Lipsman mapping and dual topology of semidirect products

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To the memory of Majdi Ben Halima

Abstract

We consider the semidirect product $G = K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional real vector space V equipped with an inner product $\langle , \rangle$. We denote by $\widehat{G}$ the unitary dual of G (note that we identify each representation $\pi \in \widehat{G}$ to its classes $[\pi]$) and by $\mathfrak{g}^\ddagger/G$ the space of admissible coadjoint orbits, where $\mathfrak{g}$ is the Lie algebra of G. It was pointed out by Lipsman that the correspondence between $\mathfrak{g}^\ddagger/G$ and $\widehat{G}$ is bijective. Under some assumption on G, we prove that the Lipsman mapping

$$\begin{array}{rcl} \Theta : \mathfrak{g}^\ddagger/G & \longrightarrow & \widehat{G} \\ \mathcal{O} & \longmapsto & \pi_{\mathcal{O}} \end{array}$$

is a homeomorphism.

1 Introduction

Let G be a second countable locally compact group and $\widehat{G}$ the unitary dual of G, i.e., the set of all equivalence classes of irreducible unitary representations of G. It is well known that $\widehat{G}$ comes equipped with the Fell topology [8, p. 426]. The description of the dual topology is a good candidate for some aspects of harmonic analysis on G (for example, see [4, 7, 20]). In such a situation, the natural and important question arises of whether the bijection between the space

Received by the editors in November 2017 - In revised form in October 2018.
Communicated by A. Valette.
2010 *Mathematics Subject Classification* : 22D10-22E27-22E45.
Key words and phrases : Lie groups, semidirect product, unitary representations, coadjoint orbits, symplectic induction.

of coadjoint orbits $\mathfrak{g}^*/G$ of G ($\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g} := Lie(G)$) and $\widehat{G}$ is a homeomorphism. For a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = exp(\mathfrak{g})$, its dual space $\widehat{G}$ is homeomorphic to the space of coadjoint orbits through the Kirillov mapping (see [16]). In the context of semidirect products $G = K \ltimes N$ of compact connected Lie group K acting on simply connected nilpotent Lie group N, then it was pointed out by Lipsman in [17], that we have again an orbit picture of the dual space of G. The unitary dual space of Euclidean motion groups is homeomorphic to the admissible coadjoint orbits [7]. This result was generalized in [4], for a class of Cartan motion groups.
According to [5, Definition 0.1], we introduce the following Definition.

Definition 1.1. *Let G be a (real) Lie group, $\mathfrak{g}$ its Lie algebra and* $\exp : \mathfrak{g} \longrightarrow G$ *its exponential map. We say that G is exponential if* $\exp(\mathfrak{g}) = G$.

In this paper, we consider the semidirect product $G = K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional real vector space V equipped with an inner product $\langle , \rangle$. In the spirit of the orbit method due to Kirillov, R. Lipsman established a bijection between a class of coadjoint orbits of G and the unitary dual $\widehat{G}$. For every admissible linear form ψ of the Lie algebra $\mathfrak{g}$ of G, we can construct an irreducible unitary representation π_ψ by holomorphic induction and according to Lipsman (see [6, p. 23]) (compare [17]), every irreducible representation of G arises in this manner. Then we get a map from the set $\mathfrak{g}^\ddagger$ of the admissible linear forms onto the dual space $\widehat{G}$ of G. Note that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' are in the same G-orbit, finally we obtain a bijection between the space $\mathfrak{g}^\ddagger/G$ of admissible coadjoint orbits and the unitary dual $\widehat{G}$.
The preceding discussion motivates our main result:

Theorem 1.2. *We assume that G is exponential. Then the Lipsman mapping*

$$\begin{array}{rcl} \Theta : \mathfrak{g}^\ddagger/G & \longrightarrow & \widehat{G} \\ \mathcal{O} & \longmapsto & \pi_{\mathcal{O}} \end{array}$$

is a homeomorphism.

The present work is organized as follows: Section 2 is devoted to the description of the unitary dual $\widehat{G}$ of G. Section 3 deals with the space of admissible coadjoint orbits $\mathfrak{g}^\ddagger/G$ of G. Theorem 1.2 is proved below in Section 4.

