Abel Convergence of the Sequence of Positive Linear Operators in $L_{p,q}(loc)$

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Abstract

In this paper, we study a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_{p,q}(loc)$ into itself with the use of Abel method which is a sequence-to-function transformation. Using the modulus of continuity for $L_{p,q}(loc)$ we also give the rate of Abel convergence of these operators.

1 Introduction

The classical Korovkin type approximation theory is essentially concerned with the approximation of real valued functions by means of positive linear operators $([1])$. It provides conditions for whether a given sequence of positive linear operators converges strongly to the identity operator in the space of continuous functions on a compact interval. These theorems exhibit a variety of test functions which guarantee that convergence property holds on the whole space provided it holds on them $([1],[9])$.

Approximation theory has many connections with theory of polynomial approximation, functional analysis, numerical solutions of differential and integral equations, summability theory, measure theory and probability theory.

Some results concerning the Korovkin type approximation in the space $L_p[a,b]$ of Lebesgue integrable functions on a compact interval may be found in [3], [5], [7].

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If the sequence of positive linear operators does not converge, then it might be beneficial to use some summability methods. The main aim of using summability methods has always been to nake a non-convergent sequence to converge. This was the motivation behind Fejer's famous theorem showing that Cesàro method being effective in making the Fourier series of a continuous periodic function to converge ([13]). The Abel method which is a nonmatrix summability method, has been used in the Korovkin type approximation of functions in the weighted space ([12]). Also L_p approximation via Abel convergence has been studied in [11].

The purpose of this paper is to use the Abel method, a sequence-to-function transformation, to study a Korovkin type approximation of a function f by means of a sequence $\{T_n(f;x)\}$ of positive linear operators acting from the locally integrable function spaces into itself. Section 2 is devoted to preliminaries and basic definitions concerning $L_{p,q}(loc)$, the locally integrable function spaces. Section 3 deals with the Korovkin type approximation with the use of Abel convergence in the space of locally integrable functions. The rate of the Abel convergence is considered in Section 4.

2 Preliminaries

First of all, we recall some notation and basic definitions used in this paper.

Let $q(x) = 1 + x^2$; $-\infty < x < \infty$. For $h > 0$, by $L_{p,q}(loc)$ we will denote the space of measurable functions f satisfying the inequality,

$$\left(\frac{1}{2h}\int_{x-h}^{x+h}|f(t)|^p\,dt\right)^{1/p} \leq M_f\, q(x) \quad , -\infty < x < \infty \tag{1.1}$$

where $p \geq 1$ and M_f is a positive constant which depends on the function f.

It is known [8] that $L_{p,q}(loc)$ is a linear normed space with norm,

$$\|f\|_{p,q} = \sup_{-\infty<x<\infty} \frac{\left(\frac{1}{2h}\int_{x-h}^{x+h}|f(t)|^p\,dt\right)^{1/p}}{q(x)} \tag{1.2}$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. To simplify the notation, we need the following. For any real numbers a and b put

$$\|f; L_p(a,b)\| := \left(\frac{1}{b-a}\int_a^b |f(t)|^p\,dt\right)^{1/p},$$

$$\|f; L_{p,q}(a,b)\| = \sup_{a<x<b} \frac{\|f; L_p(x-h, x+h)\|}{q(x)},$$

$$\|f; L_{p,q}(|x| \geq a)\| = \sup_{|x|\geq a} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}.$$

With this notation the norm in $L_{p,q}(loc)$ may be written in the form

$$\|f\|_{p,q} = \sup_{x\in\mathbb{R}} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}.$$

We recall that $L^k_{p,q}(loc)$ is the subspace of all functions $f \in L_{p,q}(loc)$ for which there exists a constant k_f such that

$$\lim_{|x|\to\infty} \frac{\|f - k_f q; L_p(x-h, x+h)\|}{q(x)} = 0.$$

As usual, if T is a positive linear operator from $L_{p,q}(loc)$ into $L_{p,q}(loc)$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f\neq 0} \frac{\|Tf\|_{p,q}}{\|f\|_{p,q}}$.

