Strong topological transitivity of some classes of operators

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Abstract

In this paper, we investigate the strong topological transitivity of some well-known classes of topologically transitive operators. Also, topological largeness of the set of all strongly topologically transitive operators on a second countable Baire locally convex space is investigated.

1 Introduction

Let X be a topological vector space and $L(X)$ be the space of all continuous linear operators on X. By an operator T, we always mean $T \in L(X)$. An operator T is said to be *hypercyclic* if there is some $x \in X$ for which $\mathrm{orb}(T,x) = \{T^n x : n \in \mathbb{N}_0\}$ is dense in X. In that case, x is called a *hypercyclic vector* for T. Here $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and $T^0 = I$, the identity operator on X. Two historic examples of hypercyclic operators are the derivative operator on $H(\mathbb{C})$ which is due to Maclane [6], and $T = 2B$, twice the backward shift on ℓ^2, due to Rolewicz [8].

The set of all hypercyclic vectors for T is denoted by $HC(T)$ and it is known that, if T is hypercyclic then $HC(T)$ is dense in X. If $HC(T) = X\backslash\{0\}$ then T is called *hypertransitive*. It is clear that T is hypertransitive if and only if T lacks nontrivial closed invariant subsets. The Read operator on ℓ^1 is an example of such operators [7].

An operator T is called *topologically transitive* if, for any nonempty open sets $U, V \subset X$, there is some $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$. It is well-known

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that, on any second countable Baire topological vector space, an operator T is topologically transitive if and only if it is hypercyclic [3]. Also, it can be readily seen that T is topologically transitive if and only if, for any nonempty open set $U \subset X$, the set $\bigcup_{n\in\mathbb{N}_0} T^n(U)$ is dense in X.

Recently, the authors in [1] have introduced a new type of transitivity which is strictly stronger than topological transitivity:

Definition 1.1. Let X be a topological vector space. An operator $T \in L(X)$ is called strongly topologically transitive if $X\backslash\{0\} \subset \bigcup_{n\in\mathbb{N}_0} T^n(U)$ for any nonempty open set $U \subset X$.

That the strong topological transitivity implies topological transitivity is apparent from the above definition. We will see in section 2, that there are topologically transitive operators which are not strongly topologically transitive. The difference of topological transitivity and strong topological transitivity can be explained in the language of equations. Given a nonzero point $x_0 \in X$ and an open set $U \subset X$, is there a solution $(n_0, y_0) \in \mathbb{N}_0 \times U$ for the equation $T^n y = x_0$? Topological transitivity of T can only give *approximate solutions* for this equation. Indeed, since $\overline{\bigcup_{n\in\mathbb{N}_0} T^n(U)} = X$ there are sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(y_k)_k$ in U such that $T^{n_k} y_k \to x_0$ as $k \to \infty$. Then for large enough k, the pair (n_k, y_k) could be considered as a desirable approximate solution for the equation $T^n y = x_0$. Now, if T is strongly topologically transitive then the mentioned equation has *exact solutions* by the definition of strong topological transitivity.

Proposition 1.2. *(Proposition* 4 *of [1]) Every strongly topologically transitive operator is surjective.*

Proposition 1.3. *(Proposition* 6 *of [1]) An invertible operator T is strongly topologically transitive if and only if T^{-1} is hypertransitive.*

Proposition 1.4. *(Theorem* 5 *of [1]) An invertible Hilbert space operator T is strongly topologically transitive if and only if T^* is strongly topologically transitive.*

Remark 1.5. The above proposition is also true for invertible operators on locally convex spaces. Suppose X is a locally convex space and $T \in L(X)$ is invertible. If $M \subset X$ is a nontrivial closed T-invariant set then it is easy to see that $M^* = \{f \in X^* : f(M) = \{0\}\}$ is a nontrivial closed invariant set for T^*. Thus, T is hypertransitive if and only if T^* is. Now, the result follows by Proposition 1.3 and the fact that $(T^*)^{-1} = (T^{-1})^*$.

