer to Question 2.8 in [21].

**Theorem 2.2.** *Let $C$ be a nonempty, closed and bounded subset of a Banach space $E$. If $T : C \longrightarrow C$ satisfying : For any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq diam(X) < \epsilon + \delta \Rightarrow diam(T(X)) < \epsilon,$$

*then $T$ has a unique fixed point.*

*Proof.* As in the proof of Theorem 3.1, we prove the existence of a non-empty $T-$invariant closed convex subset $C_\infty$ with $diam(C_\infty) = 0$, which means that $C_\infty$ is a singleton and therefore $T$ has a fixed point. For the uniqueness, we assume that there exists two different fixed points $x_0, x_1 \in C$ and we put $X = \{x_0, x_1\}$. We have $diam(X) = diam(T(X)) = \| x_0 - x_1 \|$, which is absurd, so $T$ has a unique fixed point.

Lim [16] and Suzuki [19] introduced the notion of $L-$functions and characterized Meir-Keeler contractions in metric spaces by $L-$functions. In the same way, Aghajani [2] characterized Meir-Keeler condensing operators by L-functions.

**Definition 2.2.** *([16]) A function $\varphi$ from $\mathbb{R}_+$ into itself is called $L-$function (resp. strictly $L-function$) if $\varphi(0) = 0$, $\varphi(s) > 0$, for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ ( resp. $\varphi(t) < s$), for any $t \in [s, s + \delta]$.*

**Example 2.1.** *Consider a right continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\varphi(r) < r$, for all $r > 0$. It is clear that $\varphi$ is a strictly $L-$function. In particular, we consider $\varphi(t) = kt$ for $k \in (0, 1)$.*

Similar to Proposition 1 in [16] and Theorem 2.6 in [2], we can prove the following characterization of Meir-Keeler condensing operators with respect to a measure of weak noncompactness.

**Proposition 2.1.** *Let $C$ be a nonempty and bounded subset of a Banach space $E$, $\omega$ an arbitrary measure of weak noncompactness and $T : C \to C$ a mapping. Then $T$ is a Meir-Keeler condensing operator if and only if there exists an $L-$function $\varphi$ such that*

$$\omega(T(X)) < \varphi(\omega(X)),$$

*for all $X \in \mathfrak{B}_E$ with $\omega(X) \neq 0$.*

As a consequence of Theorem 3.1 and Proposition 3.1, we obtain the following fixed point result.

**Corollary 2.1.** *Let $C$ be a nonempty, bounded, closed and convex subset of a Banach space $E$, $\omega$ an arbitrary measure of weak noncompactness and $T : C \to C$ a mapping. We suppose that $T$ is weakly sequentially continuous such that*

$$\omega(T(X)) < \varphi(\omega(X)),$$

*for $X \subseteq C$, where $\varphi$ is an $L-$function. Then, $T$ has at least one fixed point and the set of all fixed points of $T$ in $C$ is weakly compact.*

**Theorem 2.3.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and $\omega$ an arbitrary measure of weak noncompactness. Let $T : C \to C$ be a weakly sequentially continuous operator such that*

$$\omega(T(X)) \leq \varphi(\omega(X)),$$

*for $X \subseteq C$, where $\varphi$ is a strictly $L-$function. Then, $T$ has at least one fixed point and the set of all fixed points of $T$ in $C$ is weakly compact.*

*Proof.* It is enough to show that $T$ is a Meir-Keeler condensing operator. Let $\epsilon > 0$ and $\delta > 0$ such that

$$\varphi(t) < \epsilon, \ \ if \ \ \epsilon \leq t \leq \epsilon + \delta.$$

Let $X$ be a subset of $E$ such that

$$\epsilon \leq \omega(X) \leq \epsilon + \delta.$$

Thus,

$$\omega(T(X)) \leq \varphi(\omega(X)) < \epsilon.$$

The proof is concluded by Theorem 3.1.

