Pointwise version of contractibility of Banach algebras of locally compact groups*

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Abstract

In this paper, we introduce the concept of pointwise compactness for a locally compact group G, and among other results, we show that pointwise compactness of G is a necessary condition for pointwise contractibility of $L^1(G)$ in a commutative case. Also, pointwise contractibility of measure algebras in a general case is studied. Finally, applying the results, we study the pointwise contractibility of Fourier and Fourier-Stieltjes algebras in a commutative case.

1 Introduction

The notion of contractibility for Banach algebras was introduced in 1989 [10]. A Banach algebra A is called contractible if, for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is inner, that is, there exists $x \in X$ such that $Da = a \cdot x - x \cdot a$ for each $a \in A$. Various notions of contractibility have been introduced and studied for several classes of Banach algebras [9, 19] among which pointwise contractibility is the newest one [21]. A Banach algebra A is called pointwise contractible at $a \in A$ if, for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is pointwise inner at a; that is, there

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exists $x \in X$ such that $Da = a \cdot x - x \cdot a$. A is pointwise contractible, if A is pointwise contractible at a for each $a \in A$. These notions of contractibility have been studied for various classes of Banach algebras, among which the Banach algebras of locally compact groups are of particular interest. It was shown that for a locally compact group G, the group algebra $L^1(G)$ is contractible if and only if G is finite.

In [21], the author studied pointwise contractibility for group algebras of discrete groups. It is shown that pointwise contractibility of group algebra $\ell^1(G)$ implies that G is a periodic group (also known as a torsion group). Also, necessary conditions for pointwise contractibility of semigroup algebras were presented.

In this paper, we continue the study of pointwise contractibility for group algebras in a commutative case. We introduce the concept of pointwise compact locally compact groups as a generalization of periodic groups and we show that pointwise contractibility of $L^1(G)$ implies pointwise compactness of G whenever G is an abelian group. Further, we show that for a class of compactly generated locally compact abelian groups G, pointwise contractibility of $L^1(G)$ implies compactness of G. We also show that pointwise contractibility of measure algebras $M(G)$ implies that G is discrete. Applying the results, we finally investigate pointwise contractibility of Fourier and Fourier-Stieltjes algebras.

2 Pointwise contractibility of group algebras

Let G be a locally compact group, and let ϕ be the augmentation character on G, that is $\phi(f) = \int_G f d\lambda$ for each $f \in L^1(G)$, where λ is the Haar measure on G. In [21], pointwise contractibility of discrete groups are introduced. In the case G is a locally compact group, this concept is defined similarly. Recall that for given $g \in G$, G is called *left (resp. right) pointwise contractible* at g, if there exists a non-zero mean m_g in $L^1(G)$ such that $\delta_g * m_g = m_g$ (resp. $m_g * \delta_g = m_g$) and $\phi(m_g) = 1_{L^\infty(G)}(m_g) = \|m_g\| = 1$, where we consider $L^1(G)$ as a pre-dual of $L^\infty(G)$. G is called left (resp. right) *pointwise contractible* if it is *pointwise contractible* at g for each $g \in G$. Further, G is called pointwise contractible if it is both left and right pointwise contractible. As an example of pointwise contractible groups, we may mention compact groups with $m_g = 1$ for all $g \in G$. Now, this question naturally occurs:

Question 2.1. Does the family of pointwise contractible groups coincide with the family of compact groups?

In the following, we include some results and definitions from [21] for the reader's convenience.

