

Higher Order Hochschild (Co)homology of Noncommutative Algebras

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Abstract

Hochschild (co)homology and Pirashvili's higher order Hochschild (co)homology are useful tools for a variety of applications including deformations of algebras. When working with higher order Hochschild (co)homology, we can consider the (co)homology of any commutative algebra with symmetric coefficient bimodules, however traditional Hochschild (co)homology is defined for any associative algebra with not necessarily symmetric coefficient bimodules. In the present paper, we generalize higher order Hochschild (co)homology to work with associative algebras which need not be commutative and in particular, show that simplicial sets admit such a generalization if and only if they are one dimensional.

Introduction

Hochschild (co)homology was first introduced in 1945 by Hochschild in [10]. Since then, mathematicians have found Hochschild (co)homology to be incredibly useful for a variety of applications. In the past couple decades, energy has been placed into generalizing Hochschild's construction. One generalization, higher order Hochschild homology was introduced by Pirashvili in [16]. This construction assigns a chain complex to any simplicial set, provided the algebra is commutative and the bimodule is symmetric. More recently, the author has worked to generalize Pirashvili's construction to work with multimodules, modules that have more than one action (see [2]). It is the aim of the present

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paper to further generalize this construction to noncommutative algebras. In particular, we will here determine a list of all simplicial sets whose Hochschild (co)homology is defined even when working over noncommutative algebras. We do so by finding the “maximal” algebraic structure allowed by each simplicial set X_\bullet . The main result found here is that the Hochschild (co)homology of X_\bullet is defined over a noncommutative algebra A if and only if X_\bullet is a one dimensional simplicial set.

Given a field, k , k -algebra, A and an A -bimodule, M , we can associate a chain complex $C_\bullet(A, M)$, whose homology was introduced by Hochschild in [10] and is referred to as the Hochschild homology of A with coefficients in M . To define $C_\bullet(A, M)$, let

$$C_n(A, M) := M \otimes A^{\otimes n}$$

and differentials be given by

$$\delta_n = \sum_{i=0}^n (-1)^i d_i$$

where $d_i: M \otimes A^{\otimes n} \rightarrow M \otimes A^{\otimes n-1}$ are defined as follows:

$$d_i(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 1 \leq i \leq n - 1 \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}$$

It can be observed that the construction above describes a simplicial k -module associated to a simplicial model of S^1 . Furthermore, Pirashvili illustrated in [16] that we can start with any arbitrary finite simplicial set and construct a simplicial k -module whose associated homology (of the complex given by alternating face maps) gives a generalization of Hochschild homology, known as higher order Hochschild homology. For this construction we start with the assumptions that A is a commutative k -algebra and M is a symmetric A -bimodule. These assumptions are necessary as we will see from Theorem 3.1. For example S^2 (with the minimal simplicial decomposition) requires that A be commutative (see Proposition 3.1). For higher order Hochschild homology, consider the Loday functor $L(A, M)$ from the category of finite pointed sets, Γ to the category of k -modules, $k\text{-mod}$ (see [12, 6.4.4] and [16]), given by

$$L(A, M): \Gamma \rightarrow k\text{-mod}$$

$$m_+ \rightarrow M \otimes \bigotimes_{\{i \in m_+ | i \neq 0\}} A$$

for objects $m_+ = \{0, 1, \dots, m\} \in \Gamma$. (where 0 is the fixed element). For morphisms $\varphi: m_+ \rightarrow n_+ \in \Gamma$ let

$$L(A, M)\varphi(m \otimes a_1 \otimes \cdots \otimes a_m) = (b_0 m \otimes \cdots \otimes b_n)$$

where

$$b_i = \prod_{\{j \in m_+ | j \neq 0, \varphi(j)=i\}} a_j.$$

Similarly we have a functor [8]

$$\mathcal{H}(A, M): \Gamma \rightarrow k\text{-mod}$$

$$m_+ \rightarrow \text{hom}\left(\bigotimes_{\{i \in m_+ | i \neq 0\}} A, M\right)$$

where for a map $\varphi: m_+ \rightarrow n_+$ and map $f: \bigotimes_{\{i \in n_+ | i \neq 0\}} A \rightarrow M$ we have

$$\mathcal{H}(A, M)\varphi(f)(a_1 \otimes \cdots \otimes a_m) = b_0 f(b_1 \otimes \cdots \otimes b_n)$$

where

$$b_i = \prod_{\{j \in m_+ | j \neq 0, \varphi(j)=i\}} a_j.$$

It was realized by Pirashvili in [16, 3.1] that for any finite simplicial set X_\bullet , we can consider the chain complex (after defining differentials to be the sums of alternating face maps)

$$\Delta^{op} \xrightarrow{X_\bullet} \Gamma \xrightarrow{L(A, M)} k\text{-mod}$$

or similarly the cochain complex (see [8])

$$\Delta^{op} \xrightarrow{X_\bullet} \Gamma \xrightarrow{\mathcal{H}(A, M)} k\text{-mod}.$$

The resulting homologies are referred to as higher order Hochschild homology and higher order Hochschild cohomology respectively.