**Corollary 2.2.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$, $\omega$ is an arbitrary measure of weak noncompactness on $E$ and $T : C \to C$ be a mapping. If $T$ is weakly sequentially continuous and a nonlinear $D-$set contraction, then $T$ has at least one fixed point.*

In particular, we obtain the following weak version of Darbo's fixed point theorem.

**Corollary 2.3.** *[5] Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$, $\omega$ is an arbitrary measure of weak noncompactness on $E$ and let $T : C \to C$ be a weakly sequentially continuous operator such that*

$$\omega(T(X)) \leq k(\omega(X)),$$

*for $X \subseteq C$, where $k \in [0, 1[$. Then, $T$ has at least one fixed point.*

In the following, we state a Krasnoselskii's fixed point result.

**Theorem 2.4.** *Let $C$ be a nonempty closed, convex and bounded subset of a Banach space $E$ and $\omega$ be a complete and subadditive measure of noncompactness. Let $F, \ G : C \longrightarrow E$ are two weakly sequentially continuous mappings such that*

*i) $F$ is Meir-Keeler condensing,*

*ii) $G$ is weakly compact,*

*iii) for all $x \in X$, $F(x) + G(x) \in C$ .*

*Then there exists at least $x \in C$ such that $F(x) + G(x) = x$.*

*Proof.* It is clear that the mapping $F + G : C \longrightarrow C$ is well defined and that it is weakly sequentially continuous. Since $\omega$ is complete, subadditive and $G$ is weakly compact, we have

$$\omega((F+G)(X)) \leq \omega(F(X) + (G(X)) \leq \omega(F(X)) + \omega(G(X)) < \phi(X),$$

for all $X \in \mathcal{P}_{bd}(E)$, where $\phi$ is the $L-$function associated to $F$. By Corollary 3.1, there is at least $x \in C$ such that $F(x) + G(x) = x$.

The following Theorem can be considered as a result in the metric fixed point theory.

**Theorem 2.5.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$. Let $F : E \to E$ and $G : C \longrightarrow E$ are weakly sequentially continuous operators such that*

*i) $\|Fx - Fy\| \leq \Theta(\|x - y\|)$, where $\Theta$ is a (nondecreasing and right continuous) strictly $L-$ function,*

*ii) $G$ is a weakly compact,*

*iii) $T(x) = F(x) + G(x) \in C$, for all $x \in C$.*

*Then, $T$ has at least one fixed point and the set of all fixed points of $T$ in $C$ is weakly compact.*

*Proof.* Let $\omega$ be the De Blasi measure of weak noncompactness and $X$ a nonempty subset of $C$ such that $\omega(X) = d > 0$. Let $\epsilon > 0$, then there exists a weakly compact $K$ of $E$ such that $X \subseteq K + B_{d+\epsilon}(\theta)$. For any $x \in X$ there exists $y \in K$ and $z \in B_{d+\epsilon}(\theta)$ such that $x = y + z$. On the other hand, we have

$$\|Fx - Fy\| \leq \Theta(\|x - y\|) \leq \Theta(d + \epsilon).$$

It follows that

$$F(X) \subseteq F(K) + B_{\Theta(d+\epsilon)}(\theta) \subseteq \overline{F(K)}^{\omega} + B_{\Theta(d+\epsilon)}(\theta).$$

Since $F$ is weakly sequentially continuous and $K$ is weakly compact, $\overline{F(K)}^{\omega}$ is weakly compact and we get

$$\omega(F(X)) \leq \Theta(d + \epsilon).$$

Since $\theta$ is right continuous and letting $\epsilon$ tends to 0, we obtain

$$\omega(F(X)) \leq \Theta(d) = \Theta(\omega(X)).$$

Now, since $G$ is weakly compact, we get

$$\omega(T(X)) \leq \omega(F(X)) + \omega(G(X)) \leq \Theta(\omega(X)).$$

Consequently, applying corollary 3.2, we deduce the result.