2 Dual spaces of semidirect product

Throughout this paper, K will denote a connected compact Lie group acting by automorphisms on a finite dimensional real vector space $(V, \langle , \rangle)$. We write $k.v$ and $A.v$ (resp. $k.\ell$ and $A.\ell$) for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := Lie(K)$ to $v \in V$ (resp. to $\ell \in V^*$).

Now, one can form the semidirect product $G := K \ltimes V$ which is a so-called generalized motion group. As a set $G = K \times V$ and the multiplication in this group is given by

$$(k, v)(h, u) = (kh, v + k.u), \ \forall (k, v), (h, u) \in G.$$

The Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus V$ (as a vector space) and the Lie algebra structure is given by the bracket

$$[(A, a), (B, b)] = ([A, B], A.b - B.a), \ \forall (A, a), (B, b) \in \mathfrak{g}.$$

Under the identification of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{k}^* \oplus V^*$, we can express the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ as $F(A, a) = f(A) + \ell(a)$, for all $F = (f, \ell) \in \mathfrak{g}^*$ and $(A, a) \in \mathfrak{g}$. The adjoint representation Ad_G and coadjoint representation Ad_G^* of G are given respectively, by the following relations

$$\begin{aligned} Ad_G(k, v)(A, a) &= (Ad_K(k)A, k.a - Ad_K(k)A.v), \forall (k, v) \in G, (A, a) \in \mathfrak{g}, \\ Ad_G^*(k, v)(f, \ell) &= (Ad_K^*(k)f + k.\ell \odot v, k.\ell), \forall (k, v) \in G, (f, \ell) \in \mathfrak{g}^*, \end{aligned}$$

where $\ell \odot v$ is the element of $\mathfrak{k}^*$ defined by

$$\ell \odot v(A) = \ell(A.v) = -(A.\ell)(v), \forall A \in \mathfrak{k}, \ell \in V^*, v \in V.$$

Note that the map $\odot : V^* \times V \longrightarrow \mathfrak{k}^*$ defined by $(\ell \odot v)(A) = \ell(A.v)$, $v \in V$, $A \in \mathfrak{k}$ satisfies a fundamental equivariance property:

$$Ad_K^*(k)(\ell \odot v) = (k.\ell) \odot (k.v), \ k \in K.$$

Therefore, the coadjoint orbit of G passing through $(f, \ell) \in \mathfrak{g}^*$ is given by

$$\mathcal{O}^G_{(f,\ell)} = \Big\{ \Big(Ad_K^*(k)f + k.\ell \odot v, k.\ell \Big), k \in K, v \in V \Big\}. \tag{2.1}$$

For $\ell \in V^*$, we define $K_\ell := \{k \in K; k.\ell = \ell\}$ the isotropy subgroup of ℓ in K and the Lie algebra of K_ℓ is given by the vector space $\mathfrak{k}_\ell = \{A \in \mathfrak{k}; A.\ell = 0\}$. Let $\imath_\ell : \mathfrak{k}_\ell \hookrightarrow \mathfrak{k}$ be the injection map, then $\imath_\ell^* : \mathfrak{k}^* \longrightarrow \mathfrak{k}_\ell^*$ is the projection map and we have

$$\mathfrak{k}_\ell^\circ = Ker(\imath_\ell^*) \tag{2.2}$$

where $\mathfrak{k}_\ell^\circ$ is the annihilator of $\mathfrak{k}_\ell$. If we define the linear map $h_\ell : \mathfrak{k} \longrightarrow V^*$ by

$$h_\ell(A) := -A.\ell, \ \forall A \in \mathfrak{k},$$

then we have $\mathfrak{k}_\ell = Ker(h_\ell)$. The dual $h_\ell^* : V \longrightarrow \mathfrak{k}^*$ of h_ℓ is given by the relation $h_\ell^*(v)(A) = h_\ell(A)(v) = -(A.\ell)(v)$, and so $h_\ell^*(v) = \ell \odot v$, $\forall \ell \in V^*$, $\forall v \in V$. (for more details see [3, p. 2-6]).
The following is a useful lemma from [3, p. 2-6], giving a characterization of the annihilator $\mathfrak{k}_\ell^\circ$ in terms of the linear map h_ℓ.