To see how this generalizes traditional Hochschild (co)homology we consider the minimal simplicial decomposition of the pointed simplicial set $S^1_\bullet: \Delta^{op} \rightarrow \Gamma$ (with one non-degenerate 1-simplex). We have a simplicial k -module

$$\Delta^{op} \xrightarrow{S^1_\bullet} \Gamma \xrightarrow{L(A, M)} k\text{-mod}$$

which gives the Hochschild chain complex. The associated cochain complex $\Delta^{op} \xrightarrow{S^1_\bullet} \Gamma \xrightarrow{\mathcal{H}(A, M)} k\text{-mod}$ is the Hochschild cochain complex. The resulting homologies are known as Hochschild homology of A with coefficients in M and Hochschild cohomology of A with coefficients in M respectively.

Hochschild (co)homology and higher order Hochschild (co)homology have been shown to be incredibly useful tools for a variety of concepts. Hochschild cohomology and higher order Hochschild cohomology have been used to study deformations of algebras and modules (for example see [5], [6], [7], [11], [12], [17] and [18]). The ability to use higher order Hochschild cohomology for additional deformations is a primary inspiration for this work. One has to wonder if deformations of noncommutative algebras would be possible in the higher order setting and this paper seeks to provide those studying deformation theory with an answer about the types of algebras allowed.

One thing that can be noticed when considering the history of Hochschild cohomology is that traditional Hochschild (co)homology is of any associative k -algebra, A with coefficients in any A -bimodule, M , however higher order

Hochschild (co)homology was restricted to commutative k -algebras with coefficients in symmetric A -bimodules. In [2] the author generalizes the higher order Hochschild construction to have coefficients in not necessarily symmetric multimodules (see Definition 1.1). The modules that can be used depend heavily on the simplicial sets that the (co)chain complexes are built over. In this paper we aim to generalize higher order Hochschild (co)homology to work with not necessarily commutative algebras and not necessarily symmetric multimodules. In particular, we will show that simplicial sets allow such a generalization to noncommutative algebras if and only if they are one dimensional. We will do so by demonstrating the best case scenario for each simplicial set. Our construction for all arbitrary simplicial sets is in line with Pirashvili and Richter's construction in [15] (see Subsection 2.1) where they gave a description of a simplicial noncommutative circle as a functor from the simplicial category, Δ^{op} to the category of finite noncommutative sets, which allowed them to construct Hochschild (co)homology (as well as other homologies of functors) with not necessarily commutative algebras and not necessarily symmetric bimodules.

While this paper is focused on simplicial sets and Pirashvili's generalization, it is worth mentioning that there are additional equivalent generalizations. In particular, Lurie's topological chiral homology (see [13]), which is equivalent to Francis's factorization homology (see [4]) is a generalization of Hochschild homology, which associates to a n -framed manifold N and little n -cubes algebra A , a chain complex. When N is S^1 , and A is an associative algebra, it has been shown by both Lurie and Francis that the chain complex is equivalent to the traditional Hochschild chain complex. Furthermore, Ginot, Tradler and Zeinalian have shown in [9] that for a commutative algebra A and n -framed manifold N , factorization/topological chiral homology is equivalent to the higher order Hochschild chain complex. For direct constructions of these chain complexes, see [14]. In [1], Ayala and Francis have even shown that there is an equivalence between the category of little n -cubes algebras and homology theories of n -framed manifolds via factorization homology.

0.1 Organization of Paper

In Section 1, we provide background knowledge about the authors previous work on coefficient modules. In Section 2 we describe how to construct the higher order Hochschild cohomology cochain complex for algebras which are not necessarily commutative. We do so by describing the necessary characteristics a simplicial set must have in order to work with noncommutative algebras, which amounts to considering orderings on fibers of face maps. Our main theorem is provided in Section 3. It determines that the only simplicial sets that can work with noncommutative algebras are one dimensional. In Section 4 we note that the construction in this paper can be extended to higher order Hochschild cohomology of pairs of simplicial sets from [3].