**Definition 2.3.** *([21]) A mapping $T$ on a complete metric space $(E, d)$ is said to be diametrically contractive if*

$$\delta(T(A)) < \delta(A) \text{ for all closed subsets } A \text{ such that } 0 < \delta(A) < \infty.$$

*(Here $\delta(A) := sup\{d(x, y) : x, y \in A\}$ is the diameter of $A$).*

**Theorem 2.6.** *([21]) Let $C$ be a weakly compact subset of a Banach space $E$ and let $T : C \to C$ be a diametrically contractive mapping. Then $T$ has a fixed point.*

**Theorem 2.7.** *Let $C$ be a nonempty, bounded and closed (not necessary convex) subset of a Banach space $E$ and $\omega$ be an arbitrary measure of weak noncompactness on $E$. If $T : C \to C$ is a diametrically contractive operator such that*

$$\omega(T(X)) \leq \varphi(\omega(X)),$$

*for all bounded set $X \subseteq C$, where $\varphi$ is nondecreasing strictly $L-$function. Then $T$ has a fixed point.*

*Proof.* Consider the sequence $(C_n)$ defined by

$$C_0 = C \text{ and } C_n = \overline{TC_{n-1}}^{\omega}, n \geq 1.$$

If there exists an integer $N \in \mathbb{N}$ such that $\omega(C_N) = 0$, then $C_N$ is relatively weakly compact. So applying Theorem 3.6 we infer that T has a fixed point. Assume that $\omega(C_n) \neq 0$ for $n \geq 0$. By assumption we have

$$\omega(C_{n+1}) = \omega(T(C_n)) \leq \varphi(\omega(C_n)) \leq \cdots \leq \varphi^n(\omega(C)).$$

Since $\varphi$ is strictly $L-$function and nondecreasing, then $\lim_{n\to\infty} \varphi^n(t) = 0$ :

$$\omega(C_n) \to 0 \text{ as } n \to +\infty.$$

So, $(C_n)$ is a decreasing sequence of nonempty, bounded and weakly closed subsets. Consequently, we deduce that the set $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty and weakly compact. Define the maps $T : C_\infty \to C_\infty$. Thus, applying Theorem 3.6, T has a fixed point.

Next, we introduce the notion of $\mathcal{F}-$contraction operator.

**Definition 2.4.** *([13]) Let $\mathcal{F}$ be the family of all functions $\Theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following three conditions :*

($\Theta_1$) *$\Theta$ is strictly increasing;*

($\Theta_2$) *for each sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \Theta(t_n) = -\infty$;*

($\Theta_3$) *$\Theta$ is continuous.*

*A mapping $T : E \to E$ is said to be an $\mathcal{F}-$contraction if there exist $\tau > 0$ and $\Theta \in \mathcal{F}$ such that, for all $x, y \in E$,*

$$d(T(x), T(y)) > 0 \Rightarrow \tau + \Theta(d(T(x), T(y))) \leq \Theta(d(x, y)).$$

For example we consider the following functions. $\Theta(t) = ln(t)$, $\Theta(t) = ln(t) + t$, $\Theta(t) = \frac{-1}{\sqrt{t}}$ and $\Theta(t) = \ln(t^2 + t)$.

**Theorem 2.8.** *Let $(E, \|\cdot\|)$ be a Banach space and let $C$ be a nonempty closed, convex and bounded subset of $E$. Let $F : E \longrightarrow E$ and $G : C \longrightarrow E$ be two weakly sequentially continuous mappings such that*

*i) $F$ is an $\mathcal{F}-$contraction,*

*ii) $G$ is weakly compact,*

*iii) for all $x \in C$, $F(x) + G(x) \in C$.*

*Then there exists at least $x \in C$ such that $F(x) + G(x) = x$.*

*Proof.* Since $F$ is an $\mathcal{F}-$contraction, there exits $\Theta \in \mathcal{F}$ and $\tau > 0$ such that

$$\|Fx - Fy\| > 0 \Rightarrow \Theta(\|Fx - Fy\|) \leq \Theta(\|x - y\|) - \tau.$$