The following result is also considered in [8].

Theorem A. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself and satisfy the conditions

$i)$ The sequence (T_n) is uniformly bounded, that is, there exists a constant C such that $\|T_n\| \leq C$ for all n,

$ii)$ For $f_i(y) = y^i$, $i = 0, 1, 2$;

$$\lim_n \|T_n(f_i; x) - f_i(x)\|_{p,q} = 0.$$

Then

$$\lim_n \|T_n f - f\|_{p,q} = 0$$

for each function $f \in L^k_{p,q}(loc)$.

Some analogs of this theorem may be found in [2].

3 Abel Convergence of the Sequence of Positive Linear Operators

In this section ,using Abel convergence, we show that the Korovkin type approximation theorem does not hold in the whole space $L_{p,q}(loc)$ but it does hold in the subspace $L^k_{p,q}(loc)$.

Let us recall the Abel convergence.

If the series

$$\sum_{k=0}^{\infty} a_k y^k$$

converges for all $y \in (0,1)$ and

$$\lim_{y\to 1^-} (1-y) \sum_{k=0}^{\infty} a_k y^k = L \tag{2.1}$$

then we say that the sequence $a = (a_k)$ is Abel convergent to L $(see,\ e.g,\ [4], [10])$.

Since $\dfrac{1}{1-y}=\sum\limits_{k=0}^{\infty}a_k y^k$, $0<y<1$, (2.1) is equivalent to the fact that

$$\lim_{y\to 1^-}(1-y)\sum_k (a_k-L)\,y^k=0.$$

Note that any convergent sequence is Abel convergent to the same value but not conversely ([4], [10]).

Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that

$$H:=\sup_{y\in(0,1)}(1-y)\sum_n \|T_n\|_{p,q}\,y^n<\infty. \tag{2.2}$$

Then for all $f\in L_{p,q}(loc)$ and $y\in(0,1)$ the operator U_y defined by

$$U_y:=U_y(f;x):=(1-y)\sum_n T_n(f;x)\,y^n$$

is a positive linear operator from $L_{p,q}(loc)$ into itself. It is shown [8] that

$$\|T_n f\|_{p,q}\le 4\,\|f\|_{p,q}.$$

It follows from (2.1) that

$$\begin{aligned}\left\|U_y\right\|_{p,q} &= \sup_{\|f\|_{p,q}=1}(1-y)\left\|\sum_n T_n(f)\,y^n\right\|_{p,q}\\ &\le \sup_{y\in(0,1)}(1-y)\sum_n\|T_n\|_{p,q}\,y^n\\ &\le 4\sup_{y\in(0,1)}(1-y)\sum_n y^n\\ &= 4.\end{aligned}$$

Now using Abel convergence we give prove that a Korovkin type approximation theorem does not hold in $L_{p,q}(loc)$.

Theorem 1. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.2) holds and satisfies

$$\lim_{y\to 1^-}\left\|U_y(f_i)-f_i\right\|_{p,q}=0 \text{ for } i=0,1,2$$

where $f_i(t)=t^i$; $i=0,1,2$. Then there exists a function f^* in $L_{p,q}(loc)$ for which

$$\lim_{y\to 1^-}\left\|U_y(f^*)-f^*\right\|_{p,q}\neq 0.$$

Proof. We consider the sequence of operators T_n given in [8] :

$$T_n(f;x)=\begin{cases}\frac{x^2}{(x+h)^2}f(x+h) & ,x\in[(2n-1)h,(2n+1)h)\\ f(x) & ,\ otherwise.\end{cases}$$