Corollary 2.2. *[21, Corollary 2.6] Let G be a discrete group. If $\ell^1(G)$ is pointwise contractible, then G is a periodic group.*

Corollary 2.3. *[21, Corollary 2.7] Let $\mathbb{F}_2$ be the free group on two generators. Then $\ell^1(\mathbb{F}_2)$ is not pointwise contractible.*

Definition 2.4. [21, Definition 3.4] Let A be a Banach algebra, let $\pi : A\widehat{\otimes}A \to A$ be the canonical homomorphism determined by $\pi(a \otimes b) = ab$, and let $a \in A$. Then $M_a \in A\widehat{\otimes}A$ is called a pointwise diagonal at a if

$$a \cdot M_a = M_a \cdot a, \qquad \text{and} \qquad a\pi(M_a) = a.$$

Recall that $a \in A$ has a right (resp. left) unit if there exists $b \in A$ such that b is a right (resp. left) unit for a, i.e., $ab = a$ (resp. $ba = a$). a has a unit if it has a right and a left unit.

Proposition 2.5. *[21, Proposition 3.5] Let A be a Banach algebra, and let $a \in A$. Then the following statements are equivalent.*

(i) A is pointwise contractible at a.

(ii) a has a unit, and there exists a pointwise diagonal at a.

For a locally compact group G, it is well known that $L^1(G)$ is contractible if and only if G is finite. By definition, contractible Banach algebras are pointwise contractible. So the class of group algebras of finite groups is pointwise contractible. The converse is an interesting question.

Question 2.6. Is there any pointwise contractible group algebra of (abelian) locally compact group, which is not contractible?

In the sequel, we are interested to investigate pointwise contractibility of a class of locally compact abelian groups. To get our goal, we need the following result.

Proposition 2.7. *Let G be a locally compact group and suppose that $L^1(G)$ is pointwise contractible. Then for each $f \in L^1(G)$ there exists $m_f \in L^1(G)$ such that $\phi(m_f) = \|m_f\| = 1$ and $f * m_f = \phi(f)m_f$.*

Proof. One may follow the argument of ([21], Proposition 2.2) to get the result. ∎

Proposition 2.8. *Let G be a locally compact abelian group. If $L^1(G)$ is pointwise contractible, then G is pointwise contractible.*

Proof. Suppose that $L^1(G)$ is pointwise contractible and choose $f \in L^1(G)$ with $\phi(f) = 1$. By Proposition 2.7, there exists $m_1 \in L^1(G)$ such that $f * m_1 = \phi(f)m_1 = m_1$. Now, for a given $g \in G$, there exists $m_2 \in L^1(G)$ such that $\delta_g * f * m_2 = \phi(\delta_g * f)m_2 = m_2$. Set $m = m_1 * m_2$. Then we have

$$\delta_g * m = \delta_g * (f * m_1) * m_2 = (\delta_g * f * m_2) * m_1 = m_2 * m_1 = m.$$

Now, we may consider m_i as a linear functional on $L^\infty(G)$ for $i = 1, 2$. Then we see that $\hat{m}_i(1_{L^\infty(G)}) = 1_{L^\infty(G)}(m_i) = 1$ and $\|\hat{m}_i\| = \|m_i\| = 1$, where $\hat{}: L^1(G) \to L^1(G)^{**}$ is the canonical embedding. Therefore, by ([19], Proposition 1.1.2), m_i is positive. Now, we obtain

$$\hat{m}(1_{L^\infty(G)}) = 1_{L^\infty(G)}(m) = \phi(m) = \phi(m_1)\phi(m_2) = 1.$$

Further, since m_i is positive for $i = 1, 2$, it is easily seen that m is positive too. Again we apply ([19], Proposition 1.1.2) to see that $\|\hat{m}\| = \|m\| = 1$, as required. ∎

In Proposition 2.8, we give a necessary condition for pointwise contractible group algebras. Now, it is a question if pointwise contractibility could be replaced by compactness in Proposition 2.8, which looks to be less difficult than Question 2.6.

Question 2.9. Let G be a locally compact abelian group. Does pointwise contractibility of $L^1(G)$ imply that G is compact?

In the following, we will provide a partial answer to Question 2.9. To get our goal, we would like to apply the structure theorems of locally compact abelian groups. Firstly, we study pointwise contractibility of the group algebra $L^1(\mathbb{R})$.