In section 2, a necessary and sufficient condition for strong topological transitivity of an operator acting on a topological vector space is presented. Then we investigate strong topological transitivity of some well-know topologically transitive operators. We show that: the derivative operator on $H(\mathbb{C})$ is strongly topologically transitive but translation operators on $H(\mathbb{C})$ are not; multiples of backward shift on ℓ^p and c_0 are strongly topologically transitive if they are topologically transitive (or equivalently hypercyclic); the adjoint of an invertible multiplication operator on H^2 is not strongly topologically transitive but, if the multiplication operator is not invertible, we may have strongly topologically transitive adjoints; no composition operator on a Banach space of analytic functions on the disk is strongly topologically transitive. Also, we give a sufficient condition and a necessary condition for strong topological transitivity of weighted backward shifts on ℓ^p and c_0.

In the last section, we see that strong topological transitivity is preserved under similarity. In other words, if X is a topological vector space and $T \in L(X)$ is strongly topologically transitive then for any invertible operator $J \in L(X)$, the operator $J^{-1}TJ$ is also strongly topologically transitive. We use this result to prove that on a second countable Baire locally convex space X, the set of all strongly topologically transitive operators is either empty or SOT-dense in $L(X)$.

2 Main theorem and results

Theorem 2.1. *Let X be a first countable topological vector space and $T \in L(X)$ be surjective with a right inverse map S. Then T is strongly topologically transitive if and only if, for every nonzero vector $x \in X$ and any $y \in X$, there exist sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(w_k)_k$ in X such that $w_k \in KerT^{n_k}$ and $S^{n_k}x + w_k \to y$ as $k \to \infty$.*

Proof. Suppose T is strongly topologically transitive. Let $x \neq 0$ and y be arbitrary vectors in X. Pick a countable neighborhood basis $\{U_k : k = 1, 2, \cdots\}$ at y. Replacing U_k by $\bigcap_{n=1}^{k} U_n$ where necessary, we may assume that $U_1 \supset U_2 \supset U_3 \supset \cdots$. Then for each $k \in \mathbb{N}$, there is some $y_k \in U_k$ and $n_k \in \mathbb{N}_0$ such that $x = T^{n_k}y_k$. Clearly, $y_k \to y$ as $k \to \infty$. On the other hand, for each $k \in \mathbb{N}$ we have $S^{n_k}x = y_k + t_k$ for some $t_k \in \mathrm{Ker}T^{n_k}$ (it is easy to see that $S^mT^mx - x \in \mathrm{Ker}T^m$). Thus, $S^{n_k}x + w_k \to y$ as $k \to \infty$, where $w_k = -t_k$.

For the converse, assume that x is a nonzero vector in X and $U \subset X$ is a nonempty open set. Choose $y \in U$ and suppose $S^{n_k}x + w_k \to y$ for some sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(w_k)_k$ in X satisfying $T^{n_k}w_k = 0$. Then, for large enough k, we have $S^{n_k}x + w_k \in U$. If we put $z = S^{n_k}x + w_k$ then $T^{n_k}z = x$ and so $x \in T^{n_k}(U)$. ∎

Remark 2.2. Theorem 2.1 is true for all topological vector spaces. The only change that we should apply in the statement and the proof of the theorem, is to replace "sequences" by "nets". Meanwhile, we should note that the proof of the theorem is independent of the chosen right inverse for T. In other words, if R and S are different right inverse maps for T then for any $n_k \in \mathbb{N}$, $\mathrm{Ran}(R^{n_k} - S^{n_k}) \subset \mathrm{Ker}T^{n_k}$. Thus, for every $x \in X$ there is some $u_k \in \mathrm{Ker}T^{n_k}$ such that $R^{n_k}x = S^{n_k}x + u_k$. Therefore, for any nonzero $x \in X$ and any $y \in X$, $S^{n_k}x + w_k \to y$ for some sequence $(w_k)_k$ such that $w_k \in \mathrm{Ker}T^{n_k}$, if and only if $R^{n_k}x + w'_k \to y$ for some sequence $(w'_k)_k$ such that $w'_k \in \mathrm{Ker}T^{n_k}$ ($w'_k = w_k - u_k$).