It is shown in [8] that

$$\|T_n f\|_{p,q}\le 4\,\|f\|_{p,q}.$$

Now it is easy to verify that, for each $i = 0, 1, 2$ we have

$$\begin{aligned}
\left\| (1-y) \sum_n y^n T_n f_i - f_i \right\|_{p,q} &= \left\| (1-y) \sum_n y^n T_n f_i - (1-y) \sum_n f_i y^n \right\|_{p,q} \\
&= \left\| (1-y) \sum_n (T_n f_i - f_i) y^n \right\|_{p,q} \\
&\leq (1-y) \sum_n y^n \| T_n f_i - f_i \|_{p,q} \\
&\to 0 \quad (y \to 1^-).
\end{aligned}$$

Consider the following function f^* given in [8] :

$$f^*(x) = \begin{cases} x^2 & , if\ x \in \bigcup\limits_{k=1}^{\infty} [(2k-1)h, 2kh) \\ -x^2 & , if\ x \in \bigcup\limits_{k=1}^{\infty} [2kh, (2k+1)h) \\ 0 & , if\ x < 0. \end{cases}$$

Then $f^* \in L_{p,q}(loc)$ and we get

$$\begin{aligned}
\left\| (1-y) \sum_n y^n T_n f^* - f^* \right\|_{p,q} &\geq \sup_{x \in [(2k-1)h, 2(k+1)h]} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum_n y^n T_n f^* - f^* \right|^p dt \right)^{\frac{1}{p}}}{q(x)} \\
&= \sup_{x \in [(2k-1)h, 2(k+1)h]} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum_n y^n \frac{\xi^2}{(\xi+h)^2} f^*(\xi+h) - f^*(\xi) \right|^p d\xi \right)^{\frac{1}{p}}}{q(x)} \\
&\geq \frac{\left(\frac{1}{2h} \int\limits_{2kh-h}^{2kh+h} \left| (1-y) \sum_n y^n \frac{\xi^2}{(\xi+h)^2} f^*(\xi+h) - f^*(\xi) \right|^p d\xi \right)^{\frac{1}{p}}}{q(2kh)} \\
&\geq \frac{\left(\frac{1}{2h} \int\limits_{2kh-h}^{2kh+h} \left| (1-y) \sum_n y^n \frac{\xi^2}{(\xi+h)^2} \left\{ -(\xi+h)^2 \right\} - (\xi)^2 \right|^p d\xi \right)^{\frac{1}{p}}}{1 + 4k^2h^2} \\
&= \frac{\left(\frac{1}{2h} \int\limits_{2kh-h}^{2kh+h} \left| -2\xi^2 \right|^p d\xi \right)^{\frac{1}{p}}}{1 + 4k^2h^2} \\
&> \frac{\left(\frac{1}{2h} 2^p \left((2k-1)h \right)^{2p} h \right)^{\frac{1}{p}}}{1 + 4k^2h^2} \\
&= \frac{2^{1-\frac{1}{p}} (2k-1)^2 h^2}{1 + 4k^2h^2}.
\end{aligned}$$

On applying the operator $\lim\limits_{y\to 1^-}$ on both sides one can see that

$$\lim_{y\to 1^-}\left\|(1-y)\sum_n y^n T_j f^* - f^*\right\|_{p,q} \neq 0$$

Therefore the theorem is proved. This result shows that Korovkin type theorem does not hold in the whole space $L_{p,q}(loc)$.

Now we show that the above mentioned problem has a positive solution in the subspace $L^k_{p,q}(loc)$. First we give the following simple lemma.

Lemma 1. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds and satisfies

$$\lim_{y\to 1^-}\left\|U_y(f_i)-f_i\right\|_{p,q} = 0 \text{ for } i = 0,1,2$$

where $f_i(t) = t^i$; $i = 0,1,2$. Then, for any continuous and bounded function f on the real axis, we have

$$\lim_{y\to 1^-}\left\|U_y(f)-f; L_{p,q}(a,b)\right\| = 0$$

where a and b are any real numbers.