Proposition 2.10. *$L^1(\mathbb{R})$ is not pointwise contractible.*

Proof. Suppose that $L^1(\mathbb{R})$ is pointwise contractible. So, by Proposition 2.8, $\mathbb{R}$ is pointwise contractible. In particular, there exists $m \in L^1(\mathbb{R})$ such that $\delta_1 * m = m$ and note that $m \neq 0$. Since $m \in L^1(\mathbb{R})$, $\int_n^{n+1} |m(x)|dx \to 0$ as $n \to \infty$, so that $\int_0^1 |m(x)|dx = 0$. It follows that $\int_{-\infty}^{\infty} |m(x)|dx = 0$, that is $m = 0$, a contradiction. Thus $L^1(\mathbb{R})$ is not pointwise contractible. ∎

Before proceeding further, we are interested to study how pointwise contractibility behaves under some standard operators such as epimorphisms and tensor products.

Lemma 2.11. *Let A and B be Banach algebras and let $\theta : A \to B$ be a surjective homomorphism. If A is pointwise contractible, then so is B.*

Proof. Suppose that A is pointwise contractible and $b \in B$. Then there exists $a \in A$ such that $\theta(a) = b$. By Proposition 2.5, a has a unit u_a and A has a pointwise diagonal at a, say M_a. Now, it is easily seen that $(\theta \otimes \theta)(M_a)$ is a pointwise diagonal at b and $\theta(u_a)$ is a unit for b. Again we apply Proposition 2.5 to see that B is pointwise contractible at b, which completes the proof. ∎

Proposition 2.12. *Let A and B be Banach algebras, let the character space of B (resp. A) be non empty, and let $A\hat{\otimes}B$ be pointwise contractible. Then A (resp. B) is pointwise contractible.*

Proof. Suppose that ϕ is a character of B and define $\theta : A\hat{\otimes}B \to A$ with $\theta(a \otimes b) = \phi(b)a$ for each $a \in A$ and $b \in B$. Then θ is a surjective homomorphism and applying Lemma 2.11 we get A is pointwise contractible. The other part holds similarly. ∎

Corollary 2.13. *Let $n \in \mathbb{N}$, and let $\{G_i : 1 \leq i \leq n\}$ be a family of locally compact groups. Set $G = G_1 \times G_2 \times \cdots \times G_n$. If $L^1(G)$ is pointwise contractible, then so is $L^1(G_i)$ for each $1 \leq i \leq n$.*

Proof. Proposition 2.12 by the fact that $L^1(G) \simeq L^1(G_1)\hat{\otimes} \cdots \hat{\otimes}L^1(G_n)$ gives the result. ∎

Corollary 2.14. *$L^1(\mathbb{R}^n)$ is not pointwise contractible for any $n \in \mathbb{N}$.*

Let G be a locally compact group. Recall that $g \in G$ is said to be compact if $\overline{\langle g \rangle}$ is compact, where $\overline{\langle g \rangle}$ we denotes the subgroup of G topologically generated by g; see [11, Definition 9.9].

For a locally compact group G, it is known that if $L^1(G)$ is contractible, then G is finite and therefore, for given $g \in G$, $\overline{\langle g \rangle}$ is finite. Hence, each element of G is compact when $L^1(G)$ is contractible. In the sequel, we are interested to study this property in terms of pointwise contractibility. In order to make more clear our arguments, we single out in a new definition the basic property of locally compact groups we shall use.

Definition 2.15. Let G be a locally compact group. We call G is pointwise compact if each element of G is compact.

Recall that the set of all compact elements of G is also denoted by G^c in some texts, and a locally compact group G with $G^c = \{e\}$ is called compact free; see [16] for example.

As an example of pointwise compact groups, we may mention compact groups and periodic groups. In [21], it is shown that all discrete groups with pointwise contractible group algebras are periodic and therefore, pointwise compact. In the following we are interested to generalize this result to a locally compact group.

