Invariant φ-means for abstract Segal algebras related to locally compact groups

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Abstract

In this paper, for a locally compact group $\mathcal{G}$ we characterize character amenability and character contractibility of abstract Segal algebras with respect to the group algebra $L^1(\mathcal{G})$ and the generalized Fourier algebra $A_p(\mathcal{G})$. As a main result we prove that $\mathcal{G}$ is discrete and amenable if and only if some class of abstract Segal algebras in $L^1(\mathcal{G})$ is character amenable. We also prove a similar result for abstract Segal algebras in $A_p(\mathcal{G})$ and $C_0(\mathcal{G})$. Finally, under some conditions we investigate when a commutative, semisimple, Tauberian Banach algebra is an ideal in its second dual space.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A *derivation* from $\mathcal{A}$ into X is a linear map $D : \mathcal{A} \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in \mathcal{A}$. For example, the map $\mathtt{ad}_x : \mathcal{A} \to X$ defined by $\mathtt{ad}_x(a) = a \cdot x - x \cdot a$ is a derivation for all $a \in \mathcal{A}$ and $x \in X$. Such derivations are called *inner derivations*. Here, let us mention that X^*, the first dual space of X, becomes a Banach $\mathcal{A}$-bimodule in the natural way. The algebra $\mathcal{A}$ is called *amenable* if every continuous derivation $D : \mathcal{A} \to X^*$ is inner for each Banach $\mathcal{A}$-bimodule X. The notion of amenability for Banach algebras was introduced by B. E. Johnson [11].

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Let $\varphi \in \Delta(\mathcal{A})$, consisting of all non-zero characters on $\mathcal{A}$. Kaniuth, Lau and Pym in [12] and [13] introduced and studied the concept of *φ-amenability* for Banach algebras as a generalization of left amenability of Lau algebras or F-algebras [14]. In fact, $\mathcal{A}$ is called *φ-amenable* if every continuous derivation $D : \mathcal{A} \to X^*$ is inner for each Banach $\mathcal{A}$-bimodule X with the left module action given by $a \cdot x = \varphi(a)x$ for all $x \in X$ and $a \in \mathcal{A}$. Recall from [12] that $\mathcal{A}$ is φ-amenable if and only if there is a *φ-mean* in $\mathcal{A}^{**}$, i.e., a bounded linear functional m on $\mathcal{A}^*$ satisfying

$$\langle m, \varphi\rangle = 1, \quad \langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Around the same time, Monfared introduced and studied in [16] the notion of character amenability for Banach algebras. Precisely, a Banach algebra $\mathcal{A}$ is called *character amenable* if $\mathcal{A}$ is φ-amenable for all $\varphi \in \Delta(\mathcal{A})$ and it has a bounded right approximate identity. There are many characterizations for φ-amenability and character amenability of Banach algebras. For example in [12, Theorem 1.4], the authors characterized φ-amenability of Banach algebras in terms of the existence of *bounded approximate φ-means*, i.e., a bounded net (a_α) in $\mathcal{A}$ such that

$$\varphi(a_\alpha) \to 1, \quad \|aa_\alpha - \varphi(a)a_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. Also, concepts of *φ-contractibility* and *character contractibility* of $\mathcal{A}$ were laid by Hu, Monfared, and Traynor [10].

Recently, Nasr-Isfahani and the second author have introduced and studied *φ-pseudo-amenability* and *character pseudo-amenability* of $\mathcal{A}$ as a new notion of amenability based on the existence of approximate φ-means and right approximate identities (not necessarily bounded) [18].

Several authors has been studied various notions of amenability for abstract Segal algebras; see for example [1, 5, 6]. Here, we characterize som notions of amenability such as φ-amenability, character amenability, character pseudo-amenability, φ-contractibility and character contractibility of some Banach algebras. In particular, we offer some applications to the abstract Segal algebras related to group algebra and generalized Fourier-algebra of a locally compact group $\mathcal{G}$. Moreover, we investigate when certain abstract Segal algebras in $L^1(\mathcal{G})$ or $A_p(\mathcal{G})$ are ideals in their second dual spaces.

2 Character amenability of abstract Segal algebras

We start the paper with the definition of abstract Segal algebras.

Let $\mathcal{A}$ be a Banach algebra with the norm $\|\cdot\|_{\mathcal{A}}$. Then a Banach algebra $\mathcal{B}$ with the norm $\|\cdot\|_{\mathcal{B}}$ is an abstract Segal algebra with respect to $\mathcal{A}$ if

(1) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$.

(2) There exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$ for each $b \in \mathcal{B}$.

(3) There exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ for all $a, b \in \mathcal{B}$.

We recall from [2, Theorem 2.1] that $\Delta(\mathcal{B})$ and $\Delta(\mathcal{A})$ are homeomorphic.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathcal{A})$. We say that $\mathcal{A}$ is *φ-pseudo-amenable* if it has an approximate φ-mean (not necessarily bounded).

We also say that $\mathcal{A}$ is *character pseudo-amenable* if $\mathcal{A}$ has a right approximate identity and it is φ-pseudo-amenable for all $\varphi \in \Delta(\mathcal{A})$.