Proof. Since f is uniformly continuous function on any closed interval, given $\varepsilon > 0$ there exists a positive $\delta = \delta(\varepsilon)$ such that

$$|f(t)-f(x)| < \varepsilon \text{ if } |t-x| < \delta, \text{ where } x\in[a,b], t\in\mathbb{R}. \tag{2.3}$$

Also, setting $M = \sup\limits_{x\in\mathbb{R}}|f(x)|$, we can write

$$|f(t)-f(x)| < 2M \text{ if } |t-x| \geq \delta, \text{ where } x\in[a,b], t\in\mathbb{R}. \tag{2.4}$$

Combining (2.3) and (2.4) we have

$$|f(t)-f(x)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2, \tag{2.5}$$

where $-\infty < t < \infty$; $x\in[a,b]$. Let $c := \max\{|a|,|b|\}$ and using the positivity and linearity of operators T_j and (2.5) we obtain

$$\begin{aligned}
\left\|U_y(f(t);x)-f(x);L_{p,q}(a,b)\right\| &\leq \left\|U_y(|f(t)-f(x)|;x)\right\|_{p,q} \\
&\quad + |f(x)|\left\|U_y(1;x)-1\right\|_{p,q} \\
&< \left\|U_y(\varepsilon+\frac{2M}{\delta^2}(t-x)^2;x)\right\|_{p,q} \\
&\quad + M\left\|U_y(1;x)-1\right\|_{p,q} \\
&< \varepsilon + \frac{2M}{\delta^2}\left\|U_y(t^2;x)-x^2\right\|_{p,q} \\
&\quad + \frac{4Mc}{\delta^2}\left\|U_y(t;x)-x\right\|_{p,q} \\
&\quad + \left(\frac{2Mc^2}{\delta^2}+\varepsilon+M\right)\left\|U_y(1;x)-1\right\|_{p,q}.
\end{aligned}$$

Hence by the hypotheses the proof is completed.

Theorem 2. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds and satisfies

$$\lim_{y\to 1^-}\left\|U_y(f_i)-f_i\right\|_{p,q}=0 \text{ for } i=0,1,2$$

where $f_i(t)=t^i;\, i=0,1,2$. Then for any function $f\in L^k_{p,q}(loc)$ we have

$$\lim_{y\to 1^-}\left\|U_y(f)-f\right\|_{p,q}=0.$$

Proof. We follow [8] up to a certain stage. If $f\in L^k_{p,q}(loc)$ then $f-k_f q\in L^0_{p,q}(loc)$. So it is sufficient to prove the theorem for the function $f\in L^0_{p,q}(loc)$. For any $\varepsilon>0$, there exists a point x_0 such that the inequality

$$\left(\frac{1}{2h}\int_{x-h}^{x+h}|f(t)|^p\,dt\right)^{1/p}<\varepsilon q(x) \tag{2.6}$$

holds for all x, $|x|\geq x_0$. By the well known Lusin Theorem, there exists a continuous function φ on the finite interval $[-x_0-h,x_0+h]$ such that the inequality

$$\left\|f-\varphi;L_p(-x_0,x_0)\right\|<\varepsilon \tag{2.7}$$

is fulfilled. Setting

$$0<\delta<\min\left\{\frac{2h\varepsilon^p}{M^p(x_0)},h\right\}, \tag{2.8}$$

where $M(x_0)=\max\left\{\max\limits_{|x|\leq x_0+h}|\varphi(x)|,1\right\}$, we can define a continuous function g by

$$g(x)=\begin{cases}\varphi(x) & ,\, if\ |x|\leq x_0+h\\ 0 & ,\, if\ |x|\geq x_0+h+\delta\\ linear & ,\, otherwise.\end{cases}$$

Then by (2.6), (2.7), (2.8) and the Minkowski inequality, we obtain

$$\|f-g\|_{p,q}<\varepsilon \tag{2.9}$$

for any $\varepsilon>0$ (see [8]).