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.3. *Let X be a first countable topological vector space and $T \in L(X)$ be surjective with a right inverse map S. If $\bigcup_{n\in\mathbb{N}_0} \mathrm{Ker}T^n$ is dense in X and, for all nonzero vector $x \in X$, the sequence $(S^mx)_m$ is convergent then T is strongly topologically transitive.*

Proof. Let $x \neq 0$ and y be arbitrary vectors in X, and suppose that $S^m x \to z$ as $m \to \infty$. Since $\bigcup\limits_{n\in\mathbb{N}_0} \mathrm{Ker}T^n$ is dense in X, there is a sequence $(w_k)_k$ in $\bigcup\limits_{n\in\mathbb{N}_0} \mathrm{Ker}T^n$ such that $w_k \to y - z$. Assuming $w_k \in \mathrm{Ker}T^{n_k}$, it is clear that $S^{n_k}x + w_k \to y$ as $k \to \infty$, and we are done by Theorem 2.1. ∎

2.1 Derivative and translation operators

Recall that $H(\mathbb{C})$ is the space of all entire functions on $\mathbb{C}$ endowed with the topology of uniform convergence on compact sets. The derivative operator T on $H(\mathbb{C})$ is defined by $Tf = f'$. It is known that T is topologically transitive [6]. We improve this result by showing that T is strongly topologically transitive.

Theorem 2.4. *The derivative operator on $H(\mathbb{C})$ is strongly topologically transitive.*

Proof. Let T be the derivative operator on $H(\mathbb{C})$. If $S : H(\mathbb{C}) \to H(\mathbb{C})$ is the map defined by $S(h)(z) = \int_0^z h(\xi)d\xi$ then $TS = I$. Suppose f, g are any functions in $H(\mathbb{C})$ and f is not the zero function. There is a sequence of polynomials $(w_k)_k$ such that $w_k \to g$ in $H(\mathbb{C})$ and, meanwhile, If $n_k = 1 + \deg w_k$ then $T^{n_k}w_k = 0$. Therefore, $\bigcup\limits_{n\in\mathbb{N}_0} \mathrm{Ker}T^n$ is dense in $H(\mathbb{C})$. On the other hand, we prove that $S^m f \to 0$ in $H(\mathbb{C})$ and then, we are done by Corollary 2.3. To this end, it is not difficult to show that, for any $m \in \mathbb{N}$, $S^m T^m f = f - p_{m-1}(f)$. Thus, $S^m f = S^m T^m (S^m f) = S^m f - p_{m-1}(S^m f)$ which gives $p_{m-1}(S^m f) = 0$ for all $m \in \mathbb{N}$, and then we have

$$\begin{aligned}
S^m f &= \sum_{n=m}^{\infty} \frac{(S^m f)^{(n)}(0)}{n!} z^n = \sum_{n=m}^{\infty} \frac{(T^n S^m f)(0)}{n!} z^n \\
&= \sum_{n=m}^{\infty} \frac{f^{(n-m)}(0)}{n!} z^n \quad = \frac{z^m}{m!} \sum_{n=m}^{\infty} \frac{m! f^{(n-m)}(0)}{n!} z^{n-m} \\
&= \frac{z^m}{m!} \sum_{n=m}^{\infty} \frac{m!}{(n-m+1)(n-m+2)\cdots n} . \frac{f^{(n-m)}(0)}{(n-m)!} z^{n-m} \\
&= \frac{z^m}{m!} \sum_{n=0}^{\infty} \frac{m!}{(n+1)(n+2)\cdots(n+m)} . \frac{f^{(n)}(0)}{(n)!} z^n .
\end{aligned}$$

Since for all $z \in \mathbb{C}$ we have $\frac{z^m}{m!} \to 0$ in $H(\mathbb{C})$, $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{(n)!} z^n = f(z)$, and meanwhile $\frac{m!}{(n+1)(n+2)\cdots(n+m)} \leq 1$ for all $n \geq 0$, we conclude that $S^m f \to 0$ in $H(\mathbb{C})$ as $m \to \infty$. ∎

Let a be a nonzero complex number. The translation operator T_a on $H(\mathbb{C})$ is defined by $T_a f(z) = f(z + a)$. It is known that T_a is hypercyclic [4].