Lemma 2.1. *Using the previous notations, then we have the equality*

$$\mathfrak{k}_\ell^\circ = Im(h_\ell^*).$$

Here we recall briefly the description of the unitary dual of G via Mackey's little group theory (see [18]). For every non-zero linear form ℓ on V, we denote by χ_ℓ the unitary character of the vector Lie group V given by $\chi_\ell = e^{i\ell}$. Let ρ be an irreducible unitary representation of K_ℓ on some Hilbert space $\mathcal{H}_\rho$. The map

$$\rho \otimes \chi_\ell : (k, v) \longmapsto e^{i\ell(v)}\rho(k)$$

is a representation of the Lie group $K_\ell \ltimes V$ such that one induce up so as to get a unitary representation of G. We denote by $\mathcal{H}_{(\rho,\ell)} := L^2(K, \mathcal{H}_\rho)^\rho$ the subspace of $L^2(K, \mathcal{H}_\rho)$ consisting of all the maps ξ which satisfy the covariance condition

$$\xi(kh) = \rho(h^{-1})\xi(k), \forall k \in K, h \in K_\ell.$$

The induced representation

$$\pi_{(\rho,\ell)} := Ind_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell)$$

is defined on $\mathcal{H}_{(\rho,\ell)}$ by

$$\pi_{(\rho,\ell)}(k, v)\xi(h) = e^{i\ell(h^{-1}.v)}\xi(k^{-1}h)$$

where $(k, v) \in G, h \in K$ and $\xi \in \mathcal{H}_{(\rho,\ell)}$. By Mackey's theory we can say that the induced representation $\pi_{(\rho,\ell)}$ is irreducible and every infinite dimensional irreducible unitary representation of G is equivalent to one of $\pi_{(\rho,\ell)}$. Moreover, the representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho',\ell')}$ are equivalent if and only if ℓ and $\ell^{'}$ are contained in the same K-orbit and the representation ρ and $\rho^{'}$ are equivalent under the identification of the conjugate subgroups K_ℓ and $K_{\ell'}$. All irreducible representations of G which are not trivial on the normal subgroup V, are obtained by this manner. On the other hand, we denote also by τ the extension of every unitary irreducible representation τ of K on G, which is simply defined by $\tau(k, v) := \tau(k)$ for $k \in K$ and $v \in V$. Let Ω be a K-orbit in V^*. We fix $\ell \in \Omega$ and we define the subset $\widehat{G}(\Omega)$ of $\widehat{G}$ by

$$\widehat{G}(\Omega) = \left\{ Ind_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell); \rho \in \widehat{K_\ell} \right\}.$$

Then we conclude that

$$\widehat{G} = \widehat{K} \bigcup \Big(\bigcup_{\Omega \in \Lambda} \widehat{G}(\Omega) \Big)$$

where Λ is the set of the non-trivial orbits in V^*/K.

In the remainder of this paper, we shall assume that G is exponential, i.e., K_ℓ is connected for all $\ell \in V^*$ (see [5, Proposition 5.1]). Let ρ_μ be an irreducible representation of K_ℓ with highest weight μ. For simplicity, we shall write $\pi_{(\mu,\ell)}$ instead of $\pi_{(\rho_\mu,\ell)}$ and $\mathcal{H}_{(\mu,\ell)}$ instead of $\mathcal{H}_{(\rho_\mu,\ell)}$.

We close this section by presenting two results which are being used in the description of the dual topology of G. These are required for our proof of Theorem 1.2.
We denote respectively by $\mathcal{C}(K)$ and Y the space of all closed subgroups of K equipped with the compact-open topology and the set of all pairs (L, k), where $L \in \mathcal{C}(K)$ and $k \in L$. It is easily seen that Y is a closed subset of $\mathcal{C}(K) \times K$ and the subspace of continuous functions with compact support $C_0(Y)$ is a normed*-algebra with the supremum norm ($||f^*|| = ||f|| := \sup_{L \in \mathcal{C}(K)} \|\Phi_L(f)\|$), where Φ_L is defined below. The completion $A_s(K)$ of $C_0(Y)$ with respect to this norm is a Banach *-algebra called the subgroup algebra of K.
For each $L \in \mathcal{C}(K)$, the mapping $f \longmapsto f_L$ defined on $C_0(Y)$ by