Let $\mathcal{G}$ be a locally compact group with left Haar measure λ and let $L^1(\mathcal{G})$ be the group algebra of $\mathcal{G}$ endowed with the norm $\|\cdot\|_1$ and the convolution product $*$ and let $M(\mathcal{G})$ be the measure algebra of $\mathcal{G}$. We denote by $\widehat{\mathcal{G}}$ the set of all continuous homomorphisms ρ from $\mathcal{G}$ into the circle group $\mathbb{T}$, and define $\varphi_\rho \in \Delta(L^1(\mathcal{G}))$ to be the character induced by ρ on $L^1(\mathcal{G})$; that is,

$$\varphi_\rho(h) = \int_G \overline{\rho(x)} f(x)\, d\lambda(x) \quad (f \in L^1(\mathcal{G})).$$

It is known that there are no other characters on $L^1(\mathcal{G})$; that is,

$$\Delta(L^1(\mathcal{G})) = \{\varphi_\rho : \rho \in \widehat{\mathcal{G}}\};$$

see, for example [9, Theorem 23.7]. Recall that, $\mathcal{G}$ is called *amenable* if there is a bounded net $(k_\alpha) \subseteq L^1(\mathcal{G})$ such that $k_\alpha \geq 0$, $\|k_\alpha\|_1 = 1$ and $\|\delta_x * k_\alpha - k_\alpha\|_1 \to 0$ for all $x \in G$, where δ_x denotes the Dirac measure at $x \in G$; see Runde [19] for details.

Lemma 2.2. *Let $\mathcal{G}$ be a locally compact group and let $S(\mathcal{G})$ be an abstract Segal algebra with respect to $L^1(\mathcal{G})$ such that $\rho \in \widehat{\mathcal{G}}$. Then $S(\mathcal{G})$ is $\varphi_\rho|_{S(\mathcal{G})}$-pseudo-amenable if and only if $\mathcal{G}$ is amenable.*

Proof. Suppose that $\mathcal{G}$ is amenable. Then $S(\mathcal{G})$ is $\varphi_\rho|_{S(\mathcal{G})}$-amenable by [1, Corollary 3.4]. Consequently, it is $\varphi_\rho|_{S(\mathcal{G})}$-pseudo-amenable.

Conversely, suppose that $S(\mathcal{G})$ is $\varphi_\rho|_{S(\mathcal{G})}$-pseudo-amenable. Then $L^1(\mathcal{G})$ is φ_ρ-pseudo-amenable by the same argument used in the proof of [1, Proposition 2.3]. Moreover, it is easy to see that $L^1(\mathcal{G})$ is φ_1-pseudo-amenable if and only if it is φ_ρ-pseudo-amenable; this is an immediate consequence of the following equality

$$\|f * h - \varphi_\rho(f)h\|_1 = \|(\overline{\rho} f) * (\overline{\rho} h) - \varphi_1(\overline{\rho} f)(\overline{\rho} h)\|_1$$

for all $f, h \in L^1(\mathcal{G})$. Now, suppose that $(f_\alpha) \subseteq L^1(\mathcal{G})$ is an approximate φ_1-mean and let $h_\alpha = h * f_\alpha$, where $f_0 \in L^1(\mathcal{G})$ with $\varphi_1(f_0) = \int_{\mathcal{G}} f_0 = 1$. Then for each $x \in \mathcal{G}$ we have

$$\|\delta_x * h_\alpha - h_\alpha\|_1 \leq \|(\delta_x * f_0) * f_\alpha - f_\alpha\|_1 + \|f_0 * f_\alpha - f_\alpha\|_1 \to 0.$$

We may assume that $\|h_\alpha\|_1 \geq 1/2$ for all α. So, if we put

$$k_\alpha := |h_\alpha| / \|h_\alpha\|_1$$

for all α, then we have $k_\alpha \geq 0$, $\|k_\alpha\|_1 = 1$ and $\|\delta_x * k_\alpha - k_\alpha\|_1 \to 0$ for all $x \in G$; indeed,

$$\|\delta_x * |h_\alpha| - |h_\alpha|\|_1 = \||\delta_x * h_\alpha| - |h_\alpha|\|_1 \leq \|\delta_x * h_\alpha - h_\alpha\|_1 \to 0.$$

Thus, $\mathcal{G}$ is amenable. ∎

For a locally compact group $\mathcal{G}$ and $1 < p < \infty$, denote by $A_p(\mathcal{G})$ the generalized Fourier algebra of $\mathcal{G}$ as defined in [4]. Elements of $A_p(\mathcal{G})$ can be represented, nonuniquely, as $u = \sum_{i=1}^{\infty}(f_i * \check{g}_i)$, where $f_i \in L^p(\mathcal{G})$, $g_i \in L^q(\mathcal{G})$, $\frac{1}{p} + \frac{1}{q} = 1$, $\check{g}(x) = g(x^{-1})$ and $\sum_{i=1}^{\infty} \|f_i\|_p \|\|g_i\|_q < \infty$. Then

$$\|u\|_{A_p} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_p \|\|g_i\|_q : u = \sum_{i=1}^{\infty}(f_i * \check{g}_i)\right\}$$

is a norm on $A_p(\mathcal{G})$. For $p = 2$, $A_p(\mathcal{G})$ coincides with the Fourier algebra $A_2(G)$ which is denoted by $A(\mathcal{G})$. With the usual operations of pointwise addition and multiplication, $A_p(G)$ is a commutative Banach algebra and $A_p(\mathcal{G}) \subseteq C_0(\mathcal{G})$, the space of continuous functions on $\mathcal{G}$ vanishing at infinity. We also recall that the spectrum of $A_p(\mathcal{G})$ can be canonically identified with $\mathcal{G}$. More precisely, the map $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in A_p(\mathcal{G})$, is a homeomorphism from $\mathcal{G}$ onto $\Delta(A_p(\mathcal{G}))$. The

In the sequel, we characterize character amenability of abstract Segal algebras with respect to them.