Proposition 2.16. *Let G be a (not necessary abelian) locally compact group containing an open compact normal subgroup. If $L^1(G)$ is pointwise contractible, then G is pointwise compact.*

Proof. Suppose that K is an open compact normal subgroup of G, and $L^1(G)$ is pointwise contractible. Then there exists a surjective algebraic homomorphism from $L^1(G)$ to $L^1(G/K)$; see [18, Section 1.9.12] for more details. So, $L^1(G/K)$ is pointwise contractible by Lemma 2.11. Since K is an open normal subgroup of G, G/K is a discrete group and by Corollary 2.2, G/K is periodic. Therefore, for given $g \in G$, there exists $n \in \mathbb{N}$ such that $g^n \in K$. So, $\overline{\langle g^n \rangle}$ as a closed subset of K is compact. The subgroup $\langle g \rangle$ is contained in finitely many cosets of K, more precisely: $\overline{\langle g \rangle} \subseteq_{k=0}^{n-1} g^k \cdot K$. So, $\langle g \rangle$ is relatively compact. ∎

The following result is suggested by the referee, which gives a relation between pointwise contractibility and pointwise compactness of locally compact groups.

Let G be a locally compact group, and H a closed subgroup of G. Then we denote $\{\chi \in \hat{G} : \chi_{|_H} = 1\}$ by $H^{\perp}$.

Proposition 2.17. *Let G be a locally compact abelian group. If G is pointwise contractible, then G is pointwise compact.*

Proof. By contra-positive, we assume that G is not pointwise compact and prove that G is not pointwise contractible. So let $g \in G$ such that $\langle g \rangle$ is not relatively compact. Then $C := \langle g \rangle$ is a closed subgroup in G. As C is not compact, by [5, Lemma 4.2.16], C is topologically isomorphic to $\mathbb{Z}$. Let $\iota : C \to G$ denote the inclusion. Dualizing we have a continuous homomorphism $\iota^* : \hat{G} \to \hat{C}$ which is onto, and with kernel $C^{\perp}$; see also [11, Corollary 24.12]. We will use the fact that the open set $\hat{G} \backslash C^{\perp}$ is dense in $\hat{G}$.

Now, let $m_g \in L^1(G)$ be any function such that $\delta_g * m_g = m_g$. Taking Fourier transform, we have $\widehat{m_g}(\chi) \cdot \chi(g) = \widehat{m_g}(\chi)$ for every $\chi \in \hat{G}$. This implies that $\widehat{m_g}(\chi) = 0$ for every χ in the open set $\hat{G} \backslash C^{\perp}$; since $\hat{G} \backslash C^{\perp}$ is dense in $\hat{G}$, and $\widehat{m_g}$ is continuous, we have $\widehat{m_g} = 0$, hence $m_g = 0$. So, G is not pointwise contractible. ∎

Theorem 2.18. *Let G be a locally compact abelian group. If $L^1(G)$ is pointwise contractible, then the following statements hold.*

(i) G has a compact open normal subgroup.

(ii) G is pointwise compact.

(iii) Each subgroup of G topologically generated by finitely many elements is compact.

(iv) If G has a discrete subgroup N such that G/N is topologically isomorphic with $\mathbb{T}^m \times F_1$, where $m \in \mathbb{N} \cup \{0\}$ and F_1 is a finite [abelian] group, then G is topologically isomorphic to $\mathbb{T}^n \times F_2$, where n is a nonnegative integer and F_2 is a finite [abelian] group.

Proof. (i). By the first structure theorem, there exists $n \in \mathbb{N}_0$ such that $G \simeq \mathbb{R}^n \times H$, where H is a locally compact abelian group containing an open compact subgroup, see ([5], Theorem 4.2.1). Suppose that $L^1(G)$ is pointwise contractible. Then by Corollary 2.13, $L^1(\mathbb{R}^n)$ is pointwise contractible. By Corollary 2.14, this is impossible unless $n = 0$. Therefore, $G \simeq H$ and so, G has an open compact subgroup. Since G is abelian, each subgroup of G is normal which gives the result.