1 Background

In this paper we fix a field k and denote \otimes_k by \otimes . We also choose to work with finite simplicial sets and any mention of a simplicial set comes with the added assumption that the simplicial set is finite. We assume that the reader is familiar with simplicial sets and has some familiarity with Hochschild (co)homology.

1.1 Higher Order Hochschild Cohomology

If we consider the definition of higher order Hochschild homology we notice that the action of A on M tells us what happens in the M tensor factor of the codomain. The reason M is not allowed to be a nonsymmetric bimodule is because it would be difficult in general to determine which of the two actions of A on M to choose for each face map. In addition, to get a simplicial k -module it is easier to satisfy the composition laws by choosing one action (which is forced to be both a left and right action since A is commutative). In [2] the author constructs a generalized version of higher order Hochschild cohomology which allows multimodule coefficients. In order to allow more actions for the coefficient modules and still get a cosimplicial k -module, we need to make action identifications.

Before proceeding with the main theorem from [2], we provide a definition of multimodules.

Definition 1.1. *Let A be a k -algebra. A left n -multimodule is an abelian group M with n distinct left A -module structures ι_i which commute in the sense that $\iota_i(a_i)\iota_j(a_j)m = \iota_j(a_j)\iota_i(a_i)m$ for all $a_i, a_j \in A, m \in M, 1 \leq i \neq j \leq n$. Similarly, a right n -multimodule is an abelian group M with n distinct right A -module structures ρ_i which commute in the sense that $m\rho_i(a_i)\rho_j(a_j) = m\rho_j(a_j)\rho_i(a_i)$ for all $a_i, a_j \in A, m \in M, 1 \leq i \neq j \leq n$. Lastly, a left/right (l, r) -multimodule is an abelian group M which has a left l -multimodule structure as well a right r -multimodule structure, where left and right actions commute in the sense that $(\iota_i(a_i)m)\rho_j(a_j) = \iota_i(a_i)(m\rho_j(a_j))$ for all $a_i, a_j \in A, m \in M, 1 \leq i \leq l, 1 \leq j \leq r$.*

We will usually drop the n or (l, r) from our notation and simply call M a multimodule.

Remark 1.2. *There are times when the type of action (left/right) will be unknown. In these situations we let Λ_i denote such an action. When we determine whether the action is left or right (see Proposition 2.1) we can replace Λ_i with the appropriate notation (ι_i or ρ_i).*

Example 1.3. *An A -bimodule M has a left action and a right action so M is an example of an A left/right $(1,1)$ -multimodule with two actions.*

Example 1.4. *Let M and N be two A -bimodules. $M \otimes N$ has an A -bimodule structure from the left action of A on M and the right action of A on N . If in addition, A is commutative, then there is an additional module structure coming from the identified right action of A on M with the left action of A on N in the definition of the tensor product of modules. In this case, $M \otimes N$ is a multimodule with three actions.*

For motivation on how multimodules can be used in higher order Hochschild cohomology, consider the following example.

Example 1.5. Let A be a k -algebra and M be a left/right $(2, 2)$ -multimodule with left actions ι_1 and ι_2 and right actions ρ_1 and ρ_2 . We have a cosimplicial k -module with the n th dimension given by $\text{hom}_k(A^{\otimes 2n}, M)$ and coface maps are given by

$$\begin{aligned} d^0 f(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_{2n}) &= \\ \iota_1(a_1)\iota_2(a_{n+1})f(a_2 \otimes \cdots \otimes a_n \otimes a_{n+2} \otimes \cdots \otimes a_{2n}) &= \\ d^i f(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_{2n}) &= \\ f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_{n+i} a_{n+i+1} \otimes \cdots \otimes a_{2n}) & \end{aligned}$$

for $0 < i < n$

$$\begin{aligned} d^n f(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_{2n}) &= \\ f(a_1 \otimes \cdots \otimes a_{n-1} \otimes a_{n+1} \otimes \cdots \otimes a_{2n-1})\rho_1(a_n)\rho_2(a_{2n}) & \end{aligned}$$

By taking the sum of alternating coface maps as differentials, the resulting homology is the higher order Hochschild cohomology over $S^1 \vee S^1$ of A with coefficients in M .

