# On totally singular generalized quadratic forms

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**Abstract**

The algebra of similitudes of totally singular generalized quadratic forms in characteristic two is investigated. It is shown that this algebra satisfies certain functorial properties. An application of this study to central simple algebras with orthogonal involutions is also given.

## 1 Introduction

Generalized quadratic forms, introduced in [13], appear as natural extensions of quadratic forms, replacing the ground field by a division algebra with involution modulo its set of alternating elements. These forms were widely investigated in the literature in different aspects. See for example [2], [15], [16], [3], [1] and [14]. Although nonsingular generalized quadratic forms have been given more attention, singular forms have been also studied in the literature, either over fields or over division algebras with involution (see [7], [4] and [11]).

In this work we investigate some properties of totally singular generalized quadratic forms, i.e., generalized quadratic forms whose associated hermitian forms are trivial. Since these forms are trivial in characteristic different from two, our study is restricted to characteristic two. We first recall in §2 some basic definitions and properties of generalized quadratic forms. In §3, similitudes of totally singular generalized quadratic forms are studied. A similitude of a generalized quadratic form $q$ is an isometry $f : q \xrightarrow{\sim} \alpha \cdot q$, where $\alpha \in F$ is a scalar, called the multiplier of $q$. Let $(D, \theta)$ be a division algebra with involution of the first kind over a field $F$. As we will see in Proposition 3.1, for every totally

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singular generalized quadratic space $(V,q)$ over $(D,\theta)$, the set $\mathrm{Sim}(V,q)$ of similitudes of $(V,q)$ is an $F$-algebra and its multiplier map $\mu(V,q)$ is a totally singular quadratic form. This fact can be used to reduce the anisotropy of $q$ over $(D,\theta)$ to the anisotropy of $\mu(V,q)$ over $(F,\mathrm{id})$ and find several anisotropy criteria for $q$ (see Theorem 3.5). In §4, the behaviour of the pair $(\mathrm{Sim}(V,q),\mu(V,q))$ under separable extensions is studied. It is shown in Proposition 4.1 that if $K/F$ is a separable extension for which $D_K$ is a division ring, then $(\mathrm{Sim}(V_K,q_K),\mu(V_K,q_K)) \simeq (\mathrm{Sim}(V,q),\mu(V,q))_K$.

In [11] an algebra $S(A,\sigma)$ was canonically associated to every central simple algebra with orthogonal involution $(A,\sigma)$ over $F$. It was shown that this algebra carries a totally singular quadratic form $q_\sigma$ which determines certain anisotropy behaviour of $\sigma$. In the last section we show that this form can be realized as the multiplier map of similitudes of a suitable totally singular generalized quadratic space. More precisely, it is shown in Theorem 5.1 that in a representation of $(A,\sigma)$ as $(\mathrm{End}_D(V),\sigma_h)$ for some hermitian space $(V,h)$ over a division algebra with involution of the first kind $(D,\theta)$, the algebra $S(A,\sigma)$ is the algebra of similitudes of the totally singular generalized quadratic form $\varphi_\sigma : V \to D/\mathrm{Alt}(D,\theta)$ given by $\varphi_\sigma(v) = h(v,v) + \mathrm{Alt}(D,\theta)$. Also, the form $q_\sigma$ is the multiplier map $\mu(V,\varphi_\sigma)$. This result can be used to complement the characterization [11, (3.8)] of direct involutions, introduced in [5] (see Corollary 5.2).