Lemma 2.3. *Let $\mathcal{G}$ be a locally compact group, and $\mathcal{A}$ be either of the Banach algebras $L^1(\mathcal{G})$ or $A_p(\mathcal{G})$ for $1 < p < \infty$. Suppose that $\mathcal{B}$ is an abstract Segal algebra with respect to $\mathcal{A}$. Then the following statements are equivalent.*

(a) *$\mathcal{B}$ is character amenable.*

(b) *$\mathcal{B}$ is character pseudo-amenable and has a bounded right approximate identity.*

(c) *$\mathcal{B} = \mathcal{A}$ and $\mathcal{G}$ is amenable.*

Proof. That (a) implies (b) is trivial.

(b)⇒(c). First let $\mathcal{A} = A_p(\mathcal{G})$. Then it is known that $A_p(\mathcal{G})$ is φ_x-amenable for all $x \in G$; see [15, Lemma 3.1]. Thus, $\mathcal{B}$ is φ_x-amenable by [1, Proposition 2.3]. Now, if $\mathcal{B}$ has a bounded approximate identity, then $\mathcal{B} = \mathcal{A}$; see [1, Theorem 2.1]. Hence, $\mathcal{G}$ is amenable by [8, Theorem 6].

(c)⇒(a). When $\mathcal{A} = A_p(\mathcal{G})$, this follows from the fact that $\mathcal{G}$ is amenable if and only if $A_p(\mathcal{G})$ has a bounded approximate identity. For the case of $\mathcal{A} = L^1(\mathcal{G})$, it follows from Lemma 2.2 and [1, Theorem 2.1]. ∎

For $1 < p \leq \infty$, let

$$\mathcal{L}^1_p(\mathcal{G}) := L^1(\mathcal{G}) \cap L^p(\mathcal{G}), \quad \|f\|_{1,p} = \|f\|_1 + \|f\|_p$$

for all $f \in \mathcal{L}^1_p(\mathcal{G})$. Then $\mathcal{L}^1_p(\mathcal{G})$ endowed with convolution product becomes an abstract Segal algebra with respect to $L^1(\mathcal{G})$.

Theorem 2.4. *Let $\mathcal{G}$ be a locally compact group and let $1 < p < \infty$. Then the following statements are equivalent.*

(a) *$\mathcal{L}^1_p(\mathcal{G})$ equipped with the convolution product is character amenable.*

(b) *$\mathcal{G}$ is discrete and amenable.*

Proof. Suppose that $\mathcal{L}^1_p(\mathcal{G})$ is character amenable. Then $\mathcal{G}$ is amenable and $L^1(\mathcal{G}) \subseteq L^p(\mathcal{G})$ by Lemma 2.3. So, it suffices to show that $\mathcal{G}$ is discrete. For this end, let (e_α) be a bounded right approximate identity for $\mathcal{L}^1_p(\mathcal{G})$ and let u be a weak*-cluster point of (e_α) in $L^p(\mathcal{G})$. Then $f * u = f$ for all $f \in L^1(\mathcal{G})$. If $\mathcal{G}$ were not discrete, then we can find a compact neighborhood U of the identity element e of $\mathcal{G}$ such that

$$\lambda(U) < \frac{1}{2^q \|u\|_p^q}.$$

Since the indicator function χ_U of U lies in $L^p(\mathcal{G})$ for all $1 \le p \le \infty$ and $L^p(\mathcal{G})$ is a left $L^1(\mathcal{G})$-module, it follows that $\|\chi_U\|_p = \|\chi_U * u\|_p \le \|\chi_U\|_1 \|u\|_p$. Now, on the one hand,

$$\|\chi_U\|_1 \le \|\chi_U\|_p \|\chi_U\|_q,$$

which implies that $1 \le \|\chi_U\|_q \|u\|_p$. On the other hand,

$$\|\chi_U\|_q \|u\|_p = \lambda(U)^{\frac{1}{q}} \|u\|_p \le \frac{1}{2}$$

which is a contradiction. Converse is trivial. ∎

For a locally compact group $\mathcal{G}$, we denote by $C_b(\mathcal{G})$, the space of all bounded continuous complex-valued functions on $\mathcal{G}$, and we denote by $LUC(\mathcal{G})$, the space of bounded left uniformly continuous functions on $\mathcal{G}$ as defined in [9].