$$f_L(k) = f(L, k)$$

extends to a continuous *-homomorphism, which we shall call $\Phi_L : A_s(K) \longrightarrow L^1(L)$. The map Φ_L has a dense image.
Every unitary representation T of L can be lifted to a *-representation $W^{L,T}$ of $A_s(K)$ ($W^{L,T} := T \circ \Phi_L$). Let $\mathcal{A}(K)$ be the set of all pairs (L, T), where L is a closed subgroup of K and T is an unitary representation of L. Note that $Im\Phi_L$ is dense, hence the map $(L, T) \longmapsto W^{L,T}$ is one-to-one. By the inner hull-kernel topology of $\mathcal{A}(K)$ we mean that topology which makes the one-to-one mapping $(L, T) \longmapsto W^{L,T}$ a homeomorphism with respect to the inner hull-kernel topology of the space of unitary representations of $A_s(K)$. This is the only topology of $\mathcal{A}(K)$ which we shall use. An important fact worth mentioning here is that $\mathcal{C}(K)$ and $\mathcal{A}(K)$ are compact spaces (equipped with their topology) (for more details see [8, p. 429-440]).

If ρ is an element of $\widehat{K_\ell}$, then the triple $(\ell, (K_\ell, \rho))$ is called a cataloguing triple. From the notations of [2], we denote by $\pi(\ell, K_\ell, \rho)$ the induced representation $Ind_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell)$.

Referring to [2, p. 187], we have

Proposition 1. *The mapping* $(\ell, (K_\ell, \rho)) \longmapsto \pi(\ell, K_\ell, \rho)$ *is onto* $\widehat{K \ltimes V}$.

Therefore, every element in $\widehat{K \ltimes V}$ can be catalogued by elements in the topological space $\widehat{V} \times \mathcal{A}(K)$. Larry Baggett has given an abstract description of the topology of the dual space of a semidirect product of a compact group with an abelian group in terms of the Mackey parameters of the dual space (see [2, Theorem 6.2-A]). The following result provides a precise and neat description of the topology of $\widehat{K \ltimes V}$.

Theorem 2.2. *Let B be a subset of $\widehat{K \ltimes V}$ and π an element of $\widehat{K \ltimes V}$. Then π is weakly contained in B if and only if there exist: a cataloguing triple $(\ell, (K_\ell, \rho))$ for π, an element (L, T) of $\mathcal{A}(K)$, and a net $\{(\chi_n, (K_{\ell_n}, \rho_n))\}$ of cataloguing triples such that:*

- (i) *for each n, the irreducible unitary representation $\pi(\ell_n, K_{\ell_n}, \rho_n)$ of $K \ltimes V$ is an element of B;*
- (ii) *the net $\{(\ell_n, (K_{\ell_n}, \rho_n))\}$ converges to $(\ell, (L, T))$ in $\widehat{V} \times \mathcal{A}(K)$;*
- (iii) *K_ℓ contains L, and the restriction representation $Res_L^{K_\ell}(\rho)$ contains T.*

3 Admissible coadjoint orbits of semidirect products

We keep the notations of section 2. Fix a non-zero linear form $\ell \in V^*$, and we consider an irreducible representation ρ_μ of K_ℓ with highest weight μ. Then the stabilizer G_ψ of $\psi = (\mu, \ell)$ in G is given by

$$\begin{aligned} G_\psi &= \Big\{(k,v) \in G;\ (Ad_K^*(k)\mu + k.\ell \odot v, k.\ell) = (\mu,\ell)\Big\} \\ &= \Big\{(k,v) \in G;\ k \in K_\ell, Ad_K^*(k)\mu + \ell \odot v = \mu\Big\} \\ &= \Big\{(k,v) \in G;\ k \in K_\ell, \imath_\ell^*(Ad_K^*(k)\mu + \ell \odot v) = \mu\Big\} \\ &= \Big\{(k,v) \in G;\ k \in K_\ell,\ Ad_K^*(k)\mu = \mu\Big\} \end{aligned}$$

since $\imath_\ell^*(\ell \odot v) = 0$ (see Lemma 2.1). Thus, we have $G_\psi = K_\psi \ltimes V_\psi$, then ψ is aligned (see [6, p. 23]). A linear form $\psi \in \mathfrak{g}^*$ is called admissible if there exists a unitary character χ of the identity component of G_ψ such that $d\chi = i\psi_{|_{\mathfrak{g}_\psi}}$. According to Lipsman (by [6, p. 23]) (compare [17]), the representation of G obtained by holomorphic induction from (μ, ℓ) is equivalent to the representation $\pi_{(\mu,\ell)}$. Let τ_λ be an irreducible representation of K with highest weight λ, then the representation of G obtained by holomorphic induction from $(\lambda, 0)$ is equivalent to τ_λ. The coadjoint orbit of G through $(\lambda, 0) \in \mathfrak{g}^*$ is denoted by $\mathcal{O}_\lambda^G$. It is clear that $\mathcal{O}_\lambda^G$ is an admissible coadjoint orbit of G. We denote by $\mathfrak{g}^\ddagger \subset \mathfrak{g}^*$ the set of all admissible linear forms on $\mathfrak{g}$. The quotient space $\mathfrak{g}^\ddagger/G$ is called the space of admissible coadjoint orbits of G. Moreover, one can check that $\mathfrak{g}^\ddagger/G$ is the union of the set of all orbits $\mathcal{O}_{(\mu,\ell)}^G$ and the set of all orbits $\mathcal{O}_\lambda^G$.