Since $\Theta$ is continuous and satisfies $(\Theta_2)$, we have $\Theta(\mathbb{R}_+) = ]-\infty, \alpha[$, where $\alpha \in \mathbb{R} \cup \{\infty\}$. On the other hand, $\Theta$ satisfies $(\Theta_1)$ then $\Theta$ is injective and we deduce that $\Theta : \mathbb{R}_+ \to ]-\infty, \alpha[$ is invertible.
Next, we define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(t) = \Theta^{-1}(\Theta(t) - \tau), \text{ for } t > 0 \text{ and } f(0) = 0.$$

By condition $(\Theta_2)$ we have $\lim_{n\to\infty} f(t_n) = 0$ for each sequence $(t_n)$ converges to 0. Hence $f$ is well defined and continuous. So

$$\|Fx - Fy\| \leq \Theta^{-1}(\Theta(\|x - y\|) - \tau) = f(\|x - y\|).$$

Now, since $\Theta$ is strictly increasing, it is clear that $f(t) < t$ for all $t > 0$. By Theorem 3.5, the proof is concluded.

## 3 Coupled Fixed Point Results for Bivariate Meir-Keeler condensing Operators

In this section we introduce the notion of bivariate Meir-Keeler condensing operator via an arbitrary measure of weak noncompactness and we prove some coupled fixed point results.

**Definition 3.1.** *([9]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the operator $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.*

We present the weak version of Theorem 3.2 in [6].

**Theorem 3.1.** *Suppose that $\omega_1$, $\omega_2$, $\cdots$, $\omega_n$ are measures of weak noncompactness on Banach spaces $E_1$, $E_2$, $\cdots$, $E_n$, respectively. Moreover, assume that the function $F : \mathbb{R}^n_+ \to \mathbb{R}_+$ is convex and $F(x_1, \cdots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \cdots, n$. Then,*

$$\omega(X) = F(\omega_1(X_1), \omega_2(X_2), \cdots, \omega_n(X_n))$$

*defines a measure of weak noncompactness on $E_1 \times E_2 \times \cdots \times E_n$, where $X_i$ denotes the natural projections of $X$ into $E_i$ for $i = 1, 2, \cdots, n$.*

**Example 3.1.** *Let $\omega$ be a measure of weak noncompactness on a Banach space $E$, considering $F_1(x, y) = \max\{x, y\}$ and $F_2(x, y) = x + y$ for $(x, y) \in \mathbb{R}^2_+$, then conditions of Theorem 4.1 are satisfied. Therefore, $\widetilde{\omega_1}(X) = \max\{\omega(X_1), \omega(X_2)\}$ and $\widetilde{\omega_2}(X) = \omega(X_1) + \omega(X_2)$ define measures of weak noncompactness on the space $E \times E$, where $X_i$, $i = 1, 2$, denote the natural projections of $X$.*

We define the notion of a bivariate Meir-Keeler condensing operator via measure of weak noncompactness.

**Definition 3.2.** *Let $C$ be a nonempty subset of a Banach space $E$, and $\omega$ an arbitrary measure of weak noncompactness on $E$. We say that an operator $T : C \times C \to C$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq max\{\omega(X_1), \omega(X_2)\} < \epsilon + \delta \Rightarrow \omega(T(X_1 \times X_2)) < \epsilon, \tag{2}$$

*for any bounded subsets $X_1$ and $X_2$ of $C$.*

**Theorem 3.2.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and $\omega$ an arbitrary measure of weak noncompactness on $E$. If $T : C \times C \to C$ is a weakly sequentially continuous and Meir-keeler condensing, then $T$ has at least one coupled fixed point.*

*Proof.* We pose

$$G : C \times C \to C \times C, \;\; G(x,y) = (T(x,y), T(y,x)) \,.$$

We recall that $(x_n, y_n) \rightharpoonup (x, y)$ in the product $C \times C$ if and only if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. Then, it is clear that the operator $G$ is weakly sequentially continuous. According to example 4.1, let

$$\widetilde{\omega}(X) = max\{\omega(X_1), \omega(X_2)\},$$

for any bounded subset $X \subseteq E \times E$. Let $\epsilon > 0$ and $\delta(\epsilon) > 0$. If $X$ is a bounded subset of $C \times C$ such that

$$\widetilde{\omega}(X) < \epsilon + \delta(\epsilon),$$

then

$$max\{\omega(X_1), \omega(X_2)\} < \epsilon + \delta(\epsilon).$$

Further, we have

$$\begin{aligned} \widetilde{\omega}(G(X)) &\leq \widetilde{\omega}(T(X_1 \times X_2) \times T(X_2 \times X_1)) \\ &= \max\{\omega(T(X_1 \times X_2), \omega(T(X_2 \times X_1))\} \\ &< \epsilon. \end{aligned}$$