Theorem 2.5. *Let $\mathcal{G}$ be a locally compact group. Then the following statements are equivalent.*

(a) *$\mathcal{L}^1_\infty(\mathcal{G})$ equipped with the convolution product is character amenable.*
(b) *$\mathcal{L}^1_\infty(\mathcal{G})$ equipped with the convolution product is character-pseudo amenable.*
(c) *$\mathcal{G}$ is discrete and amenable.*

Proof. The implication (a)⇒(b) is trivial.

(b)⇒(c). Suppose that $\mathcal{L}^1_\infty(\mathcal{G})$ is character-pseudo amenable. Then, $\mathcal{L}^1_\infty(\mathcal{G})$ has a right approximate identity such as (e_α). Therefore, $\|f * e_\alpha - f\|_\infty \to 0$ for all $f \in \mathcal{L}^1_\infty(\mathcal{G})$. Moreover, $f * e_\alpha \in LUC(\mathcal{G})$ for all α and consequently $f \in LUC(\mathcal{G})$. Hence, $\mathcal{L}^1_\infty(\mathcal{G}) \subseteq LUC(\mathcal{G})$. Now, let A be a measurable subset of $\mathcal{G}$ with $\lambda(A) < \infty$. Then the indicator function χ_A of A lies in $\mathcal{L}^1_\infty(\mathcal{G})$ and so $\chi_A \in LUC(\mathcal{G})$ which implies that A is an open subset in $\mathcal{G}$. Thus $\mathcal{G}$ is discrete. Amenability of $\mathcal{G}$ follows from Lemma 2.2. Finally, (c)⇒(a) is trivial. ∎

Let X be a closed subspace of $L^\infty(\mathcal{G})$ with $C_0(\mathcal{G}) \subseteq X \subseteq LUC(\mathcal{G})$ which is left invariant; i.e. $\ell_x f \in X$ for all $f \in X$ and $x \in G$. Then $L^1(\mathcal{G}) \cap X$ with the convolution product and the norm $\|\cdot\|_{1,\infty}$ is an abstract Segal algebra with respect to $L^1(\mathcal{G})$.

Theorem 2.6. *Let $\mathcal{G}$ be a locally compact group and let X be a closed and left invariant subspace of $L^\infty(\mathcal{G})$ such that $C_0(\mathcal{G}) \subseteq X \subseteq LUC(\mathcal{G})$. Then the following statements are equivalent.*

(a) *$L^1(\mathcal{G}) \cap X$ endowed with the convolution product is character amenable.*
(b) *$\mathcal{G}$ is discrete and amenable.*

Proof. (a)⇒(b). Suppose that $L^1(\mathcal{G}) \cap X$ is character amenable. Then $L^1(\mathcal{G}) \cap X = L^1(\mathcal{G})$ and $\mathcal{G}$ is amenable by Lemma 2.3. Thus $L^1(\mathcal{G}) \subseteq X$. Now, consider the inclusion map $\iota : L^1(\mathcal{G}) \to X$. Then ι is continuous by the closed graph theorem. Suppose on the contrary that $\mathcal{G}$ is not discrete. Then we can choose a sequence $\{U_n\}$ of open neighborhoods of the identity element e of $\mathcal{G}$ with compact closure in $\mathcal{G}$ such that $\lambda(U_n) \to 0$. Let f_n be a continuous function vanishing outside U_n, $f_n(e) = 1$ and $0 \leq f_n(x) \leq 1$ for all $x \in G$. Let $g_n := \lambda(U_n)^{-1} f_n$ and note that $g_n \in L^1(\mathcal{G}) \cap X$. Moreover, $\|g_n\|_\infty \to \infty$ and $\|g_n\|_1 \leq 1$ for all $n \in \mathbb{N}$. This contradicts the continuity of ι.

(b)⇒(a). Now, suppose that $\mathcal{G}$ is discrete and amenable. Then $L^1(\mathcal{G})$ is character amenable by Lemma 2.3. Moreover, $L^1(\mathcal{G}) = L^1(\mathcal{G}) \cap X$. Hence, the norms $\|\cdot\|_{1,\infty}$ and $\|\cdot\|_1$ are equivalent on $L^1(\mathcal{G}) \cap X$ by the open mapping theorem. Thus $L^1(\mathcal{G}) \cap X$ is character amenable. ∎

It is not hard to see that $L^1(\mathcal{G}) \cap C_0(\mathcal{G})$ with pointwise product and the norm $\|\cdot\|_{1,\infty}$ is an abstract Segal algebra with respect to $C_0(\mathcal{G})$.

Lemma 2.7. *Let $\mathcal{G}$ be a locally compact group. Then the following statements are equivalent.*

(a) $L^1(\mathcal{G}) \cap C_0(\mathcal{G})$ *endowed with the pointwise product has a bounded approximate identity.*