## 2 Generalized quadratic forms

Throughout this paper, $F$ denotes a field of characteristic two.
Let $A$ be a central simple algebra over $F$. An involution on $A$ is a map $\sigma : A \to A$ with $\sigma^2 = \mathrm{id}$ such that $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for every $x, y \in A$. If $\sigma|_F = \mathrm{id}$ we say that $\sigma$ is of *the first kind*. For an algebra with involution $(A,\sigma)$ over $F$ and a field extension $K/F$ the map $\sigma_K := \sigma \otimes \mathrm{id}$ is an involution on $A_K := A \otimes K$. We denote the pair $(A_K, \sigma_K)$ by $(A,\sigma)_K$. An involution of the first kind on $A$ is called *symplectic* if it becomes adjoint to a symmetric alternating bilinear form after scalar extension to a splitting field of $A$. Otherwise, it is called *orthogonal*. The *set of alternating elements* of an algebra with involution $(A,\sigma)$ is defined as

$$\mathrm{Alt}(A,\sigma) = \{a - \sigma(a) \mid a \in A\}.$$

Let $(D,\theta)$ be a division algebra with involution of the first kind over $F$ and let $V$ be a finite-dimensional right vector space over $D$. A *hermitian form* on $V$ is a bi-additive map $h : V \times V \to D$ satisfying $h(u\alpha, v\beta) = \theta(\alpha)h(u,v)\beta$ and $h(v,u) = \theta(h(u,v))$ for all $u, v \in V$ and $\alpha, \beta \in D$. The form $h$ is called *nonsingular* if $h(u,v) = 0$ for all $v \in V$ implies that $u = 0$.

A *(generalized) quadratic form* on $V$ is a map $q : V \to D/\mathrm{Alt}(D,\theta)$ such that

$(i)$ $q(v\alpha) = \theta(\alpha)q(v)\alpha$ for every $v \in V$ and $\alpha \in D$;

$(ii)$ there exists a hermitian form $h_q$ on $V$ such that

$$q(u+v) - q(u) - q(v) = h_q(u,v) + \mathrm{Alt}(D,\theta) \quad \text{for all } u, v \in V.$$

In this case, we say that $(V, q)$ is a *quadratic space* over $(D, \theta)$. According to [1, (1.1)], the form $h_q$ is uniquely determined by $q$. The form $q$ is called *totally singular* if $h_q$ is trivial. We say that $q$ is *isotropic* if $q(v) = 0$ for some nonzero vector $v \in V$ and *anisotropic* otherwise.

Let $(V, q)$ be a totally singular quadratic space over $(F, \mathrm{id})$. Then there exists a decomposition $q \simeq \varphi \perp \rho$, where $\varphi$ is an anisotropic totally singular quadratic form and $\rho$ is the zero form. Moreover, the form $\varphi$ is uniquely determined, up to isometry. We call $\varphi$ the *anisotropic part* of $q$ and we denote it by $q_{\mathrm{an}}$. Set

$$Q(q) = \{q(v) \mid v \in V\}.$$

Then $Q(q) \subseteq F$ is a vector space over $F^2$. Note that $Q(q_{\mathrm{an}}) = Q(q)$. Also, the following result is easily verified.

**Lemma 2.1.** *Let $q$ and $q'$ be totally singular quadratic forms over $(F, \mathrm{id})$. Then $q \simeq q'$ if and only if $\dim_F q = \dim_F q'$ and $Q(q) = Q(q')$.*

A quadratic space $(V, q)$ over $(F, \mathrm{id})$ is called a *quasi-Pfister form* if there exists a bilinear Pfister form $\mathfrak{b} : V \times V \to F$ for which $q(v) = \mathfrak{b}(v, v)$ for all $v \in V$ (see [6, §10] for more details).

**Proposition 2.2.** *An anisotropic totally singular quadratic form $q$ over $(F, \mathrm{id})$ is a quasi-Pfister form if and only if $Q(q)$ is a field.*

*Proof.* The 'only if' implication can be found in [6, (10.4)]. The converse follows from [7, (8.5)]. ∎

## 3 Similitudes of totally singular quadratic forms

In this section, we fix $(D, \theta)$ as a division algebra with involution of the first kind over $F$ and $(V, q)$ as a totally singular quadratic space over $(D, \theta)$.