Now we can find a point $x_1>x_0$ such that

$$q(x_1)>\frac{M(x_0)}{\varepsilon} \text{ and } g(x)=0 \text{ for } |x|>x_1, \tag{2.10}$$

where $M(x_0)$ is defined above. Then by (2.7), (2.8), (2.9) and the definition of g

and Lemma 1 we get

$$\begin{aligned}
\left\|U_y(f;x)-f(x)\right\|_{p,q} &= \left\|U_y(f(t)-g(t)+g(t);x)-f(x)-g(x)+g(x)\right\|_{p,q} \\
&\leq \left\|U_y(f-g)\right\|_{p,q}+\left\|U_y g-g\right\|_{p,q}+\|f-g\|_{p,q} \\
&\leq \varepsilon(1-y)\sum_n y^n\|T_n\|_{p,q}+\varepsilon+\left\|U_y g-g\right\|_{p,q} \\
&\leq \varepsilon\left((1-y)\sum_n y^n\|T_n\|_{p,q}+1\right)+\left\|U_y g-g;L_{p,q}(-x_1,x_1)\right\| \\
&\quad+\left\|U_y g-g;L_{p,q}(|x|\geq x_1)\right\| \\
&\leq \varepsilon\left((1-y)\sum_n y^n\|T_n\|_{p,q}+2\right) \\
&\quad+\left\|U_y g;L_{p,q}(|x|\geq x_1)\right\|. \qquad (2.11)
\end{aligned}$$

Since $|g(x)|\leq M(x_0)$ for all $x\in\mathbb{R}$, we can write

$$\begin{aligned}
\left\|U_y g;L_{p,q}(|x|\geq x_1)\right\|_{p,q} &\leq M(x_o)\left\|U_y 1;L_{p,q}(|x|\geq x_1)\right\| \\
&\leq M(x_o)\left\|U_y 1-1;L_{p,q}(|x|\geq x_1)\right\| \\
&\quad+M(x_o)\left\|1;L_{p,q}(|x|\geq x_1)\right\| \\
&\leq M(x_o)\left\|U_y 1-1\right\|_{p,q}+\frac{M(x_o)}{q(x_1)}.
\end{aligned}$$

Considering hypothesis and (2.10) we get by (2.11) that

$$\lim_{y\to 1^-}\left\|U_y f-f\right\|_{p,q}=0$$

which proves the theorem.

In the whole space $L_{p,q}(loc)$ we have the following

Theorem 3. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds and satisfy

$$\lim_{y\to 1^-}\left\|U_y(f_i)-f_i\right\|_{p,q}=0 \text{ for } i=0,1,2$$

where $f_i(t)=t^i;\ i=0,1,2$. Then for any functions $f\in L_{p,q}(loc)$ we have

$$\lim_{y\to 1^-}\left(\sup_{x\in\mathbb{R}}\frac{\left\|U_y f-f;L_p(x-h,x+h)\right\|_{p,q}}{q^*(x)}\right)=0$$

where q^* is a weight function such that $\lim_{|x|\to\infty}\frac{1+x^2}{q^*(x)}=0$.

Proof. By hypothesis, given $\varepsilon>0$, there exists x_0 such that for all x with $|x|\geq x_0$ we have

$$\frac{1+x^2}{q^*(x)}<\varepsilon. \qquad (3.2)$$

Let $f \in L_{p,q}(loc)$. Then we get

$$
\begin{aligned}
\alpha_y \quad : \quad &= \left\| U_y f - f; L_p(|x| > x_0) \right\| \\
&= \sup_{|x|>x_0} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum\limits_n y^n T_n f - f \right|^p dt \right)^{1/p}}{1+x^2} \\
&\leq \sup_{x\in\mathbb{R}} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum\limits_n y^n T_n f \right|^p dt \right)^{1/p}}{1+x^2} + \sup_{x\in\mathbb{R}} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} |f|^p dt \right)^{1/p}}{1+x^2} \\
&\leq \left\| (1-y) \sum_n y^n T_n f \right\|_{p,q} + \|f\|_{p,q} \\
&\leq (1-y) \sum_n y^n \|T_n f\|_{p,q} + \|f\|_{p,q} \\
&\leq \|f\|_{p,q} \left((1-y) \sum_n y^n \|T_n\|_{p,q} + 1 \right) < N, \text{ say.}
\end{aligned}
$$