(b) $\mathcal{G}$ *is compact.*

Proof. (a)⇒(b). Suppose that e_α is a bounded approximate identity for $L^1(\mathcal{G}) \cap C_0(\mathcal{G})$. If $\mathcal{G}$ were not compact, then we can find a sequence $\{U_n\}$ of open neighborhoods of the identity e with compact closure such that $\lambda(U_n) \to \infty$. Choose a symmetric neighborhood V of e with compact closure. Set

$$f_n := \frac{1}{\lambda(V)} \chi_{U_n V} * \chi_V.$$

Then it is easy to see that $f_n \in C_c(\mathcal{G})$, the space of all continuous functions with compact support, and $f_n|_{U_n} = 1$. By assumption for each $n \in \mathbb{N}$ there is a α_n such that $\|e_\alpha f_n - f_n\|_\infty \leq \frac{1}{2}$ for all $\alpha \geq \alpha_n$. In particular, $|e_\alpha(x) - 1| \leq \frac{1}{2}$ for all $x \in U_n$. Therefore,

$$\|e_\alpha\|_{1,\infty} \geq \|e_\alpha\|_1 = \int_{U_n} |e_\alpha(x)| dx \geq \frac{1}{2}\lambda(U_n) \to \infty,$$

which is a contradiction. The implication (b)⇒(a) is trivial. ∎

Recall that $\Delta(C_0(\mathcal{G})) = \{\varphi_x : x \in G\}$ and $C_0(\mathcal{G})$ is φ_x-amenable for all $x \in G$. As a consequence of Lemma 2.7 we have the following result.

Theorem 2.8. *Let $\mathcal{G}$ be a locally compact group. Then the following statements are equivalent.*

(a) $L^1(\mathcal{G}) \cap C_0(\mathcal{G})$ *endowed with the pointwise product is character amenable.*

(b) $\mathcal{G}$ *is compact.*

Let G be a locally compact group and let

$$\mathcal{A}_p^1(\mathcal{G}) := L^1(\mathcal{G}) \cap A_p(\mathcal{G}), \quad |||f||| = \|f\|_1 + \|f\|_{A_p}$$

for all $f \in \mathcal{A}_p^1(\mathcal{G})$. Then $\mathcal{A}_p^1(\mathcal{G})$ with this norm and the convolution product is an abstract Segal algebra with respect to $L^1(\mathcal{G})$; see [7] for details.

Now, we have the following result whose proof is omitted, since it can be proved in the same direction of Theorem 2.6.

Theorem 2.9. *Let $\mathcal{G}$ be a locally compact group and let $1 < p < \infty$. Then the following statements are equivalent.*

(a) $\mathcal{A}_p^1(\mathcal{G})$ *endowed with the convolution product is character amenable.*

(b) $\mathcal{G}$ *is discrete and amenable.*

Recall from [7] that $\mathcal{A}_p^1(\mathcal{G})$ with the norm $|||\cdot|||$ and the pointwise product is an abstract Segal algebra with respect to $A_p(\mathcal{G})$.

Theorem 2.10. *Let $\mathcal{G}$ be a locally compact group and let $1 < p < \infty$. Then the following statements are equivalent.*

(a) $\mathcal{A}_p^1(\mathcal{G})$ *endowed with the pointwise product is character amenable.*

(b) $\mathcal{G}$ *is compact.*

Proof. Suppose that $\mathcal{G}$ is compact. Then it is clear that the constant function 1 is the identity of $A_p^1(\mathcal{G})$. Moreover, $A_p^1(\mathcal{G})$ is φ_x-amenable for all $x \in G$ by [15, Lemma 3.1] and [1, Proposition 2.3].

For the converse, we can apply the same argument for the proof of Lemma 2.7. ■

Recall that if $\mathcal{A}$ is a Banach algebra, then $\mathcal{A}^{**}$, the second dual of $\mathcal{A}$ with the first Arens multiplication $\odot$ which is defined as follows is a Banach algebra

$$(m \odot n)(f) = m(n \cdot f), \quad (n \cdot f)(a) = n(f \cdot a), \quad (f \cdot a)(b) = f(ab)$$

for all $m, n \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$.

Theorem 2.11. *Let $\mathcal{G}$ be a locally compact group and let $S(\mathcal{G})$ be an abstract Segal algebra with respect to $L^1(\mathcal{G})$. Then $S(\mathcal{G})^{**}$ endowed with the first Arens product is character amenable if and only if $\mathcal{G}$ is finite.*

Proof. Suppose that $S(\mathcal{G})^{**}$ is character amenable. Then $S(\mathcal{G})$ is character amenable by [10, Theorem 3.8]. In particular, it has a bounded right approximate identity. Thus $S(\mathcal{G}) = L^1(\mathcal{G})$ by [2, Theorem 1.2]. The rest of the proof follows from [10, Theorem 3.10]. ■

Using a same argument for the proof of the above theorem and applying [10, Theorem 3.12] we have the following result.

Theorem 2.12. *Let $\mathcal{G}$ be a locally compact group and let $SA(\mathcal{G})$ be an abstract Segal algebra with respect to $A(\mathcal{G})$. Then $SA(\mathcal{G})^{**}$ endowed with the first Arens product is character amenable if and only if $\mathcal{G}$ is finite.*