We now give the main theorem of [2] which tells us how many actions the coefficient modules for higher order Hochschild cohomology each simplicial set is allowed by identifying certain actions. Note that when A is commutative, any left action also satisfies the definition of a right action. The module below is just assumed to be a multimodule.

Theorem 1.6. [2, 1.1] Let A be a commutative algebra, X_\bullet be a simplicial set, and M be a multimodule, with a (left or right) module structure $\Lambda_{(i,n)}^\sigma$ for each $n \geq 0, 0 \leq i \leq n, \sigma \in X_n$. Then there is a cosimplicial k -module $(A, M, X)^\bullet$ which has n -cosimplices

$$(A, M, X)^n = \text{hom}_k(k \otimes_k \bigotimes_{\substack{\sigma \in X_n \\ \sigma \neq *}} A, M),$$

coface maps

$$d_n^i f(1 \otimes_k \bigotimes_{\substack{\sigma \in X_{n+1} \\ \sigma \neq *}} a_\sigma) = \prod_{\substack{\sigma \in X_{n+1} \\ d_i(\sigma) = *}} (\Lambda_{(i,n)}^\sigma(a_\sigma)) \cdot f(1 \otimes_k \bigotimes_{\substack{\Omega \in X_n \\ \Omega \neq *}} \prod_{\substack{\sigma \in X_{n+1} \\ d_i(\sigma) = \Omega}} a_\sigma)$$

and codegeneracy maps

$$s_n^i f(1 \otimes_k \bigotimes_{\substack{\sigma \in X_{n+1} \\ \sigma \neq *}} a_\sigma) = f(1 \otimes_k \bigotimes_{\substack{\Omega \in X_{n+1} \\ \Omega \neq *}} 1 \cdot \prod_{\substack{\sigma \in X_n \\ s_i(\sigma) = \Omega}} a_\sigma)$$

provided that the structure maps $\Lambda_{(i,n)}^\sigma$ satisfy the following properties:

- i) $\Lambda_{(j,n+1)}^\sigma = \Lambda_{(i,n+1)}^\sigma$ if $\sigma \neq *, d_i(\sigma) = d_j(\sigma) = *$ and the dimension of σ is at least 2 and $i < j$.

- ii) $\Lambda_{(j,n+1)}^\sigma = \Lambda_{(j-1),n}^\Omega$ if $d_i(\sigma) = \Omega, d_j(\sigma) = *, d_{j-1}(\Omega) = *$ and the dimension of σ is at least 2.
- iii) $\Lambda_{(i,n)}^\Omega = \Lambda_{(j-1),n}^\mu$ if $d_i(\Omega) = *, d_{j-1}(\mu) = *$ and there exists a σ of dimension at least 2 where $d_j(\sigma) = \Omega, d_i(\sigma) = \mu$ and $i < j$.
- iv) $\Lambda_{(i,n)}^\Omega = \Lambda_{(i,n+1)}^\sigma$ if $d_i(\sigma) = *, d_i(\Omega) = *, d_j(\sigma) = \Omega$ and the dimension of σ is at least 2.

where $\Lambda_{(i,n)}^\sigma(a)$ represents the $\Lambda_{(i,n)}^\sigma$ action of $a \in A$ whenever $0 \leq i \leq n, \sigma \in X_{n+1}$ and $d_i(\sigma) = *$.

2 Not Necessarily Commutative Algebras

We would like to develop higher order Hochschild (co)homology to accept non-commutative algebras, but in order to do so we need to provide an order in which to multiply elements in the equation from Theorem 1.6. In the following section we will determine algebras and modules allowed for higher order Hochschild cohomology and leave it to the reader to check that the same holds for homology.

Definition 2.1. Let X_\bullet be a simplicial set and $f: X_n \rightarrow X_m$ be a map of sets. By f -ordering of X_n , we mean an ordering of the simplices in each of the f -fibers: $S = \{\sigma_i | f(\sigma_i) = \tau\}$ for some $\tau \in X_m$.

Now if we consider all d_i -orderings of X_n we see that this gives a way to multiply elements of our algebra for each d_i if A is not necessarily commutative (where products of elements in each tensor factor are multiplied with elements represented by smaller simplices on the left and elements represented by larger simplices on the right). The main issue is that in order for the associated cosimplicial k -module to satisfy the cosimplicial identities, we need the d_i -orderings to have some compatibilities. We start with the following remark.