Hence we have α_y is bounded. By Lusin's theorem we can find a continuous function φ on $[-x_0 - h, x_0 + h]$ such that

$$\left\| f - \varphi; L_p(-x_0 - h, x_0 + h) \right\| < \varepsilon. \tag{3.3}$$

Now we consider the function G in [8] given by

$$
G(x) := \begin{cases} \varphi(-x_0 - h) & , \ x \leq -x_0 - h \\ \varphi(x_0) & , \ |x| < x_0 + h \\ \varphi(x_0 + h) & , \ x \geq x_0 + h. \end{cases}
$$

We see that G is continuous and bounded on the whole real axis. Now let $f \in L_{p,q}(loc)$ and we get that

$$
\begin{aligned}
\beta_y \quad : \quad &= \left\| U_y f - f; L_{p,q}(-x_0, x_0) \right\| \\
&= \left\| U_y(f - G); L_{p,q}(-x_0, x_0) \right\| + \left\| U_y G - G; L_{p,q}(-x_0, x_0) \right\| \\
&\quad + \left\| f - G; L_{p,q}(-x_0 - h, x_0 + h) \right\| \\
&\leq (1-y) \sum_n y^n \|T_n\|_{p,q} \left\| (f - G); L_{p,q}(-x_0 - h, x_0 + h) \right\| \\
&\quad + \left\| U_y G - G; L_{p,q}(-x_0, x_0) \right\| + \left\| f - G; L_{p,q}(-x_0 - h, x_0 + h) \right\| \\
&\leq \left\| f - G; L_{p,q}(-x_0 - h, x_0 + h) \right\| \left((1-y) \sum_n y^n \|T_n\|_{p,q} + 1 \right) \\
&\quad + \left\| U_y G - G; L_{p,q}(-x_0, x_0) \right\|.
\end{aligned}
$$

Hence by the hypothesis and Lemma 1 we have

$$\lim_{y\to 1^-} \beta_y = 0. \tag{3.4}$$

On the other hand, a simple calculation shows that

$$\begin{aligned}
\gamma_y \quad : &= \left\| (1-y) \sum_n y^n T_n f - f \right\|_{p,q^*} \\
&< \sup_{|x|<x_0} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum_n y^n T_n f - f \right|^p dt \right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
&\quad + \sup_{|x|\geq x_0} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| (1-y) \sum_n y^n T_n f - f \right|^p dt \right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
&= \beta_y \sup_{|x|<x_0} \frac{q(x)}{q^*(x)} + \alpha_y \sup_{|x|\geq x_0} \frac{q(x)}{q^*(x)} \\
&\quad < \beta_y q(x_0) + \varepsilon \alpha_y.
\end{aligned} \tag{3.5}$$

It follows from (3.2), (3.3) , (3.4) , (3.5) and Lemma 1 that

$$\begin{aligned}
\gamma_y &< q(x_0) \left\| f - G; L_{p,q}(-x_0 - h, x_0 + h) \right\| \left((1-y) \sum_n y^n \left\| T_n \right\|_{p,q} + 1 \right) \\
&\quad + q(x_0) \left\| U_y G - G; L_{p,q}(-x_0, x_0) \right\| + \varepsilon N \\
&= K\varepsilon + q(x_0) \left\| U_y G - G; L_{p,q}(-x_0, x_0) \right\|
\end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. By Lemma 1 we get

$$\lim_{y\to 1^-} \left(\sup_{x\in\mathbb{R}} \frac{\left\| U_y f - f; L_p(x-h, x+h) \right\|_{p,q}}{q^*(x)} \right) = 0.$$

4 Rates of Abel convergence in $L_{p,q}(loc)$

In this section, using the modulus of continuity, we study rates of convergence in $L_{p,q}(loc)$.