3 Character contractibility

Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathcal{A})$, and recall that a *$\varphi$-diagonal* for $\mathcal{A}$ is an element $\mathbf{m} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\varphi(\pi(\mathbf{m})) = 1 \quad \text{and} \quad a \cdot \mathbf{m} = \varphi(a)\mathbf{m}$$

for all $a \in \mathcal{A}$. Here and in the sequel π always denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$, specified by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is called *φ-contractible* if there is a φ-diagonal $\mathbf{m} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. Also, $\mathcal{A}$ is called *character contractible* if it has a right identity and it is φ-contractible for all $\varphi \in \Delta(\mathcal{A})$. These notions was introduced by Hu, Monfared, and Traynor [10]. Before, we give the next result, recall from [3] that a Banach algebra $\mathcal{A}$ is a left (resp. right) ideal in $\mathcal{A}^{**}$ if and only if, for every $a \in \mathcal{A}$, the operator $\lambda_a : b \mapsto ba$ (resp. $\rho_a : b \mapsto ab$) is weakly compact on $\mathcal{A}$. We also recall that when $\mathcal{A}$ is commutative then for each $a \in \mathcal{A}$ the letter $\widehat{a}$ refers to the Gelfand transform of a which is defined on $\Delta(\mathcal{A})$ endowed with the Gelfand topology by $\widehat{a}(\varphi) := \varphi(a)$ for all $\varphi \in \Delta(\mathcal{A})$. As usual, we say that $\mathcal{A}$ is Tauberian if the set of all $a \in \mathcal{A}$ such that $\widehat{a}$ has compact support is norm dense in $\mathcal{A}$. Note that this condition is obviously fulfilled if $\mathcal{A}$ has an identity or it is a C*-algebra. Moreover, Tauberian Banach algebras form an important and large class of Banach algebras related to locally compact groups.

Theorem 3.1. *Let $\mathcal{A}$ be a commutative, semisimple, Tauberian Banach algebra. Then the following statements are equivalent.*

(a) *$\mathcal{A}$ is a left ideal in its second dual and it is φ-amenable for all $\varphi \in \Delta(\mathcal{A})$.*

(b) *$\mathcal{A}$ is φ-contractible for all $\varphi \in \Delta(\mathcal{A})$.*

(c) *$\Delta(\mathcal{A})$ is discrete with respect to the Gelfand topology.*

Proof. (a)⇒(b). We note that if (a) holds, then $n := m \odot a_0$ is a φ-mean in $\mathcal{A}$, where m is a φ-mean in $\mathcal{A}^{**}$ and $a_0 \in \mathcal{A}$ with $\varphi(a_0) = 1$. Therefore, $n \otimes n$ is a φ-diagonal in $\mathcal{A}\widehat{\otimes}\mathcal{A}$.

(b)⇒(c). Let φ be an arbitrary element of $\Delta(\mathcal{A})$. We will show $\{\varphi\}$ is an open subset of $\Delta(\mathcal{A})$ with respect to the Gelfand topology. Since $\mathcal{A}$ is φ-contractible, there is an element $\mathbf{m} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\widehat{\pi(\mathbf{m})}(\varphi) = 1 \quad \text{and} \quad a \cdot \mathbf{m} = \varphi(a)\mathbf{m}.$$

This implies that $\widehat{\pi(\mathbf{m})}(\varphi) = 1$ and $a\pi(\mathbf{m}) = \widehat{a}(\varphi)\pi(\mathbf{m})$ for all $a \in \mathcal{A}$. Thus, we conclude that $\widehat{\pi(\mathbf{m})}$ is the the indicator function at φ. That is, $\{\varphi\}$ is open in $\Delta(\mathcal{A})$ with respect to the Gelfand topology.

(c)⇒(a). Let a be an arbitrary element of $\mathcal{A}$ and consider the operator $\lambda_a : b \mapsto ba$ on $\mathcal{A}$. Since $\mathcal{A}$ is Tauberian, there exist a sequence (a_n) in $\mathcal{A}$ norm convergent to a such that each $\widehat{a_n}$ has compact and hence finite support, say K_n. By Shilov's idempotent theorem for each $\varphi \in \Delta(\mathcal{A})$, there is an idempotent a_φ in $\mathcal{A}$ such that $\widehat{a_\varphi}$ equals the indicator function at φ. This implies that a_φ is a φ-mean in $\mathcal{A}$ and that the operator λ_{a_φ} is one-dimensional. Moreover, $a_n = \sum_{\varphi \in K_n} \widehat{a_n}(\varphi) a_\varphi$.

Therefore, each λ_{a_n} has finite rank. Since (λ_{a_n}) is norm convergent to λ_a, we obtain that λ_a is a compact operator. It follows that $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$. ■

In the sequel, we present some examples to motivate and complement the main result of this section.

Example 3.2. Let $\mathcal{X}$ be a compact metric space with metric d and let $0 < \alpha < 1$. Then $\mathrm{Lip}_\alpha(\mathcal{X})$ denotes the set of all Lipschitz functions of order one on $\mathcal{X}$; that is, all continuous complex-valued functions f on $\mathcal{X}$ for which

$$p_\alpha(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)^\alpha} :\ x,y \in \mathcal{X}, x \neq y\right\}$$

is finite. Moreover, $\mathrm{lip}_\alpha(\mathcal{X})$ denotes the set of all functions in $\mathrm{Lip}_\alpha(\mathcal{X})$ satisfying

$$\frac{|f(x)-f(y)|}{d(x,y)^\alpha} \longrightarrow 0 \quad \text{as} \quad d(x,y) \longrightarrow 0.$$