Definition 2.2. For any two maps $f: X_n \rightarrow X_m$ and $g: X_m \rightarrow X_k$, given an f -ordering on X_n and a g -ordering on X_m , there is an induced ordering on X_n called the composition ordering from gf . For simplices σ_i and σ_j in X_n we say that $\sigma_i < \sigma_j$ in the composition ordering from gf if and only if $\sigma_i < \sigma_j$ in the f -ordering or $f(\sigma_i) < f(\sigma_j)$ in the g -ordering.

Definition 2.3. Given a simplicial set X_\bullet with a choice of simplicial orderings for each composition of face maps, we say that X_\bullet admits a multiplicative ordering if for each composition of face maps $f = d_{i_k} \cdots d_{i_1}: X_n \rightarrow X_{n-k}$ and $g = d_{i_{j+k}} \cdots d_{i_{j+1}}: X_{n-k} \rightarrow X_{n-j-k}$ the composition ordering from gf agrees with the gf -ordering for all gf -fibers whose image under gf is not the basepoint.

Our goal is to generalize Theorem 1.6 to noncommutative algebras. Thus far we have considered orderings that will allow us to multiply elements in the cosimplicial module $(A, M, X)^\bullet$. Notice that we have not considered how this will affect module actions. To take this into account, we make use of the following definition.

Definition 2.4. A symmetric multimodule is a multimodule for which the actions are both left and right actions at the same time.

Later, we will see that symmetry is not always necessary (see Remark 3.5), but when the simplicial set is arbitrary, symmetry is a necessity (see Section 2.2).

Theorem 2.5. Let X_\bullet be a simplicial set, A be an associative algebra and M a symmetric multimodule. The collection of modules and maps $(A, M, X)^\bullet$, d, s of Theorem 1.6 defines a cosimplicial k -module for any (not necessarily commutative) algebra A and any multimodule M as in Theorem 1.6 (satisfying the same conditions) if and only if X admits a multiplicative ordering.

Proof. From [2] we see that if A is a commutative algebra, then $(A, M, X)^\bullet$ exists. To see that $(A, M, X)^\bullet$ still exists when A is not necessarily commutative and X_\bullet admits a multiplicative ordering, we first notice that the orderings of X_\bullet give an ordering for the multiplication in each tensor factor for each coface map in $(A, M, X)^\bullet$. It is straightforward to see that the compatibility of orderings, coming from the fact that X_\bullet admits a multiplicative ordering forces the composition of coface maps to be well defined with respect to the order in which elements in each tensor factor are to be multiplied. For the other direction, notice if $(A, M, X)^\bullet$ exists, then each coface map provides an ordering for the associated face maps in X_\bullet . We see that these orderings must be compatible for $(A, M, X)^\bullet$ to be a well defined cosimplicial k -module, which implies that X_\bullet admits a multiplicative ordering. ■

2.1 Relationship to Pirashvili and Richter's construction

In [15], Pirashvili and Richter use a similar approach when defining functor homology. Starting with the category of pointed noncommutative finite sets $\Gamma(as)$, they show that the traditional Hochschild chain complex is given by the composition of functors

$$L(A, M) \circ \hat{C}: \Delta^{op} \rightarrow \Gamma(as) \rightarrow k\text{-mod}$$

where $\Gamma(as)$ is defined to be the category of pointed finite sets Γ with the property that maps have a total ordering on preimages and \hat{C} is a lifting of the pointed simplicial circle $C: \Delta^{op} \rightarrow \Gamma$.

This construction is very similar to the way the present paper considers simplicial sets which admit a multiplicative ordering. In fact, any simplicial set X_\bullet which has a lifting to $\hat{X}_\bullet: \Delta^{op} \rightarrow \Gamma(as)$ will admit a multiplicative ordering and thus has the property that $(A, M, X)^\bullet$ is a cosimplicial k -module for any associative algebra A . The main difference between the approach in [15] and the approach here is that it is conceivable that X_\bullet can admit a multiplicative ordering without inducing f -orderings on the subsets of X_n whose image is the basepoint. This does not guarantee a lifting $\hat{X}_\bullet: \Delta^{op} \rightarrow \Gamma(as)$. We show that the absence of an f -ordering on such subsets requires M to have a specific type of action in the next subsection. In addition, the main goal of Pirashvili and Richter was to construct a generalization of Hochschild and cyclic homology, while the goal in the

present paper is to provide a description of simplicial sets which still work with noncommutative algebras. We determine an exact list of simplicial sets in Section 3 by using this approach.