We now turn to introducing some notation and basic definitions to obtain the rate of convergence of the operators given in Theorem 3.

Also, we consider the following modulus of continuity:

$$w(f,\delta) = \sup_{|x-y|\leq\delta} |f(y) - f(x)|,$$

where δ is a positive constant, $f \in L_{p,q}(loc)$ and $q(x) = 1 + x^2$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(loc)$,

$$w(f,\delta) \leq (1 + [c])\, w(f,\delta),$$

where $[c]$ is defined to be the greatest integer less than or equal to c $(see\ [6])$.

To obtain our main results we first need the following lemma.

Lemma 2. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds.Then for each $y \in (0,1)$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have

$$\begin{aligned}\left\|U_y f - f; L_{p,q}(a,b)\right\| &\leq w(f;\delta)\left\|U_y f_0 - f_0\right\|_{p,q} \\ &\quad +2w(f;\delta) + C_1\left\|U_y f_0 - f_0\right\|_{p,q}\end{aligned}$$

where $f_0(t) = 1, \varphi_x(t) := (t-x)^2, C_1 = \sup\limits_{a<x<b} |f(x)|$ and $\delta := \alpha_k^{(n)} = \sqrt{\left\|U_y\varphi_x\right\|_{p,q}}$.

Proof. Let f be any continuous and bounded function on the real axis, and let $x \in [a,b]$ be fixed. Using linearity and monotonicity of T_n and for any $\delta > 0$, by modulus of continuity, we get

$$\begin{aligned}\left|U_y(f;x) - f(x)\right| &\leq U_y\left(w\left(f, \frac{|t-x|}{\delta}\delta\right), x\right) \\ &\quad + |f(x)|\left|U_y(f_0;x) - f_0(x)\right| \\ &\leq w(f,\delta)\left|U_y(f_0;x) - f_0(x)\right| + w(f,\delta) \\ &\quad + \frac{w(f,\delta)}{\delta^2}\left|U_y\varphi_x\right| + |f(x)|\left|U_y(f_0;x) - f_0(x)\right|.\end{aligned}$$

Now let $C_1 = \sup\limits_{a<x<b} |f(x)|$ and $\delta := \alpha_k^{(n)} = \sqrt{\left\|U_y\varphi_x\right\|_{p,q}}$. Then we have

$$\begin{aligned}\left\|U_y f - f; L_{p,q}(a,b)\right\| &\leq w_q(f,\delta)\sup_{a<x<b} q(x)\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} + w(f,\delta) \\ &\quad + \frac{w(f,\delta)}{\left(\sqrt{\left\|U_y\varphi_x\right\|_{p,q}}\right)^2}\left\|U_y\varphi_x\right\|_{p,q} \\ &\quad + \left\|U_y(f_0;x) - f_0(x)\right\|_{p,q}\sup_{a<x<b}|f(x)| \\ &= w(f,\delta)\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} + 2w(f,\delta) \\ &\quad + C_1\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q}.\end{aligned}$$

Theorem 4. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds. Assume that for each $y \in (0,1), \delta > 0$ and for each continuous and bounded function f on the real line, the following conditions hold:

$$(i)\ \lim_{y\to 1^-}\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} = 0,$$

$$(ii)\ \lim_{y\to 1^-} w(f,\delta) = 0.$$

Then we have

$$\lim_{y\to 1^-}\left\|U_y f - f; L_{p,q}(a,b)\right\| = 0.$$

Proof. Using Lemma 2 and considering (i) and (ii), we immediately get

$$\lim_{y\to 1^-}\left\|U_y f - f; L_{p,q}(a,b)\right\| = 0$$

for all continuous and bounded functions on the real axis.