Then $\mathrm{Lip}_\alpha(\mathcal{X})$ endowed with the pointwise operations and the norm $\|f\|_\alpha := \|f\|_{\sup} + p_\alpha(f)$ is a commutative, semisimple and Tauberian Banach algebra, such that $\mathrm{lip}_\alpha(\mathcal{X})$ is a closed subalgebra of $\mathrm{Lip}_\alpha(\mathcal{X})$. The map $x \to \varphi_x$, where $\varphi_x(f) = f(x)$ for $f \in \mathrm{Lip}_\alpha(\mathcal{X})$, is a homeomorphism from $\mathcal{X}$ onto $\Delta(\mathrm{Lip}_\alpha(\mathcal{X}))$ endowed with the Gelfand topology. Similarly, $\Delta(\mathrm{lip}_\alpha(\mathcal{X}))$ can be identified with $\mathcal{X}$. If $\mathcal{X}$ is finite, then $\mathrm{lip}_\alpha(\mathcal{X}) = \mathrm{Lip}_\alpha(\mathcal{X})$ is semisimple and finite-dimensional and hence is an ideal in its second dual. Therefore, by [13, Example 5.3] and Theorem 3.1, we have the following characterizations.

(a) $\mathrm{Lip}_\alpha(\mathcal{X})$ is an ideal in its second dual.
(b) $\mathrm{Lip}_\alpha(\mathcal{X})$ is φ_x-amenable for all $x \in \mathcal{X}$.
(c) $\mathrm{Lip}_\alpha(\mathcal{X})$ is φ_x-contractible for all $x \in \mathcal{X}$.
(d) $\mathcal{X}$ is finite.

Note that a same result is true for $\mathrm{lip}_\alpha(\mathcal{X})$.

Lemma 3.3. *Let $\mathcal{A}$ be a commutative Banach algebra and let $\mathcal{B}$ be an abstract Segal algebra in $\mathcal{A}$ such that the linear span of the set $\mathcal{AB} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$ is dense in $\mathcal{B}$. Then the following hold.*

(a)*If $\mathcal{A}$ is Tauberian, then $\mathcal{B}$ is also Tauberian.*

(b) *If $\mathcal{A}$ is semisimple and Tauberian and it is φ-amenable for all $\varphi \in \Delta(\mathcal{A})$, then $\mathcal{A}$ is a left ideal in its second dual if and only if $\mathcal{B}$ is a left ideal in its second dual.*

Proof. (a). First note that, for arbitrary $b \in \mathcal{B}$, by assumption, there are $u_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, ..., n)$ with $\|b - \sum_{i=1}^n u_i b_i\|_\mathcal{B} < \varepsilon/2$. Since $\mathcal{A}$ is Tauberian, it follows that for each $1 \le i \le n$ there is $v_i \in \mathcal{A}$ such that $\widehat{v_i}$ has compact support and $\|u_i - v_i\|_\mathcal{A} < \varepsilon/(2nM\|b_i\|_\mathcal{B})$. Thus

$$\left\|\sum_{i=1}^n u_i b_i - \sum_{i=1}^n v_i b_i\right\|_\mathcal{B} \le \sum_{i=1}^n M\|u_i - v_i\|_\mathcal{A}\, \|b_i\|_\mathcal{B} < \varepsilon/2.$$

Therefore,

$$\left\|b - \sum_{i=1}^n v_i b_i\right\|_\mathcal{B} \le \left\|b - \sum_{i=1}^n u_i b_i\right\|_\mathcal{B} + \left\|\sum_{i=1}^n u_i b_i - \sum_{i=1}^n v_i b_i\right\|_\mathcal{B} < \varepsilon.$$

It is clear that $\sum_{i=1}^{n} v_i b_i \in \mathcal{B}$ and $\widehat{\sum_{i=1}^{n} v_i b_i} = \sum_{i=1}^{n} \widehat{v_i}\widehat{b_i}$ has compact support as required.

(b). Suppose that $\mathcal{A}$ is semisimple and Tauberian which is φ-amenable for all $\varphi \in \Delta(\mathcal{A})$ and that the linear span of the set $\mathcal{AB}$ is dense in $\mathcal{B}$. Then, from part (a) and the correspondence between $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, we obtain respectively, that $\mathcal{B}$ is semisimple and Tauberian. Moreover, it is shown in [1] that $\mathcal{A}$ is φ-amenable [resp. φ-contractible] if and only if $\mathcal{B}$ is $\varphi|_{\mathcal{B}}$-amenable [resp. $\varphi|_{\mathcal{B}}$-contractible]. Therefore, in light of Theorem 3.1, $\mathcal{A}$ is a left ideal in its second dual if and only if $\mathcal{B}$ is a left ideal in its second dual. ■

Next, we examine the abstract Segal algebra in generalized Fourier algebra.