Let $\mathrm{End}_D(V)$ be the endomorphism algebra of $V$ over $D$. For $g_1, g_2 \in \mathrm{End}_D(V)$ we denote the composition $g_1 \circ g_2$ by $g_1 g_2$. A *similitude* of $(V, q)$ is a map $g \in \mathrm{End}_D(V)$ for which there exists $\alpha \in F$ such that $q(g(v)) = \alpha q(v)$ for every $v \in V$. The element $\alpha \in F$ is called the *multiplier* of $g$. The set of all similitudes of $(V, q)$ is denoted by $\mathrm{Sim}(V, q)$. Let

$$\mu(V, q) : \mathrm{Sim}(V, q) \to F$$

be the *multiplier map* which assigns to every $g \in \mathrm{Sim}(V, q)$ its multiplier. Hence, $q(g(v)) = \mu(V, q)(g)q(v)$ for every $g \in \mathrm{Sim}(V, q)$ and $v \in V$. We will simply denote $\mathrm{Sim}(V, q)$ by $\mathcal{S}$ and $\mu(V, q)$ by $\mu$ if no confusion arises.

**Proposition 3.1.** *The set $\mathcal{S}$ is an $F$-subalgebra of $\mathrm{End}_D(V)$ and the pair $(\mathcal{S}, \mu)$ is a totally singular quadratic space over $(F, \mathrm{id})$ satisfying $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)$ for all $g_1, g_2 \in \mathcal{S}$.*

*Proof.* Let $g_1, g_2 \in \mathcal{S}$ with respective multipliers $\alpha_1, \alpha_2 \in F$. Then

$$q(g_1 g_2(v)) = \alpha_1 q(g_2(v)) = \alpha_1 \alpha_2 q(v).$$

Also, as $(V,q)$ is totally singular we have

$$q((g_1+g_2)(v)) = q(g_1(v)) + q(g_2(v)) = (\alpha_1+\alpha_2)q(v).$$

Hence, $g_1g_2$ and $g_1+g_2$ are similitudes of $(V,q)$ with multipliers $\mu(g_1g_2) = \mu(g_1)\mu(g_2)$ and

$$\mu(g_1+g_2) = \mu(g_1) + \mu(g_2). \tag{1}$$

Further, if $g \in \mathcal{S}$ and $\alpha \in F$, then

$$q(\alpha g(v)) = \alpha^2 q(g(v)) = \alpha^2 \mu(g) q(v) \quad \text{for all } v \in V.$$

Hence, $\alpha g \in \mathcal{S}$ and $\mu(\alpha g) = \alpha^2\mu(g)$. Using this and (1), one concludes that $(S,\mu)$ is a totally singular quadratic space over $(F, \mathrm{id})$, proving the result. ■

**Lemma 3.2.** *The set $Q(\mu)$ is a subfield of $F$ containing $F^2$. In particular, $\mu_{\mathrm{an}}$ is a quasi-Pfister form.*

*Proof.* Since $\mu(\alpha\,\mathrm{id}) = \alpha^2$ for every $\alpha \in F$ we have $F^2 \subseteq Q(\mu)$. By Proposition 3.1, $\mu(g_1+g_2) = \mu(g_1)+\mu(g_2)$ and $\mu(g_1g_2) = \mu(g_1)\mu(g_2)$ for all $g_1, g_2 \in \mathcal{S}$, hence $Q(\mu)$ is a subring of $F$. Let $0 \neq \alpha \in Q(\mu)$. Choose a similitude $g \in \mathcal{S}$ with $\alpha = \mu(g)$. Then $\mu(\alpha^{-1}g) = \alpha^{-2}\alpha = \alpha^{-1}$, hence $\alpha^{-1} \in Q(\mu)$. This proves that $Q(\mu)$ is a field. The second statement follows from Proposition 2.2. ■

In view of Proposition 3.1, the map $\mu : \mathcal{S} \to F$ is a ring homomorphism. We denote the kernel of $\mu$ by $\mathcal{I}(V,q)$. Hence, $\mathcal{I}(V,q)$ is an ideal of $\mathcal{S}$. Note that

$$\mathcal{I}(V,q) = \{g \in \mathcal{S} \mid q(g(v)) = 0 \text{ for all } v \in V\} = \mathrm{Hom}_D(V, \ker q).$$

In particular, $\mathcal{I}(V,q)$ is a right ideal of $\mathrm{End}_D(V)$. We will simply denote $\mathcal{I}(V,q)$ by $\mathcal{I}$.