(ii). Applying (i) and Proposition 2.16 gives the result. One also may apply Proposition 2.8 and Proposition 2.17 to get the result.

(iii). Suppose that H is a subgroup of G topologically generated by $g_1, \cdots, g_n \in G$. Then $H = \overline{\langle g_1 \rangle} \cdots \overline{\langle g_n \rangle}$, and so H as a product of compact groups is compact.

(iv). By [11, Theorem 9.4], G is topologically isomorphic to $\mathbb{T}^n \times \mathbb{R}^k \times \mathbb{Z}^l \times F_2$, where n, k, l are nonnegative integers and F_2 is a finite [abelian] group. Now, applying Corollary 2.13, to see that $L^1(\mathbb{R}^k)$ and $L^1(\mathbb{Z}^l)$ are pointwise contractible. By Corollary 2.14, we see that $k = 0$. Furthermore, as $\mathbb{Z}^l$ is not periodic, by Corollary 2.2, $L^1(\mathbb{Z}^l)$ is not pointwise contractible unless $l = 0$, which gives the result. ∎

Now, we would like to investigate pointwise contractibility of some certain classes of locally compact group algebras. In particular, we provide a partial answer to Question 2.9.

Proposition 2.19. *Let G be a compactly generated locally compact abelian group. If $L^1(G)$ is pointwise contractible, then G is compact.*

Proof. By [11, 9.26(b)], the set of all compact elements of G is a compact subgroup of G, which is indeed the whole of G when $L^1(G)$ is pointwise contractible; see Theorem 2.18(ii). Therefore G is a compact group. ∎

Corollary 2.20. *Let G be a connected locally compact abelian group. Then pointwise contractibility of $L^1(G)$ implies that G is compact.*

Proof. Since a connected group is compactly generated, the result follows from Proposition 2.19. ∎

Now, we are interested to know how pointwise contractibility behaves under special subgroups.

Proposition 2.21. *Let G be a locally compact abelian group. If $L^1(G)$ is pointwise contractible, then the following statements hold.*

(i) The connected component of the identity in G is compact.

(ii) Each compactly generated open subgroup of G is compact.

(iii) For each compact symmetric neighborhood U of the identity in G, there exists $n \in \mathbb{N}$ such that U^n is a compact subgroup of G.

Proof. (i). Suppose that G_e is the connected component of the identity in G. Since $L^1(G)$ is pointwise contractible, by Theorem 2.18 we see that each element of G (and specially, each element of G_e) is compact. On the other hand, by [11, Theorem 9.14] G_e is topologically isomorphic with $\mathbb{R}^n \times E$, where n is a nonnegative integer and E is a compact connected abelian group. So, each element of $\mathbb{R}^n$ is compact, which is possible only when $n = 0$. Hence $G_e \simeq E$ is a compact group.

(ii). Suppose that H is a compactly generated open subgroup of G. Then by [11, Theorem 9.14] , H is topologically isomorphic with $\mathbb{Z}^m \times F$, where m is a nonnegative integer and F is a compact abelian group. A similar argument shows that $m = 0$ and so, H is compact.

(iii). Suppose that U is a symmetric compact neighborhood of the identity in G. Then $H = \cup_{n=1}^{\infty} U^n$ is a compactly generated subgroup of G. Furthermore, by [11, Theorem 5.7], we see that H is an open subgroup of G. So, applying (ii) shows that H is compact, and (iii) follows. ∎

Let A be an amenable Banach algebra. Recall that A is called C-amenable with $C \geq 1$ if there exists an approximate diagonal for A which is bounded by C. Further, amenability constant of A is denoted by $AM(A)$ which is defined as $AM(A) := \inf\{C : A \text{ is C-amenable}\}$. For more details, see [14, 15]. Also, for a locally compact group G, by $L^1_0(G)$ we denote the augmentation ideal of G that is $L^1_0(G) = \{f \in L^1(G) : \int_G f d\lambda = 0\}$. In the sequel, we give some results on disconnected groups.