Example 3.4. Let $\mathcal{G}$ be a locally compact group with identity e and let $1 < p < \infty$. Then $A_p(\mathcal{G})$ is a commutative, semisimple, and Tauberian Banach algebra. In [15, Lemma 3.1], it is proved that $A_p(\mathcal{G})$ is φ_x-amenable for all $x \in \mathcal{G}$. Note that $m \in A_p(\mathcal{G})$ is a φ_x-mean if and only if $\ell_{x^{-1}} m$ is a φ_e-mean in $A_p(\mathcal{G})$, where $\ell_{x^{-1}}$ is the left translations by x^{-1} on $A_p(\mathcal{G})$.
(i) If $SA_P(\mathcal{G})$ is an abstract Segal algebra in $A_p(\mathcal{G})$ such that the linear span of $A_p(\mathcal{G})SA_p(\mathcal{G})$ is dense in $SA_p(\mathcal{G})$, then $SA_p(G)$ is Tauberian and so by Theorem 3.1 the following statements are equivalent.

(a) $SA_p(\mathcal{G})$ is an ideal in its second dual.
(b) $SA_p(\mathcal{G})$ is φ_x-contractible for all $x \in G$.
(c) $SA_p(\mathcal{G})$ is φ_x-contractible for some $x \in G$.
(d) $\mathcal{G}$ is discrete.

(ii) If $MA_p(\mathcal{G})$ is the space of pointwise bounded multipliers of $A_p(\mathcal{G})$ equipped with the multiplier norm

$$\|v\|_M = \sup\left\{\|vu\|_{A_p(\mathcal{G})} : u \in A_p(G), \|u\|_{A_p(\mathcal{G})} \leq 1\right\};$$

that is, those (necessarily continuous and bounded) functions v on G such that $vA_p(\mathcal{G}) \subseteq A_p(\mathcal{G})$. It is well-known that $A_p(\mathcal{G}) \subseteq MA_p(\mathcal{G})$ and $\|u\|_M \leq \|u\|_{A_p(\mathcal{G})}$ for all $u \in A_p(\mathcal{G})$. We denote by $A_p^M(\mathcal{G})$ the $\|\cdot\|_M$-closure of $A_p(\mathcal{G})$ in $MA_p(\mathcal{G})$. Then, $A_p^M(\mathcal{G})$ with pointwise product is a commutative, semisimple, and Tauberian Banach algebra containing $A_p(\mathcal{G})$ as an abstract Segal algebra. Therefore, by Theorem 3.1 the above characterizations are true for $A_p^M(\mathcal{G})$ to be an ideal in its second dual.

Before giving the following result, recall that a locally compact group $\mathcal{G}$ is called amenable if $L^1(\mathcal{G})$ is φ_1-amenable. The following result on unimodular locally compact groups is from Mustafayev [17].

Theorem 3.5. [17, Corollary 3] *Let $\mathcal{G}$ be a unimodular locally compact group. Then $\mathcal{L}_p^1(\mathcal{G})$ endowed with the convolution product is an ideal in its second dual if and only if $\mathcal{G}$ is compact.*

Here, we will now show that this result is also valid for the class of amenable groups.

Theorem 3.6. *Let $\mathcal{G}$ be a locally compact group. Then $\mathcal{L}^1_p(\mathcal{G})$ endowed with the convolution product is an ideal in its second dual and $\mathcal{G}$ is amenable if and only if $\mathcal{G}$ is compact.*

Proof. Suppose that $\mathcal{G}$ is compact. Then $\mathcal{L}^1_p(\mathcal{G}) = L^p(\mathcal{G})$ and consequently $\mathcal{L}^1_p(\mathcal{G})$ is an ideal in its second dual.

Conversely, Suppose that $\mathcal{L}^1_p(\mathcal{G})$ is a left ideal in its second dual. Since $\mathcal{G}$ is amenable, it follows from [1, Proposition 2.3], that $\mathcal{L}^1_p(\mathcal{G})$ has a bounded approximate φ_1-mean, say (f_α). Let m be any weak* cluster point of (f_α) in $(\mathcal{L}^1_p(\mathcal{G}))^{**}$. Then $\varphi_1(m) = 1$ and for each $f \in \mathcal{L}^1_p(\mathcal{G})$ we have $f \odot m = \varphi_1(f)m$ by the weak* continuity of the map $n \mapsto f \odot n$ on $(\mathcal{L}^1_p(\mathcal{G}))^{**}$. Take $g_0 := (m \odot f_0)$, where $f_0 \in \mathcal{L}^1_p(\mathcal{G})$ with $\varphi_1(f_0) = 1$ and note that $g_0 \in \mathcal{L}^1_p(\mathcal{G})$. Furthermore,

$$\varphi_1(g_0) = \varphi_1(m \odot f_0) = \varphi_1(m) = 1,$$

and for each $f \in \mathcal{L}^1_p(\mathcal{G})$,

$$\begin{aligned} f * g_0 &= f * (m \odot f_0) \\ &= (f \odot m) * f_0 \\ &= \varphi_1(f)(m \odot f_0) \\ &= \varphi_1(f) g_0. \end{aligned}$$

That is, g_0 is a φ_1-mean in $\mathcal{L}^1_p(\mathcal{G})$. Consequently $\mathcal{G}$ is compact by [1, Theorem 3.3]. ∎

Theorems 3.5 and 3.6 lead us to the following problem.

Problem 3.7. *Is Theorem 3.6 true for each locally compact group $\mathcal{G}$? That is, must $\mathcal{G}$ be compact when $\mathcal{L}^1_p(\mathcal{G})$, endowed with the convolution product, is an ideal in its second dual?*

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