Theorem 5. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself such that (2.1) holds. Assume that

$$\lim_{y\to 1^-}\left\|U_y(f_i;x) - f_i(x)\right\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. If

$$(i)\ \lim_{y\to 1^-}\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} = 0,$$
$$(ii)\ \lim_{y\to 1^-} w(G,\delta) = 0$$

where G is given in the proof of Theorem 3. Then, for $f \in L_{p,q}(loc)$, we have

$$\lim_{y\to 1^-}\left(\sup_{x\in\mathbb{R}}\frac{\left\|U_y f - f; L_p(x-h,x+h)\right\|}{q^*(x)}\right) = 0$$

where q^* is a weight function such that $\lim_{|x|\to\infty}\frac{1+x^2}{q^*(x)} = 0$.

Proof. It is known from Theorem 3 that

$$\begin{aligned} u_k^{(n)} &< q(x_0)\left\|f - G; L_{p,q}(-x_0-h, x_0+h)\right\|\left((1-y)\sum_n y^n\left\|T_n\right\|_{p,q} + 1\right) \\ &\quad + q(x_0)\left\|U_y G - G; L_{p,q}(-x_0,x_0)\right\| + \varepsilon N \\ &= K\varepsilon + q(x_0)\left\|U_y G - G; L_{p,q}(-x_0,x_0)\right\| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. Then by Lemma 2 and Theorem 4 we get

$$\begin{aligned} u_k^{(n)} &\leq K\varepsilon + q(x_0)\, w(G;\delta)\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} + 2q(x_0)\, w(G;\delta) \\ &\quad + q(x_0)\, C_1^{'}\left\|U_y(f_0;x) - f_0(x)\right\|_{p,q} \end{aligned}$$

where $C_1^{'} := \sup_{-x_0<x<x_0}|G(x)|$ and the proof is completed.

References

[1] F. Altomare and M. Campiti, Korovkin Type Approximation Theory and Its Applications, de Gruyter, Berlin, (1994).

[2] N. S. Bayram, Strong Summation Process in Locally Integrable Function Spaces, Hacettepe Journal of Mathematics and Statistics 45 (3) (2016), 683-694.

[3] S. J. Bernau, Theorems of Korovkin type for L_p spaces, Pacific J. Math. 53 (1974), 11-19.

[4] J. Boos, Classical and Modern Methods in Summability, Oxford Univ. Press, (2000).

[5] K. Donner, Korovkin theorems in L_p spaces, J. Functional Analysis 42 (1981), 12-28.

[6] O. Duman and C. Orhan, Rates of A- statistical convergence of operators in the space of locally integrable functions, Appl. Math. Letters, (2008), 431-435.

[7] V. K. Dzyadik, On the approximation of functions by linear positive operators and singular integrals, Mat. Sbornik 70 (112) (1966), 508-517 (in Russian).

[8] A. D. Gadjiev, R. O. Efendiyev and E. İbikli, On Korovkin's type theorem in the space of locally İntegrable functions, Czech. Math. J.,53 (128) (2003), 45-53.

[9] P. P. Korovkin, Linear Operators and The Theory of Approximation, India, Delhi, (1960).

[10] R. E Powell and S. M. Shah, Summability Theory and Its Applications, Van Nostrand Reinhold Company, London, (1972).

[11] İ. Sakaoğlu, L_p approximation via Abel convergence, Bull. Belg. Math. Soc. Simon Stevin 22 (2) (2015), 271-279.

[12] M. Unver, Abel transforms of positive linear operators on weighted spaces, Bull. Belg. Math. Soc. Simon Stevin 21 (5) (2014), 813-822.

[13] A. Zygmund, Trigonometric Series, Cambridge University Press, 1979.

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