**Proposition 3.3.** *The quotient ring $\mathcal{S}/\mathcal{I}$ is a field. In particular, $\mathcal{I}$ is a maximal ideal of $\mathcal{S}$.*

*Proof.* Since $\mathcal{S}/\mathcal{I} \simeq \mathrm{Im}\,\mu \subseteq F$, the quotient ring $\mathcal{S}/\mathcal{I}$ is commutative. Let $\alpha \in \mathrm{Im}\,\mu$ be a nonzero element and write $\alpha = \mu(g)$ for some $g \in \mathcal{S} \setminus \mathcal{I}$. Then $\alpha^{-1} = \mu(\alpha^{-1}g) \in \mathrm{Im}\,\mu$. Hence, $\mathcal{S}/\mathcal{I} \simeq \mathrm{Im}\,\mu$ is a field. ■

We denote the field $\mathrm{Sim}(V,q)/\mathcal{I}(V,q)$ by $\overline{\mathrm{Sim}}(V,q)$. Define a map $\bar{\mu}(V,q) : \bar{\mathcal{S}} \to F$ via

$$\bar{\mu}(V,q)(g+\mathcal{I}) = \mu(g).$$

Since the restriction $\mu|_{\mathcal{I}}$ is trivial, the map $\bar{\mu}(V,q)$ is well-defined. We will simply denote $\overline{\mathrm{Sim}}(V,q)$ by $\bar{\mathcal{S}}$ and $\bar{\mu}(V,q)$ by $\bar{\mu}$.

**Lemma 3.4.** *The pair $(\bar{\mathcal{S}}, \bar{\mu})$ is an anisotropic totally singular quadratic form over $(F, \mathrm{id})$ and $\bar{\mu} \simeq \mu_{\mathrm{an}}$. In particular, $\bar{\mu}$ is a quasi-Pfister form.*

*Proof.* Clearly, $\bar{\mu}$ is an anisotropic totally singular quadratic form. The rest statements of the result follow from the equalities $Q(\bar{\mu}) = Q(\mu) = Q(\mu_{\mathrm{an}})$ and Lemma 3.2. ■

**Theorem 3.5.** *The following statements are equivalent.*

(1) *$q$ is anisotropic.*

(2) *$\mu$ is anisotropic.*

(3) $\mathcal{I} = \{0\}$.

(4) $\mu \simeq \bar{\mu}$.

(5) *$\mathcal{S}$ is a field.*

*Furthermore, if these conditions are satisfied, then identifying $F$ with a subfield of $\mathcal{S}$, $\mathcal{S}/F$ is a purely inseparable extension of exponent one, i.e., $g^2 \in F$ for all $g \in \mathcal{S}$.*

*Proof.* The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from the equality $\mathcal{I} = \mathrm{Hom}_D(V, \ker q)$. The equivalence (3) $\Leftrightarrow$ (4) is evident and (3) $\Leftrightarrow$ (5) follows from Proposition 3.3. To prove the last statement of the result, let $g \in \mathcal{S}$ and set $\alpha = \mu(g)$. Then $\mu(g^2) = \alpha^2 = \mu(\alpha)$ by Proposition 3.1, hence $\mu(g^2 + \alpha) = 0$. It follows that $g^2 = \alpha \in F$, because $\mu$ is anisotropic. ■