2.2 Module actions

Before getting to the main theorem which determines precisely which simplicial sets allow noncommutative algebras, we will consider what multimodule actions arise if A is noncommutative. For a commutative algebra A and A -module M , it can be seen that any left action is also by definition a right action (we will denote such an action as an *lr action*), but for noncommutative algebras, coefficient modules need not have lr actions. To see what actions on coefficient modules need not be lr actions we consider the following proposition.

Proposition 2.1. *Let A be a (not necessarily commutative) algebra, X_\bullet be a simplicial set with a multiplicative ordering, M be a multimodule with a module structure $\Lambda_{(i,n)}^\sigma$ for each $n \geq 0, 0 \leq i \leq n$, and $\sigma \in X_n$. For the cosimplicial k -module $(A, M, X)^\bullet$ suppose there exists $\sigma < \tau \in X_n$ with $d_{i_t} \cdots d_{i_1}(\sigma) = d_{i_t} \cdots d_{i_1}(\tau) = \omega \neq *$ with $d_{i_r} \cdots d_{i_{t+1}}(\omega) = *$ and $d_{i_{r-1}} \cdots d_{i_{t+1}}(\omega) \neq *$ then $\Lambda_{(i_r, n-r)}^{d_{i_{r-1}} \cdots d_{i_{t+1}}(\omega)}$ is necessarily a*

- *left action if there exists a set of maps $d_{j_r} \cdots d_{j_1} = d_{i_r} \cdots d_{i_1}$ with the property that $d_{j_l} \cdots d_{j_1}(\sigma) = *$ but $d_{j_l} \cdots d_{j_1}(\tau) \neq *$ for some $l < r$.*
- *right action if there exists a set of maps $d_{j_r} \cdots d_{j_1} = d_{i_r} \cdots d_{i_1}$ with the property that $d_{j_l} \cdots d_{j_1}(\tau) = *$ but $d_{j_l} \cdots d_{j_1}(\sigma) \neq *$ for some $l < r$.*
- *lr action if $\Lambda_{(i_r, n-r)}^{d_{i_{r-1}} \cdots d_{i_{t+1}}(\omega)}$ is both a right and left action.*

Before providing the proof, consider the following example.

Example 2.6. *For the minimal simplicial decomposition of S^1 , let σ be the nondegenerate one-dimensional simplex. We have that the following properties are satisfied:*

$$d_1 s_0 \sigma = d_1 s_1 \sigma,$$

$$d_1 d_1 s_0 \sigma = *,$$

$$d_1 d_1 = d_1 d_2,$$

$$d_2 s_1 \sigma \neq *$$

and

$$d_2 s_0 \sigma = *.$$

If $s_0 \sigma < s_1 \sigma$ in the d_1 ordering we get that $\Lambda_{(1,0)}^\sigma$ needs to be a left action by the properties above. Similarly, we can see that $\Lambda_{(0,0)}^\sigma$ needs to be a right action.

Proof of Proposition 2.1. Let σ, τ and ω be the simplices from the proposition above. We will prove the proposition for a right action, but note that an analogous proof works for left actions. We can assume that there exists $1 \leq t < s < r$ so that

$$d_{j_s} \cdots d_{j_1}(\sigma) = *$$

$$d_{j_{s-1}} \cdots d_{j_1}(\sigma) \neq *$$

$$d_{j_t} \cdots d_{j_1}(\tau) = *$$

$$d_{j_{t-1}} \cdots d_{j_1}(\tau) \neq *$$

and from [2] $\Lambda_{(j_s, n-s)}^{d_{j_{s-1}} \cdots d_{j_1}(\sigma)} = \Lambda_{(j_t, n-t)}^{d_{j_{t-1}} \cdots d_{j_1}(\tau)} = \Lambda_{(i_r, n-r)}^{d_{j_{r-1}} \cdots d_{j_{k+1}}(\omega)}$ which we will simply denote as Λ , but $d^{j_1} \cdots d^{j_r} f(-) = d^{i_1} \cdots d^{i_r} f(-)$ and among other elements, we see that on the left we have a_σ followed by a_τ acting on $f(-)$ and on the right we have $a_\sigma a_\tau$ acting on $f(-)$ so we have $(f(-)a_\sigma)a_\tau = f(-)(a_\sigma a_\tau)$ so Λ is a right action. ■

This gives whether an action Λ is left/right or an lr action from the perspective of each simplex. Recall from Theorem 1.6 that there are many action identifications so to determine if Λ need only be left/right we need to actually consider all action identifications i.e. if $\Lambda^\sigma = \Lambda^\tau$ and Λ^σ is a left action while Λ^τ is a right action, then they are both the same lr action.