Proposition 2.5. *Translation operators are not strongly topologically transitive.*

Proof. Since translation operators are invertible ($T_a^{-1} = T_{-a}$) and the constant map $f(z) = 1$ ($z \in \mathbb{C}$) can not be a hypercyclic vector for any translation operator, we conclude that no translation operator is hypertransitive and so, no translation operator is strongly topologically transitive by Proposition 1.3. ∎

2.2 Multiples of backward shift and weighted backward shifts

The backward shift operator on $X = \ell^p$ $(1 \leq p < \infty)$ or c_0 is defined by $Be_0 = 0$ and $B(e_n) = e_{n-1}$ $(n = 1, 2, 3, \cdots)$, where (e_n) is the canonical basis of X. Clearly, if $|\lambda| \leq 1$ then $\|\lambda B\| \leq 1$ and so λB can not be hypercyclic. It is known that λB is hypercyclic if $|\lambda| > 1$ [8]. We show that this condition gives the strong topological transitivity of λB. In other words, λB is strongly topologically transitive if and only if it is hypercyclic.

Proposition 2.6. *The operator $T = \lambda B$ on $X = \ell^p$ $(1 \leq p < \infty)$ or c_0 is strongly topologically transitive if $|\lambda| > 1$.*

Proof. If $S : X \to X$ is defined by $Se_n = \frac{1}{\lambda} e_{n+1}$ $(n = 0, 1, 2, 3, \cdots)$ then $TS = I$. Suppose $x \neq 0$ and $y = (b_0, b_1, \cdots)$ are arbitrary vectors in X. Choose $(n_k) = (k)$, the full sequence of natural numbers, and $w_k = (b_0, b_1, \cdots, b_{k-1}, 0, 0, 0, \cdots)$. Then $w_k \in \mathrm{Ker}T^k$ and $w_k \to y$. Thus, $\bigcup\limits_{n \in \mathbb{N}_0} \mathrm{Ker}T^n$ is dense in X. On the other hand, it is clear that $\|S^m x\| \to 0$ as $m \to \infty$. Therefore, the result follows by Corollary 2.3. ∎

The weighted backward shift B_W on $X = \ell^p$ $(1 \leq p < \infty)$ or c_0 is defined by $B_W(e_0) = 0$, and $B_W(e_n) = w_n e_{n-1}$ $(n = 1, 2, 3, \cdots)$ where $W = (w_n)$ is a bounded sequence of positive numbers. It is known that B_W is hypercyclic if and only if $\limsup\limits_{k \to \infty} (w_1 w_2 \cdots w_k) = \infty$ [9].

Now, we give a sufficient condition for strong topological transitivity of B_W. Before it, we present the following remark.

Remark 2.7. In Theorem 2.1, it suffices to check the condition $S^{n_k}(x) + w_k \to y$ for those vectors y coming from a dense subset of X. Indeed, assume that $\overline{D} = X$ and the condition holds for any $0 \neq x \in X$ and all $y \in D$. Now, consider a vector $y' \in X \backslash D$ with a neighborhood basis $\{U_k : k = 1, 2, \cdots, \}$ such that $U_1 \supset U_2 \supset U_3 \supset \cdots$. Then for any $j \in \mathbb{N}$, there is some $y_j \in D \cap U_j$ and so $y_j \to y'$. On the other hand, for any y_j, there are sequences $(n_{jk})_k$ and $(w_{jk})_k$ such that $w_{jk} \in \mathrm{Ker}T^{n_{jk}}$ and $S^{n_{jk}}(x) + w_{jk} \to y_j$ as $k \to \infty$. Thus, for any j, there is some $k(j)$ such that $S^{n_{jk(j)}}(x) + w_{jk(j)} \in U_j$. Now, it is clear that $S^{n_{jk(j)}}(x) + w_{jk(j)} \to y'$ as $j \to \infty$.

Proposition 2.8. *Let $X = c_0$ or ℓ^p $(1 \leq p < \infty)$ and B_W be a hypercyclic weighted backward shift on X. Then B_W is strongly topologically transitive if there is a positive number r such that $w_n \cdots w_{n+k-1} \geq r$ for every $n, k \in \mathbb{N}$.*

Proof. For simplicity in notations, we put $T = B_W$ and $M_n^k = w_n \cdots w_{n+k-1}$. Then we are assuming that $M_n^k \geq r$ for all $n, k \in \mathbb{N}$. If $S : X \to X$ is defined by $Se_n = \frac{1}{w_{n+1}} e_{n+1}$ $(n = 0, 1, 2, 3, \cdots)$ then $TS = I$. Let $x = (a_0, a_1, \cdots)$ be a nonzero vector and $y = (b_0, b_1, \cdots)$ be a hypercyclic vector for T (regarding Remark 2.7, we choose the dense set $D = HC(T)$). Then there is a sequence $(n_k)_k$ such that