Thus, from Theorem 3.1, $G$ has at least one fixed point in $C \times C$. Then, $T$ has at least one coupled fixed point.

Next, we prove a coupled fixed point theorem using strictly $L-$functions.

**Theorem 3.3.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$, $\omega$ an arbitrary measure of weak noncompactness on $E$ and $\varphi$ is strictly $L-$function. Suppose that $T : C \times C \to C$ is a weakly sequentially continuous operator satisfying*

$$\omega(T(X_1 \times X_2)) \leq \frac{1}{2}\varphi(\omega(X_1) + \omega(X_2)),$$

*for any subset $X_1, X_2$ of $C$. Then $T$ has at least one coupled fixed point.*

*Proof.* Similar to the proof of the previous theorem, we define a mapping $G : C \times C \to C \times C$ as

$$G(x,y) = (T(x,y), T(y,x))$$

which is a weakly sequentially continuous map. Moreover,

$$\widetilde{\omega}(X) = \omega(X_1) + \omega(X_2)$$

defines a measure of weak noncompactness on $E \times E$, where $X_i$, $i = 1, 2$, denote the natural projections of $X$. Now, let $X \subset C \times C$ be any nonempty subset. Then, we obtain

$$\begin{aligned}
\widetilde{\omega}(G(X)) &\leq \widetilde{\omega}(T(X_1 \times X_2) \times T(X_2 \times X_1)) \\
&= \omega(T(X_1 \times X_2) + \omega(T(X_2 \times X_1)) \\
&\leq \frac{1}{2}\varphi(\omega(X_1) + \omega_1(X_2)) + \frac{1}{2}\varphi(\omega(X_2) + \omega_2(X_1)) \\
&= \varphi(\widetilde{\omega}(X)).
\end{aligned}$$

Therefore, $G$ has a coupled fixed point.

**Theorem 3.4.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $F, G : E \times E \to E$ be weakly sequentially continuous operators. We assume that*

*i) $\|F(x,y) - F(u,v)\| \leq \frac{1}{2}\Theta(\|x-u\| + \|y-v\|)$, where $\Theta$ is a (nondecreasing and right*

*continuous) strictly $L-$ function,*

*ii) $G$ is weakly compact,*

*iii) $F(x,y) + G(x,y) \in C$, for all $(x,y) \in C \times C$.*

*Then, $T$ has at least one coupled fixed point.*

*Proof.* We pose $\widetilde{F}, \widetilde{G} : C \times C \to C \times C$ defined by

$$\widetilde{F}(x,y) = (F(x,y), F(y,x)) \text{ and } \widetilde{G}(x,y) = (G(x,y), G(y,x)).$$

We equipped $E \times E$ by the norm $\| (x,y) \| = \| x \| + \| y \|$.
Let $X_1$ and $X_2$ are nonempty subsets of $C$, for all $(x,y), (u,v) \in X_1 \times X_2$ we have

$$\begin{aligned}
\|\widetilde{F}(x,y) - \widetilde{F}(u,v)\| &= \|(F(x,y), F(y,x)) - (F(u,v), F(v,u))\| \\
&= \|F(x,y) - F(u,v)\| + \|F(y,x) - F(v,u)\| \\
&\leq \frac{1}{2}\Theta(\|x-u\| + \|y-v\|) + \frac{1}{2}\Theta(\|x-u\| + \|y-v\|)\} \\
&= \Theta(\|x-u\| + \|y-v\|).
\end{aligned}$$

On the other hand, it is clear that $\widetilde{G}$ is weakly compact. Then, by Theorem 3.5, $\widetilde{F} + \widetilde{G}$ has a fixed point which is a coupled fixed point of $F + G$.