3 Simplicial sets that work

In general, given a simplicial set X_\bullet , it seems that it should be a daunting task to determine if X_\bullet admits a multiplicative ordering. Unfortunately, it turns out that very few simplicial sets admit a multiplicative ordering. We start by showing which simplicial sets do not. For a motivating example, we consider a simplicial model for S^2

Proposition 3.1. *The minimal simplicial decomposition of S^2 with one nondegenerate 0-simplex and one nondegenerate 2-simplex does not admit a multiplicative ordering.*

Proof. Consider the 4-simplices $[00112], [00122], [01122], [01112]$. Each of these simplices is carried to the 2-simplex $[012]$ by the composition of face maps $d_2 d_1 = d_1 d_3$. We have a d_1 -ordering for the sets $\{[00112], [01112]\}$ and $\{[00122], [01122]\}$, but there is also a d_3 -ordering for the sets $\{[00112], [00122]\}$ and $\{[01122], [01112]\}$. It can be seen that no ordering of these four simplices for $d_2 d_1 = d_1 d_3$ will agree with both the d_3 -ordering and the d_1 -ordering. ■

One might guess that if S^2 does not admit a multiplicative ordering, then it may not be possible for any simplicial set of dimension equal or greater than 2 to admit a multiplicative ordering. As one might expect, we can only work with one dimensional simplicial sets (simplicial sets X whose only nondegenerate simplices are in X_0 or X_1). This brings us to the main theorem.

Theorem 3.1. *Let X_\bullet be a simplicial set, then X_\bullet admits a multiplicative ordering if and only if X_\bullet is a one dimensional simplicial set.*

Remark 3.2. *The implication here is that higher order Hochschild (co)homology of a noncommutative algebra A only works if the simplicial set is one dimensional.*

Before proving Theorem 3.1, let us consider examples of some one dimensional simplicial sets which do admit a multiplicative ordering. We start by describing a “nicer” ordering on fibers of face maps.

Definition 3.3. *We say that a pointed simplicial set X_\bullet has a cyclic ordering if for every $n \geq 1$, each set $X_n \setminus \{*\}$ has an ordering with the property that if $\sigma, \tau \in X_n \setminus \{*\}$ ($n \geq 2$) with $\sigma < \tau$ then whenever $d_i(\sigma) \neq *$ and $d_i(\tau) \neq *$, $d_i(\sigma) \leq d_i(\tau)$ for all $0 \leq i \leq n$*

Remark 3.4. *We can see that a cyclic ordering on any simplicial set X_\bullet also provides a multiplicative ordering on X_\bullet by using the same orderings on X_\bullet for all face maps (notice that in both a cyclic ordering and multiplicative ordering we don’t need an ordering on the fibers over the basepoint). By considering Figure 1 below, we can also identify whether an action is left, right or lr.*

To motivate a proof that all one dimensional simplicial sets admit a multiplicative ordering, we start with the following.

Proposition 3.2. *Let X_\bullet be the minimal simplicial decomposition of $\bigvee_{i \in I} S^1$, then X_\bullet has a cyclic ordering.*

Proof. First notice that X_\bullet has one 1-simplex for each copy of S^1 and one degenerate 1-simplex for the basepoint $*$. The 1-simplices of $X_1 \setminus \{*\}$ can be ordered in any way, so all we need to do is order X_n for larger n . This can be done by giving a cyclic ordering for each sub-simplicial set S^1_\bullet . One such ordering is

$$\begin{aligned}
 S^1_1 &: [01] \\
 S^1_2 &: [001] < [011] \\
 S^1_3 &: [0001] < [0011] < [0111] \\
 S^1_4 &: [00001] < [00011] < [00111] < [01111] \quad \blacksquare
 \end{aligned}$$

We call the ordering above a cyclic ordering because we can actually order all of the S^1 n -simplices (including $*$ clockwise around a circle as is done in Figure 1 (starting with $*$ = $[0\dots 0]$ = $[1\dots 1]$) and see that face maps d_i have the property that $d_i(\sigma) = d_i(\tau)$ if and only if σ and τ are in the $n + 1 - i$ and $n + 2 - i$ places around the circle. From this, we can imagine that face maps essentially squeeze adjacent simplices together and do not change the order.