$T^{n_k}y \to x$. Now, if we put $v_k = (b_0, b_1, \cdots, b_{n_k-1}, 0, 0, 0, \cdots)$ then $T^{n_k}v_k = 0$, and

$$\begin{aligned} \|S^{n_k}x + v_k - y\| &= \|(0,0,0,\cdots,0, \frac{1}{M_1^{n_k}}a_0 - b_{n_k}, \frac{1}{M_2^{n_k}}a_1 - b_{n_k+1}, \cdots)\| \\ &= \|(\frac{1}{M_1^{n_k}}(a_0 - M_1^{n_k}b_{n_k}), \frac{1}{M_2^{n_k}}(a_1 - M_2^{n_k}b_{n_k+1}), \cdots)\| \\ &\leq \frac{1}{r}\|T^{n_k}y - x\| \to 0. \end{aligned}$$

Hence, T is strongly topologically transitive by Theorem 2.1 and Remark 2.7. ∎

Example 2.9. Let B_W be the operator with the weight sequence

$$2, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots.$$

Then B_W is hypercyclic since $\limsup_{k\to\infty} M_1^k = \infty$. Also, we have $M_n^k \geq \frac{1}{8}$ for all $n, k \in \mathbb{N}$. Thus, B_W is strongly topologically transitive on ℓ^p (for all $p \in [1, \infty)$) and c_0.

The next proposition gives a necessary condition for strong topological transitivity of B_W.

Proposition 2.10. *Let $X = c_0$ or ℓ^p $(1 \leq p < \infty)$ and B_W be a weighted backward shift on X. If there is an increasing sequence $(m_k)_k$ in $\mathbb{N}$ such that $\sum_{k=1}^{\infty}(M_{m_k}^k)^p < \infty$ then B_W is not strongly topologically transitive on c_0 and ℓ^p. The weaker condition $\lim_{k\to\infty} M_{m_k}^k = 0$ is also sufficient when $X = c_0$.*

Proof. Let U be the open unit ball of X and $x = (a_0, a_1, \cdots)$ be the vector in X defined by

$$a_n = \begin{cases} 1 & n = 0 \\ M_{m_k}^k & n = m_k \ (k = 1, 2, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $x \notin U = T^0(U)$. Now, let $z = (c_n)_n \in U$ and fix $k \geq 1$. Then $T^k z = (M_1^k c_k, M_2^k c_{k+1}, \cdots)$ and the m_k'th coordinate of $T^k z$ is $M_{m_k}^k c_{k-1+m_k}$ which is not equal to $M_{m_k}^k$, the m_k'th coordinate of x. So, $x \neq T^k z$ and hence $x \notin \bigcup_{k=0}^{\infty} T^k(U)$ since k and z were arbitrary. ∎

Example 2.11. Let $W = (w_n)_n$ be the sequence

$$2, 2, \frac{1}{2}, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots.$$

Then B_W is hypercyclic, but, since there is an increasing sequence $(m_k)_k$ $(m_1 = 3, m_2 = 7, m_3 = 13, \cdots)$ such that $M_{m_k}^k = \frac{1}{2^k}$ for all $k \geq 1$, by the above proposition B_W is not strongly topologically transitive on ℓ^p (for all $p \in [1, \infty)$) and c_0.

2.3 Multiplication and composition operators

Recall that the Hardy Hilbert space H^2 is the set of all $f \in H(\mathbb{D})$ for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

Let $\phi \in H(\mathbb{D})$ be a bounded analytic function and M_ϕ be the multiplication operator on H^2 defined by $M_\phi f = \phi f$. It is known that M_ϕ is never hypercyclic but M_ϕ^*, the adjoint of M_ϕ, is hypercyclic if and only if ϕ is non-constant and $\phi(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$ [5]. Note that M_ϕ^* is invertible if and only if (M_ϕ is invertible and M_ϕ is invertible if and only if) ϕ is bounded from bellow, i.e., there is a positive number r such that $|\phi(z)| > r$ for all $z \in \mathbb{D}$. In that case, $(M_\phi^*)^{-1} = M_{\frac{1}{\phi}}^*$. By using Proposition 1.4, we can give the following result.