## 4 Application

Let $E$ be a Banach space with the norm $\| \, . \, \|$, $E^*$ its dual and $I = [0, T]$, $T > 0$. In this section, we apply our results to prove the existence of solutions for the following system of functional integral equations of Volterra type :

$$\begin{cases} x(t) = f(t, x(t), y(t)) + \int_0^t g(t, x(s), y(s))ds \\ \\ y(t) = f(t, y(t), x(t)) + \int_0^t g(t, y(s), x(s))ds \end{cases} \tag{3}$$

for $t \in I$, where $f : I \times E \times E \to E$ and $g : I \times E \times E \to E$.
We denote by $C(I, E)$ the Banach space of all continuous functions from $I$ to $E$ endowed with the sup-norm $\| \, . \, \|_\infty$ defined by $\| x \|_\infty = \sup\{\| x(t) \|, t \in I\}$, for each $x \in C(I, E)$.
Let $(E \times E, \|\|_{C(I,E)^2})$ be the product space endowed with the norm

$$\| (x, y) \|_{C(I,E)^2} = \| x \|_\infty + \| y \|_\infty .$$

The integral in system (3) is the Pettis integral and the solutions of (3) are considered in the Banach space $X = C(I, E)$. We consider this system under the following assumptions :

$(H_1)$ The function $f$ is weakly sequentially continuous and there exists a (nondecreasing and right continuous) strictly $L-$function $\Theta$ such that

$$\| f(t, x, y) - f(t, u, v) \| \leq \frac{1}{2}\Theta(\| x - u \| + \| y - v \|), \tag{4}$$

for all $(x, y)$, $(u, v) \in E \times E$ and all $t \in I$.

$(H_2)$ For each $t \in I$, $g_t = g(t, \cdot, \cdot)$ is weakly completely continuous (i.e. weakly sequentially continuous and weakly compact).

$(H_3)$ For each continuous $x, y : I \to E$, $g(., x(.), y(.))$ is Pettis integrable on $I$.

$(H_4)$ For any $r > 0$, there exists $h_r \in L^1(I)$ with $\| g(t, x, y) \| \leq h_r(t)$ for all $t \in I$ and all $(x, y) \in E \times E$ with $\| (x, y) \| \leq r$. We let

$$M_r = \int_0^T h_r(s)ds.$$

$(H_5)$ There exists $r > 0$ such that

$$\frac{1}{2}\Theta(r) + \| f(0, 0) \| + M_r \leq r.$$

We need the following results in the sequel.

**Theorem 4.1.** *(I. Dobrokov [11]). Let $I$ be a compact Hausdorff space and $E$ be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(I,E)$ and $x \in C(I,E)$. Then, $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $x$ if and only if $(x_n(t))_{n\in\mathbb{N}}$ is weakly convergent to $x(t)$ for each $t \in I$.*

**Theorem 4.2.** *[14]. Let $f : I \to E$ be a mapping satisfying the following conditions :*

*(1) there exists a sequence of Pettis integrable functions $(f_n)$ weakly convergent to $f$,*

*(2) there exists $h \in L^1(I)$ such that $\| f_n \| \leq h$, for each $n \in \mathbb{N}$.*

*Then $f$ is Pettis integrable and $\int_0^T f_n(s)ds$ converges weakly to $\int_0^T f(s)ds$.*

**Theorem 4.3.** *[20]. A subset $\mathcal{H}$ in $C(I,E)$ is relatively weakly compact if and only if :*

*(i) $\mathcal{H}$ is weakly equicontinuous on $I$;*

*(ii) for each $t \in I$, the subset $\mathcal{H}(t) = \{f(t);\ f \in \mathcal{H}\}$ is relatively weakly compact in $E$.*

**Theorem 4.4.** *Assume that hypotheses $(H_1) - (H_5)$ hold. Then the integral system (3) has a solution in $C(I,E) \times C(I,E)$.*

*Proof.* For all $x, y \in C(I,E)$, we put

$$F(x,y)(t) = f(t, x(t), y(t)),$$

$$G(x,y)(t) = \int_0^t g(t, x(s), y(s))ds.$$

Let $S = \{x \in X;\ \| x \| \leq r\}$. Notice that $S$ is closed, convex and bounded subset of $X$. We will show that operators $F, G$ satisfy all assumptions of Theorem 4.4.