We conclude this section by recalling needed results. Let L be a closed subgroup of $K_\ell \subset K$ with Lie algebra $\mathfrak{l}$. Let T_K, T_{K_ℓ} and T_L be maximal tori respectively in K, K_ℓ and L such that $T_L \subset T_{K_\ell} \subset T_K$. Their corresponding Lie algebras are denoted by $\mathfrak{t}_\mathfrak{k}$, $\mathfrak{t}_\ell$ and $\mathfrak{t}_\mathfrak{l}$. We denote by W_K, W_{K_ℓ} and W_L the Weyl groups of K, K_ℓ and L associated respectively to the tori T_K, T_{K_ℓ} and T_L. Notice that every element $\lambda \in P_K$ takes pure imaginary values on $\mathfrak{t}_\mathfrak{k}$, where P_K is the integral weight lattice of T_K. Hence such an element $\lambda \in P_K$ can be considered as an element of $(i\mathfrak{t}_\mathfrak{k})^*$. Let C_K^+ be a positive Weyl chamber in $(i\mathfrak{t}_\mathfrak{k})^*$, and we define the set P_K^+ of dominant integral weights of T_K by $P_K^+ := P_K \cap C_K^+$. For $\lambda \in P_K^+$, denote by $\mathcal{O}_\lambda^K$ the K-coadjoint orbit passing through the vector $-i\lambda$. It was proved by Kostant in [15], that the projection of $\mathcal{O}_\lambda^K$ on $\mathfrak{t}_\mathfrak{k}^*$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, and that is the convex hull of $-i(W_K.\lambda)$. For the same manner, we fix a positive Weyl chamber C_L^+ in $\mathfrak{t}_\mathfrak{l}^*$ and we define the set P_L^+ of dominant integral weights of T_L.

Also we denote by $\imath_\mathfrak{l}^*$ the $\mathbb{C}$-linear extension of both the natural projection of $\mathfrak{k}^*$ onto $\mathfrak{l}^*$ and the natural projection of $\mathfrak{t}_\mathfrak{k}^*$ onto $\mathfrak{t}_\mathfrak{l}^*$. Consider the irreducible representations $\rho_\mu \in \widehat{K_\ell}$ and $\pi_\nu \in \widehat{L}$ with respective highest weights $\mu \in P_{K_\ell}^+$ and $\nu \in P_L^+$. Let q be the restriction of $\imath_\mathfrak{l}^*$ to $\mathfrak{k}_\ell^*$. We have the following results.

Lemma 3.1. *If $\nu = q(s.\mu)$ with $s \in W_{K_\ell}$, then π_ν occurs in the restriction representation $Res_L^{K_\ell}(\rho_\mu)$.*

We refer to [1], for the proof of this Lemma.

Let $\mathcal{O}_\mu^{K_\ell}$ and $\mathcal{O}_\nu^L$ be the coadjoint orbits of K and L passing through $-i\mu$ and $-i\nu$, respectively. According to Guillemin and Sternberg (see, [9, 10]) (compare [11]), we have the following result.

Lemma 3.2. *If the restriction representation $Res_L^{K_\ell}(\rho_\mu)$ contains π_ν, then the orbit $\mathcal{O}_\nu^L$ is contained in $q(\mathcal{O}_\mu^{K_\ell})$.*

4 Main results

Let us now return to the context and notations of the previous sections. Now, for each irreducible representation ρ_μ of K_ℓ with highest weight μ and a non-zero linear form ℓ on V, we associate the representation $\pi_{(\mu,\ell)}$ of G and its corresponding cataloguing triple $(\ell, (K_\ell, \rho_\mu))$. Also for an irreducible representation τ_λ of K with highest weight λ, we denote by $(0, (K, \tau_\lambda))$ the cataloguing of the trivial extension of τ_λ to G.