Remark 3.5. *In the above ordering, $\Lambda^{[0\dots 01]}$ is a left action, while $\Lambda^{[01\dots 1]}$ is a right action for each copy of S^1 . In this case the set of possible actions permitted by $\bigvee_{i \in I} S^1$ is twice as large as I , with half of the actions right and half of the actions left. This demonstrates a fact that we are already aware of—traditional Hochschild cohomology has the ability to work with non-commutative algebras and not necessarily symmetric bimodules as coefficient modules.*

We now proceed with the proof of Theorem 3.1.

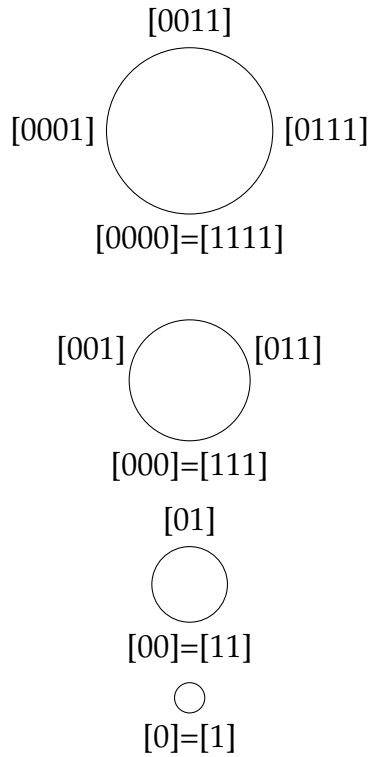


Figure 1: Cyclic ordering for S_{\bullet}^1

Proof of Theorem 3.1. Suppose X_{\bullet} is n -dimensional where $n \geq 2$ and let σ be a nondegenerate simplex whose dimension is greater than or equal to 2. Following a similar argument to the one in the proof of Proposition 3.1 we see that the simplices $s_2s_0\sigma, s_3s_0\sigma, s_3s_1\sigma, s_1s_1\sigma$ are carried to σ via $d_2d_1 = d_1d_3$. We have a d_1 -ordering for the sets $\{s_2s_0\sigma, s_1s_1\sigma\}$ and $\{s_3s_0\sigma, s_3s_1\sigma\}$ but we also have a d_3 -ordering for the sets $\{s_2s_0\sigma, s_3s_0\sigma\}$ and $\{s_3s_1\sigma, s_1s_1\sigma\}$. As in the proof of Proposition 3.1 we see that this will not allow X_{\bullet} to admit a multiplicative ordering. Now to see that any one dimensional simplicial set admits a multiplicative ordering, we actually show that any one dimensional simplicial set has a cyclic ordering. Notice that we can represent any simplex as the composition of degeneracy maps on a 1-simplex. Let us denote each simplex as follows: let $s_0s_3s_1s_1\sigma$ be denoted by $[001111]_{\sigma}$ if σ is a nondegenerate 1-simplex and $[000000]_{\sigma}$ otherwise. By ordering the 1-simplices, we can induce an order on the n -simplices by first ordering by subscript, so $[- \dots -]_{\sigma} < [- \dots -]_{\tau}$ if $\sigma < \tau$ in X_1 . We then order alphabetically. For example, if $\sigma < \tau$ in X_1 then $[001]_{\sigma} < [011]_{\sigma} < [001]_{\tau} < [011]_{\tau}$ in X_2 . It is straight forward that this provides X_{\bullet} with a cyclic ordering. ■

Remark 3.6. *This proof shows that given a simplicial set of dimension 1, one can always find a cyclic ordering.*

4 Secondary Hochschild cohomology and pairs of simplicial sets

In [17] Staic introduced secondary Hochschild cohomology which was used to study B -algebra structures on $A[t]$ given k -algebras A and B with a map $\varepsilon: B \rightarrow A$. In [3] the author and Staic show that secondary Hochschild cohomology is a version of higher order Hochschild cohomology by generalizing Hochschild cohomology to pairs of simplicial sets $X_\bullet \subseteq Y_\bullet$.

To extend noncommutativity to pairs of algebras A and B we first consider the simplicial set Y_\bullet . If Y_\bullet is one dimensional, then A and B can both be noncommutative. If Y_\bullet is not one dimensional B must be commutative. In the second case, we then consider the simplicial set X_\bullet . If X_\bullet is one dimensional, then A need not be commutative, however $\varepsilon(B)$ must be in the center of A . For module coefficients, we simply consider where the action comes from in the simplicial set.

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