**Step I** : We show that $F : X \times X \to X$ is well defined. Let $x, y \in X$ and $t, t' \in I$. We have

$$\begin{aligned} \|F(x,y)(t) - F(x,y)(t')\| &= \|f(t,x(t),y(t)) - f(t',x(t'),y(t'))\| \\ &\leq \frac{1}{2}\Theta(\|x(t) - x(t')\| + \|y(t) - y(t')\|), \end{aligned}$$

since $\Theta$ is right continuous and $\Theta(0) = 0$, we deduce that $F(x,y) \in C(I,E)$.

**Step II** : We show that $G : S \times S \to X$ is well defined and weakly sequentially continuous.. Let $t, t' \in I$ with $t > t'$. Without loss of generality, assume that

$G(x,y)(t) - G(x,y)(t^{'}) \neq 0$. By the Hahn-Banach theorem, there exists $\phi \in E^*$, such that $\| \phi \| = 1$ and

$$\begin{aligned}
\|G(x,y)(t) - G(x,y)(t^{'})\| &= | \phi(G(x,y)(t) - G(x,y)(t^{'})) | \\
&= | \phi(\int_0^t g(s,x(s),y(s))ds - \int_0^{t^{'}} g(s,x(s),y(s))ds) | \\
&= | \phi(\int_{t'}^t g(s,x(s),y(s))ds) | \\
&= | \int_{t'}^t \phi(g(s,x(s),y(s)))ds | \\
&\leq \int_{t'}^t | \phi(g(s,x(s),y(s))) | ds \\
&\leq \int_{t'}^t \| g(s,x(s),y(s)) \| ds \\
&\leq \int_{t'}^t h_r(s)ds.
\end{aligned}$$

Since $h_r \in L^1(I)$, then $G(x,y)$ is continuous.

Next we show that $G$ is weakly sequentially continuous. Let $(x_n, y_n)$ be any sequence in $S \times S$ weakly convergent to $(x,y) \in S \times S$, then $(x_n, y_n)$ is bounded and by Dobrokov's Theorem, we get

$$\forall t \in I, (x_n, y_n)(t) \rightharpoonup (x,y)(t).$$

Fix $t \in I$, since $(x_n, y_n)(s) \rightharpoonup (x,y)(s)$ for each $s \in [0,t]$, then by assumption $(H_2)$, the set $\{g(s, x_n(s), y_n(s)),\ n \in \mathbb{N}\}$ is relatively weakly compact for each $s \in [0,t]$. Since the weak topology on $E^{[0,t]}$ coincides with the product of weak topologies, then from Tychonoff's theorem, the set

$$\{g(.,x_n,y_n),\ n \in \mathbb{N}\} = \prod_{s\in[0,t]} \{g(s,x_n(s),y_n(s))),\ n \in \mathbb{N}\}$$

is relatively weakly compact in $E^{[0,t]}$. Hence, there exists a subsequence, for simplicity we note also the sequence $(g(.,x_n,y_n))$ such that $g(.,x_n,y_n) \rightharpoonup g(.,x,y)$. By assumption $(H_4)$ and Theorem 5.2, we get

$$\int_0^t g(s,x_n(s),y_n(s))ds \rightharpoonup \int_0^t g(s,x(s),y(s))ds.$$

Since $(G(x_n,y_n))$ is bounded, then we can again apply the Dobrokov's Theorem to get

$$G(x_n,y_n) \rightharpoonup G(x,y).$$

So, $G$ is weakly sequentially continuous.