We easily find the following remark:

Remark 4.1. *If we have the following convergence*

$$\ell_n \longrightarrow \ell \tag{4.1}$$

$$K_{\ell_n} \longrightarrow L \tag{4.2}$$

where L is a subgroup of K, then K_ℓ contains L.

To study the convergence in the quotient space $\mathfrak{g}^\ddagger/G$, we need to the following result.

Lemma 4.2. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e., a subset V in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(V)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}_n^G)_n$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}^G$ in $\mathfrak{g}^*/G$ if and only if for any $l \in \mathcal{O}^G$, there exist $l_n \in \mathcal{O}_n^G$, $n \in \mathbb{N}$, such that $l = \lim_{n \longrightarrow +\infty} l_n$.*

A proof of this Lemma can be found in [6, p. 17].

Now, we may prove the following propositions.

Proposition 4.3. *Let $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ be a sequence in $\mathfrak{g}^\ddagger/G$.*
If $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$, then $(\pi_{(\mu^n,\ell_n)})_n$ converges to $\pi_{(\mu,\ell)}$ in $\widehat{G}$.

Proof. Referring to [3, Theorem 10.1], we show that the coadjoint orbit $\mathcal{O}^G_{(\mu,\ell)}$ is always obtained by symplectic induction from the coadjoint orbit $M = \mathcal{O}^H_{(\mu,\ell)}$ of $H := K_\ell \ltimes V$ passing through $(\mu, \ell) \in \mathfrak{k}^*_\ell \oplus V^*$ ($\mathfrak{k}_\ell \ltimes V := Lie(H)$), i.e.,

$$\mathcal{O}^G_{(\mu,\ell)} \;=\; M_{ind} := J^{-1}_{\widetilde{M}}(0)/H, \tag{4.3}$$

where $J_{\widetilde{M}} : \widetilde{M} = M \times T^*G \longrightarrow \mathfrak{k}^*_\ell \ltimes V^*$ is the momentum map of $\widetilde{M}$ and the zero level set $J^{-1}_{\widetilde{M}}(0)$ is given by

$$J^{-1}_{\widetilde{M}}(0) = \Big\{ \Big((Ad^*_K(k)\mu, \ell), g, (Ad^*_K(k)\mu + \ell \odot v, \ell) \Big),\ k \in K_\ell, g \in G, v \in V \Big\}.$$

Let φ_M be the action of H on M, hence H acts on $\widetilde{M} = M \times T^*G$ by $\varphi_{\widetilde{M}}$ as follows

$$\varphi_{\widetilde{M}}(h)(m, g, f) \;=\; \Big(\varphi_M(h)(m), gh^{-1}, Ad^*_H(h) f \Big), \tag{4.4}$$

for all $h \in H, (m, g, f) \in M \times T^*G$. By identifying $\mathfrak{g}^*$ with the left-invariant 1-forms on G, we can write $T^*G \cong G \times \mathfrak{g}^*$.
Let us assume that the sequence of admissible coadjoint orbits $\big(\mathcal{O}^G_{(\mu^n,\ell_n)}\big)_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$. By compactness of $\mathcal{A}(K)$ there exists a subsequence of subgroup representations $\{(K_{\ell_{n_m}}, \rho_{\mu^{n_m}})\}_m$, which converges to (L, π_ν) in $\mathcal{A}(K)$ (where ν is the highest weight of π_ν). Now, using Lemma 4.2 and by combining (4.3) with (4.4), then we deduce that there exist sequences $k_m, h_m \in K_{\ell_{n_m}}$, $v_m, w_m \in V$, and $g_m \in G$ such that the sequence $(\phi_m)_m$ defined by

$$\begin{aligned} \phi_m \;&=\; \varphi_{\widetilde{M}}(k_m, v_m)\big((Ad^*_K(h_m)\mu^{n_m}, \ell_{n_m}), g_m, (Ad^*_K(h_m)\mu^{n_m} \\ &+\; \ell_{n_m} \odot w_m, \ell_{n_m})\big) \\ &=\; \Big(Ad^*_K(k_m h_m)\mu^{n_m} + \imath^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m), \ell_{n_m}\big), g_m(k_m, v_m)^{-1}, \\ &\quad\ (Ad^*_K(k_m h_m)\mu^{n_m} + Ad^*_K(k_m)(\ell_{n_m} \odot w_m) + \ell_{n_m} \odot v_m, \ell_{n_m})\Big) \end{aligned}$$