Corollary 2.22. *Let G be a locally compact abelian group with finitely many connected components. If $L^1(G)$ is pointwise contractible, then $AM(L^1_0(G)) \leq 2$.*

Proof. Suppose that $L^1(G)$ is pointwise contractible. Since G has finitely many connected components, G/G_e is finite. Now, by Proposition 2.21(i), pointwise contractibility of $L^1(G)$ implies that G_e is compact. So, G is compact, and by applying [20], we see that for compact locally compact group G, $AM(L^1_0(G)) \leq 2$, as required. ∎

As an example of a locally compact abelian group with finitely many connected components, we may consider the multiplicative group $(\mathbb{R}^\times, \cdot) = (\mathbb{R}\backslash\{0\}, \cdot)$ which has two connected components. Since $\mathbb{R}^\times$ is not compact, it is not pointwise contractible by Corollary 2.22. Also, one may use the fact that $\mathbb{R}^\times$ is not pointwise compact and apply Proposition 2.16 to see that $L^1(\mathbb{R}^\times)$ is not pointwise contractible.

Now, we apply the results to arbitrary locally compact group algebras.

Proposition 2.23. *Let G be a locally compact group and let G' be the commutator subgroup of G. If $L^1(G)$ is pointwise contractible and N is a closed normal subgroup of G with $G' \subseteq N$, then the following statements hold.*

(i) G/N is pointwise compact.

(ii) If G has finitely many connected components, then G/N is compact.

(iii) If N is compact and G has finitely many connected components, G is compact.

Proof. Since there exists a surjective homomorphism from $L^1(G)$ to $L^1(G/N)$, we see that $L^1(G/N)$ is pointwise contractible. Further, it is easily seen that G/N is a locally compact abelian group. Now, (i) follows from Theorem 2.18.

(ii). Suppose that G has finitely many connected components. Then so has G/N. Then, the result follows from Proposition 2.19.

(iii). By (ii), G/N is compact. Then G is compact by [11, Theorem 5.25]. ∎

The following result investigates pointwise contractibility of measure algebras of locally compact groups.

Proposition 2.24. *Let G be a locally compact group. If $M(G)$ is pointwise contractible, then G is a discrete group.*

Proof. Suppose that G is a non-discrete locally compact group. By [3], there exists a closed complement left ideal I in $M(G)$ such that $\overline{I^2} \neq I$. On the other hand, by [21, Theorem 4.1(i)], as $M(G)$ is pointwise contractible, each complemented left ideal I of $M(G)$ has a right unit. In particular, $\overline{I^2} = I$, a contradiction. So, G is a discrete group, as required. ∎

We end this section by giving some examples.

Example 2.25. (i). Suppose that $G = SL(2, \mathbb{R}) \times \mathbb{R}^\times$, where we consider G with a subspace topology of $\mathbb{R}^4 \times \mathbb{R}$. Then G/G' is isomorphic with $\mathbb{R}^\times$. Since $\mathbb{R}^\times$ is not pointwise compact, $L^1(G)$ is not pointwise contractible by Proposition 2.23. Also, one may use Proposition 2.24 to see that $M(G)$ is not pointwise contractible.

(ii). Suppose that $n \geq 3$, and set $G = SO(n)$. Obviously, G with relative topology is a compact non-discrete group and therefore, by Proposition 2.24, $M(G)$ is not pointwise contractible, where we note that $SO(n)$ has a copy of $\mathbb{F}_2$. Further, we see that $M(G_d) = \ell^1(G_d)$, where G_d is the same group equipped with the discrete group. Then by [21, Corollory 2.7], $M(G_d)$ is not pointwise contractible.

3 Pointwise contractibility of Fourier and Fourier-Stieltjes algebras

Let G be a locally compact group, $\hat{G}$ the character group of G, $B(G)$ the Fourier-Stieltjes algebras of G and $A(G)$ the Fourier algebra of G. Recall that if G is abelian, from Bochner's theorem it follows that $B(G)$ is the Banach space of the Fourier transform u of the measure $\mu \in M(\hat{G})$, with the norm $\|u\| = \|\mu\|$. Further, in this case, the Gelfand transform coincides with the Fourier transform.