**Step III** : We will show that $G(S\times S)$ is relatively weakly compact. To do this, it is enough to prove that $G(S\times S)(t)$ is relatively weakly compact and $G(S\times S)$ is weakly equicontinuous. To see this, let $(z_n)$ be a sequence in $G(S\times S)$ and let $(x_n, y_n)$ be a sequence in $S\times S$ such that $z_n(t) = G(x_n, y_n)(t)$, where

$$z_n(t) = \int_0^t g(s, x_n(s), y_n(s))ds, \text{ for all } t \in I.$$

Fix $t \in I$. As in step $II$, $\{g(., x_n, y_n),\ n \in \mathbb{N}\}$ is relatively weakly compact in $E^{[0,t]}$. Hence, there exists a subsequence $(g(., x_{n_k}, y_{n_k}))$ such that $g(., x_{n_k}, y_{n_k}) \rightharpoonup g(., x, y)$. By assumption $(H_4)$ and Theorem 5.2, we deduce that

$$z_{n_k}(t) = \int_0^t g(s, x_{n_k}(s), y_{n_k}(s))ds \rightharpoonup z(t) = \int_0^t g(s, x(s), y(s))ds.$$

Hence $\{z_n(t),\ n \in \mathbb{N}\}$ is relatively weakly compact which implies that $G(S\times S)(t)$ is relatively weakly compact.

Next we show that $G(S\times S)$ is weakly equicontinuous. Let $\epsilon > 0$; $(x, y) \in S\times S$; $\phi \in E^*$ with $\| \phi \| = 1$; $t, t' \in I$ such that $t > t'$ and $t - t' \leq \epsilon$, we have

$$\begin{aligned}
\mid \phi(G(x,y)(t) - G(x,y)(t')) \mid &= \mid \phi\Big(\int_0^t g(s,x(s),y(s))ds - \int_0^{t'} g(s,x(s),y(s))ds\Big) \mid \\
&= \mid \phi\Big(\int_{t'}^t g(s,x(s),y(s))ds\Big) \mid \\
&= \mid \int_{t'}^t \phi(g(s,x(s),y(s)))ds \mid \\
&\leq \int_{t'}^t \mid \phi(g(s,x(s),y(s))) \mid ds \\
&\leq \int_{t'}^t \| (g(s,x(s),y(s))) \| ds \\
&\leq \int_{t'}^t h_r(s)ds.
\end{aligned}$$

Since $h_r \in L^1(0, T]$, it follows that $G(S\times S)$ is weakly equicontinuous. By Theorem 5.3, we deduce that $G(S\times S)$ is relatively weakly compact.

**Step IV** : For arbitrary fixed $t \in I$, we have

$$\begin{aligned}
\|(T(x,y))(t)\| &= \|F(x,y)(t) + G(x,y)(t)\| \\
&= \|f(t,x(t),y(t)) - f(t,0,0) + f(t,0,0) + \int_0^t g(t,s,x(s))ds\| \\
&\leq \|f(t,x(t),y(t)) - f(t,0,0\| + \|f(t,0,0)\| + \int_0^T \|g(t,s,x(s))\|ds \\
&\leq \frac{1}{2}\Theta(\|x(t)\| + \|y(t)\|) + \|F(0,0)(t)\| + M_r. \qquad (5)
\end{aligned}$$

By assumption $(H_1)$, the function $\Theta$ is nondecreasing, then we obtain

$$\begin{aligned} \|(T(x,y))\| &\leq \frac{1}{2}\Theta(\|x\| + \|y\|) + \|F(0,0)\| + M_r \\ &\leq \frac{1}{2}\Theta(\|(x,y)\| + \|F(0,0)\| + M_r. \end{aligned} \tag{6}$$

Thus, by assumption $(H_5)$, we infer that $T$ is a mapping from $S \times S$ into $S$. On the other hand,

$$\|F(x,y) - F(u,v)\| \leq \frac{1}{2}\Theta(\|x-u\| + \|y-v\|).$$

Then by Theorem 4.4, $T$ has at least one coupled fixed point.

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