converges to $\big((\mu, \ell), e_G, (\mu, \ell)\big)$. It follows that

$$\ell_{n_m} \;\longrightarrow\; \ell \tag{4.5}$$

and

$$Ad^*_K(k_m h_m)\mu^{n_m} + \imath^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m) \;\longrightarrow\; \mu \tag{4.6}$$

as $n \longrightarrow +\infty$. By compactness of K we may assume that $(k_m h_m)_m$ converges to an element $k \in K_\ell$. Using the fact that $\imath^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m) = 0$, we obtain from (4.6) that

$$\mu^{n_m} \;=\; Ad^*_K(k^{-1})\mu \tag{4.7}$$

for m large enough. On the other hand, we have $Ad^*_K(k^{-1})\mu = s.\mu$ for some s in the Weyl group W_{K_ℓ} (see [12, p. 285]). Hence $\mu^{n_m} = s.\mu$ for m large enough. From the fact that the mapping $(K_\ell, \rho_\mu) \longmapsto \rho_\mu$ is continuous (see, [8, p. 429-440]), we get that $\nu = s.\mu$. By Lemma 3.1, it follows that $\pi_\nu \in Res^{K_\ell}_L(\rho_\mu)$. Comparing to Theorem 2.2 we obtain the desired result. ∎

Proposition 4.4. *If the sequence* $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ *converges to* $\mathcal{O}^G_\lambda$ *in* $\mathfrak{g}^\ddagger/G$, *then* $(\pi_{(\mu^n,\ell_n)})_n$ *converges to* τ_λ *in* $\widehat{G}$.

Proof. We use the same arguments and proceedings as in the proof of Proposition 4.3. ∎

Proposition 4.5. *We have* $(\mathcal{O}^G_{\lambda^n})_n$ *converges to* $\mathcal{O}^G_\lambda$ *in* $\mathfrak{g}^\ddagger/G$ *if and only if* $(\tau_{\lambda^n})_n$ *converges to* τ_λ *in* $\widehat{G}$.

Proof. Suppose that $(\mathcal{O}^G_{\lambda^n})_n$ converges to $\mathcal{O}^G_\lambda$ in $\mathfrak{g}^\ddagger/G$, then there exists $(k_n)_n \subset K$ such that

$$Ad^*_K(k_n)\lambda^n \longrightarrow \lambda \text{ as } n \longrightarrow +\infty. \tag{4.8}$$

By compactness of K we may assume that $(k_n)_n$ converges to $k \in K$. Then we obtain $\lambda^n = Ad^*_K(k^{-1})\lambda$ for n large enough. Hence there exists $w \in W_K$ such that $Ad^*_K(k^{-1}) = w.\lambda$ for n large enough. It follows that $\lambda^n = w.\lambda$ for n large enough. Since the weights λ^n and λ are contained in the set iC^+_K and since each W_K-orbit in $\mathfrak{k}^*$ intersects the closure $\overline{iC^+_K}$ in exactly one point, it follows that $\lambda^n = \lambda$ for n large enough and this means that $(\tau_{\lambda^n})_n$ converges to τ_λ.
Conversely, assume that $(\tau_{\lambda^n})_n$ converges to τ_λ. Since K is compact, then $\widehat{K}$ is a discrete space and we obtain $\tau_{\lambda^n} = \tau_\lambda$ for n large enough. Hence $\lambda^n = \lambda$ for n large enough. Applying Lemma 4.2, it follows that $(\mathcal{O}^G_{\lambda^n})_n$ converges to $\mathcal{O}^G_\lambda$ in $\mathfrak{g}^\ddagger/G$. ∎
We summarize the above results into.

Theorem 4.6. *The Lipsman mapping*

$$\begin{aligned}\Theta : \mathfrak{g}^\ddagger/G &\longrightarrow \widehat{G}\\ \mathcal{O} &\longmapsto \pi_{\mathcal{O}}\end{aligned}$$

is continuous.

It remains to prove:

Theorem 4.7. *The inverse of the Lipsman mapping*

$$\begin{aligned}\Theta^{-1} : \widehat{G} &\longrightarrow \mathfrak{g}^\ddagger/G\\ \pi &\longmapsto \mathcal{O}_\pi\end{aligned}$$

is continuous.