Proposition 2.12. *For any invertible multiplication operator M_ϕ on H^2, the adjoint operator M_ϕ^* is not strongly topologically transitive.*

But, if M_ϕ is not invertible then M_ϕ^* may have the chance to be strongly topologically transitive.

Proposition 2.13. *If $\phi(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ then M_ϕ^* is strongly topologically transitive if and only if $|\lambda| > 1$.*

Proof. If $|\lambda| \leq 1$ then $\phi(\mathbb{D})$ does not intersect $\partial\mathbb{D}$ which shows that M_ϕ^* is not hypercyclic and hence it can not be strongly topologically transitive. For the converse, assume that $|\lambda| > 1$. Identifying ℓ^2 and H^2 in the usual way, the operator M_ϕ^* for $\phi(z) = \lambda z$ corresponds to the operator $\overline{\lambda}B$. Hence, it follows from Proposition 2.6 that M_ϕ^* is strongly topologically transitive. ∎

Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic function and X be a Banach space of analytic functions on $\mathbb{D}$. The composition operator C_ϕ on X is defined by $C_\phi(f) = f \circ \phi$. The well-known examples of such spaces are the Hardy spaces H^p $(1 \leq p \leq \infty)$ and the weighted Bergman spaces A_α^p $(1 \leq p < \infty,\ \alpha > -1)$.

Proposition 2.14. *There is no strongly topologically transitive composition operator on a Banach space X of analytic functions on the disk.*

Proof. If C_ϕ is strongly topologically transitive then it is clear that ϕ is not a constant map. Hence, C_ϕ is one-to-one. On the other hand, C_ϕ is surjective by Proposition 1.2 and hence C_ϕ is invertible. Then C_ϕ^{-1} is hypertransitive by Proposition 1.3, but this is not true since the constant function $f(z) = 1$ $(z \in \mathbb{C})$ is not a hypercyclic vector for C_ϕ^{-1}. ∎

3 The size of the set of strongly topologically transitive operators

Let X be a topological vector space. For an operator $T \in L(X)$, the similarity orbit of T which is denoted by Sim(T) is the set $\{J^{-1}TJ : J \in L(X), J$ invertible$\}$. In the next proposition, we show that if T is strongly topologically transitive then so is every operator in Sim(T).

Proposition 3.1. *Assume that X is a topological vector space, $T \in L(X)$ is strongly topologically transitive, and J is any invertible operator on X. Then $J^{-1}TJ$ is strongly topologically transitive.*

Proof. Let S be a right inverse map of T and $0 \neq x, y$ be arbitrary vectors in X. By Theorem 2.1, for the nonzero vector Jx and the vector Jy there exist sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(w_k)_k$ in X with $T^{n_k}w_k = 0$ $(k = 1, 2, \cdots)$ such that $S^{n_k}(Jx) + w_k - Jy \to 0$ as $k \to \infty$. Now, $J^{-1}SJ$ is a right inverse map of $J^{-1}TJ$ and if we put $w'_k = J^{-1}w_k$ then $w'_k \in \mathrm{Ker}(J^{-1}TJ)^{n_k}$. Then $(J^{-1}SJ)^{n_k}(x) + w'_k - y = J^{-1}\big(S^{n_k}(Jx) + w_k - Jy\big) \to 0$ as $k \to \infty$ and hence $J^{-1}TJ$ is strongly topologically transitive by Theorem 2.1. ■

If the underlying topological vector space is second countable and Baire then strong topological transitivity implies hypercyclicity. On the other hand, it is known that on a locally convex space X, the similarity orbit of any hypercyclic operator is SOT-dense in $L(X)$ [2, Proposition 2.20]. Denote by $L_{stt}(X)$ the set of all strongly topologically transitive operators on X. By the above proposition, if T is strongly topologically transitive then Sim(T) $\subset L_{stt}(X)$ and hence we have the following result.

Corollary 3.2. *If X is a second countable Baire locally convex space then $L_{stt}(X)$ is either empty or SOT-dense in $L(X)$.*

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