In this section, we apply the previous results to investigate pointwise contractibility of Fourier and Fourier-Stieltjes algebras in a commutative case. The following result is pointwise version of [17, Theorem 4.1].

Proposition 3.1. *Let G be a locally compact abelian group. If $A(G)$ is pointwise contractible, then for each closed subgroup H of G, $A(H)$ is pointwise contractible.*

Proof. Suppose that H is a closed subgroup of G. As $A(G) \simeq L^1(\hat{G})$, we see that $L^1(\hat{G})$ is pointwise contractible. Furthermore, by [11, Theorem 24.11], $\hat{H} \simeq \hat{G}/H^{\perp}$. As there exists a surjective algebraic homomorphism from $L^1(\hat{G})$ to $L^1(\hat{G}/H^{\perp})$, Lemma 2.11 shows that $L^1(\hat{G}/H^{\perp})$ is pointwise contractible. This shows that $A(H) \simeq L^1(\hat{H})$ is pointwise contractible, as required. ∎

Proposition 3.2. *Let G be a locally compact abelian group. If $A(G)$ is pointwise contractible, then G is totally disconnected.*

Proof. Suppose that $A(G)$ is pointwise contractible. By Proposition 3.1, $A(G_e)$ is pointwise contractible. On the other hand, $A(G_e) \simeq L^1(\hat{G}_e)$. Therefore, by Theorem 2.18, $\hat{G}_e$ is pointwise compact. Now, by [11, Corolloary 24.18], G_e is totally disconnected, which implies that G_e is the trivial group. Thus G is totally disconnected. ∎

Theorem 3.3. *Let G be a locally compact compactly generated abelian group. If $A(G)$ is pointwise contractible, then there exists a positive integer $n \in \mathbb{N}$, and a totally disconnected compact group F such that $G \simeq \mathbb{Z}^n \times F$.*

Proof. By [11, Theorem 9.8], there exists positive integers m, n and a compact abelian group F such that $G \simeq \mathbb{R}^m \times \mathbb{Z}^n \times F$. Suppose that $A(G)$ is pointwise contractible. As $A(G) \simeq L^1(\hat{G})$, we see that $L^1(\hat{G})$ is pointwise contractible. On the other hand, $\hat{G} \simeq \mathbb{R}^m \times \mathbb{T}^n \times H$, where $H = \hat{F}$ is a discrete group. Then by Theorem 2.18, $\hat{G}$ has to be pointwise compact. So, $m = 0$. Further, the connected component of G is isomorphic to the connected component of F which is a trivial group by Proposition 3.2. So, F is totally disconnected. ∎

The following result immediately follows from Proposition 3.2.

Corollary 3.4. *Let G be a locally compact abelian group. Suppose that G has finitely many connected components. Then the following statements are equivalent:*

(i) *$A(G)$ is pointwise contractible.*

(ii) *$A(G)$ is contractible.*

(iii) *G is finite.*

Corollary 3.5. *Let G be a locally compact abelian group. If the Fourier-Stieltjes algebra $B(G)$ is pointwise contractible, then G is compact and totally disconnected group. Especially, G is profinite.*

Proof. Suppose that $B(G)$ is pointwise contractible. By Proposition 2.24, we see that $\hat{G}$ is discrete which implies that G is a compact group. So, by [6, Example 3.6], $A(G) = B(G)$, and Proposition 3.2 implies that G is totally disconnected. ∎

Corollary 3.6. *Let G be a locally compact abelian group which satisfies one of the following conditions:*

(i) G is discrete.

(ii) G is almost connected.

Then the following statements are equivalent:

(i) $B(G)$ is pointwise contractible.

(ii) $B(G)$ is contractible.

(ii) G is finite.

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