Proof. Let $(\pi_{\mu^n,\ell_n})_n$ be a sequence in $\widehat{G}$, such that $(\pi_{(\mu^n,\ell_n)})_n$ converges to $\pi_{(\mu,\ell)}$. According to Baggett's result (Theorem 2.2), then there exist a cataloguing triple $(\ell, (K_\ell, \rho_\mu))$ for $\pi_{(\mu,\ell)}$, an element (L, π_ν) of $\mathcal{A}(K)$ and a sequence $\{(\ell_n, (K_{\ell_n}, \rho_{\mu^n}))\}_n$ for which we have:

1. The sequence $\{(\ell_n, (K_{\ell_n}, \rho_{\mu^n}))\}_n$ converges to $\{(\ell, (L, \pi_\nu))\}$ in $V^* \times \mathcal{A}(K)$;
2. K_ℓ contains the subgroup L;
3. The representation π_ν occurs in the restriction $Res_L^{K_\ell}(\rho_\mu)$.

From (3), we can write also

$$\lim_{n \longrightarrow +\infty} \rho_{\mu^n} \in Res_L^{K_\ell}(\rho_\mu). \tag{4.9}$$

Using (4.9), we deduce by Lemma 3.2 that there exists $p \in K_\ell$ such that

$$\mu^n = q(Ad_K^*(p))\mu$$

for n large enough. On the other hand we use the fact that the mapping $(L, (K_\ell, \rho_\mu)) \longmapsto Res_L^{K_\ell}(\rho_\mu)$ is continuous (see, [8, Theorem 3.2]), then (4.9) implies that

$$\lim_{n \longrightarrow +\infty} \rho_{\mu^n} \in \lim_{n \longrightarrow +\infty} Res_{K_{\ell_n}}^{K_\ell}(\rho_\mu) \tag{4.10}$$

Applying Lemma 3.2 to (4.10), then there exists $h_n \in K_\ell$ such that

$$\lim_{n \longrightarrow +\infty} \mu^n = \lim_{n \longrightarrow +\infty} \imath_{\ell_n}^*(Ad_K^*(h_n)\mu).$$

Let $\beta_n := \imath_{\ell_n}^*(Ad_K^*(h_n))\mu$, $(n \in \mathbb{N})$. In view of Lemma 2.1, there exists $w_n \in V$ such that

$$\beta_n + \ell_n \odot w_n = Ad_K^*(h_n)\mu. \tag{4.11}$$

Then

$$\lim_{n \longrightarrow +\infty} \mu^n = \lim_{n \longrightarrow +\infty} (Ad_K^*(h_n)\mu - \ell_n \odot w_n) \tag{4.12}$$

$$= q(Ad_K^*(p))\mu. \tag{4.13}$$

By assuming that $(h_n)_n$ converges to $h \in K_\ell$, we check that the sequence $(\ell_n \odot w_n)_n$ converges in $\mathfrak{k}^*$. Hence (4.12) becomes as follows

$$\lim_{n \longrightarrow +\infty} (Ad_K^*(h^{-1})\mu^n + h^{-1}.\ell_n \odot h^{-1}.w_n) = \mu \tag{4.14}$$

Now, we fix (k, v) in G and for each $n \in \mathbb{N}$, we put

$$(k_n, v_n) := (kh^{-1}, kh^{-1}.w_n + v) \in G.$$

We can easily see that $(k_n.\ell_n)_n$ converges to $k.\ell$ and according to (4.14) we see that the sequence $(\alpha_n)_n$ defined by

$$\alpha_n = Ad_K^*(k_n)\mu^n + k_n.\ell_n \odot v_n = Ad_K^*(k)\mu + kh^{-1}.\ell_n \odot v$$

converges to the element $Ad_K^*(k)\mu + k.\ell \odot v$. We conclude by Lemma 4.2, that the sequence of the admissible coadjoint orbits $\mathcal{O}^G_{(\mu^n,\ell_n)}$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$. If $(\pi_{(\mu^n,\ell_n)})_n$ converges to τ_λ, then it is very similar to see that $\mathcal{O}^G_{(\mu^n,\ell_n)}$ converges to $\mathcal{O}^G_\lambda$. This completes the proof of the Theorem. ∎

We have finished the proof of the main result (Theorem 1.2).

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