## 4 Functoriality

We continue to assume that $(D, \theta)$ is a division algebra with involution of the first kind over $F$ and $(V, q)$ is a totally singular quadratic space over $(D, \theta)$. Let $K/F$ be a field extension such that $D_K$ is a division ring. Then there exists a quadratic space $(V, q)_K := (V_K, q_K)$ over $(D, \theta)_K$, where $V_K = V \otimes_F K$ and the map $q_K : V_K \to D_K / \mathrm{Alt}(D_K, \theta_K)$ satisfies $q_K(v \otimes \alpha) = \alpha^2 q(v)$ for every $v \in V$ and $\alpha \in K$. It can be shown that the multiplier map $\mu(V, q)$ is not functorial, in the sense that the isometry $\mu(V_K, q_K) \simeq \mu(V, q)_K$ does not generally hold (see Remark 5.3 below). However, for separable extensions we have the following result.

**Proposition 4.1.** *Let $K/F$ be a separable field extension such that $D_K$ is a division ring. Then there exist a $K$-algebra isomorphism $\mathrm{Sim}(V_K, q_K) \simeq \mathrm{Sim}(V, q) \otimes_F K$ and an isometry*

$$(\mathrm{Sim}(V_K, q_K), \mu(V_K, q_K)) \simeq (\mathrm{Sim}(V, q), \mu(V, q))_K.$$

*Proof.* Since

$$\mathrm{Sim}(V_K, q_K) \subseteq \mathrm{End}_{D_K}(V_K) \simeq \mathrm{End}_D(V) \otimes_F K,$$

we may identify $\mathrm{Sim}(V_K, q_K)$ with a subalgebra of $\mathrm{End}_D(V) \otimes_F K$. Clearly, $f \otimes \alpha \in \mathrm{Sim}(V_K, q_K)$ for every $f \in \mathrm{Sim}(V, q)$ and $\alpha \in K$. Conversely, let $f \in \mathrm{Sim}(V_K, q_K)$ and set $\lambda = \mu(V_K, q_K)(f) \in K$. Write

$$f = \sum_{i=1}^{m} g_i \otimes \lambda_i,$$

where $g_i \in \mathrm{End}_D(V)$ and $\lambda_i \in K$ for $i = 1, \cdots, m$. Since $K/F$ is separable, there exists a field $L$ with $F \subseteq L \subseteq K$ such that $[L : F] < \infty$ and $\lambda, \lambda_1, \cdots, \lambda_m \in L$.

Choose $\eta \in L$ such that $L = F(\eta)$ and set $n = [L : F]$. Then the set $\{1, \eta, \cdots, \eta^{n-1}\}$ is a basis of $L$ over $F$ and the map $f$ can be rewritten as

$$f = \sum_{i=0}^{n-1} f_i \otimes \eta^i,$$

where $f_0, \cdots, f_{n-1} \in \mathrm{End}_D(V)$. We show that $f_i \in \mathrm{Sim}(V, q)$ for every $i = 0, \cdots, n-1$, which implies that

$$f \in \mathrm{Sim}(V, q) \otimes_F L \subseteq \mathrm{Sim}(V, q) \otimes_F K.$$

Since $L/F$ is separable and $\operatorname{char} F = 2$, we have $L = F(\eta^2)$. Hence, the set $\{1, \eta^2, \cdots, \eta^{2(n-1)}\}$ is also a basis of $L$ over $F$. Write $\lambda = \sum_{i=0}^{n-1} \alpha_i \eta^{2i}$ for some $\alpha_0, \cdots, \alpha_{n-1} \in F$. Then for every $v \in V$ we have

$$\begin{aligned}\sum_{i=0}^{n-1} \alpha_i q(v) \eta^{2i} &= \lambda q(v) = \lambda q_K(v \otimes 1) = q_K(f(v \otimes 1)) \\ &= q_K\Big(\sum_{i=0}^{n-1} f_i(v) \otimes \eta^i\Big) = q_L\Big(\sum_{i=0}^{n-1} f_i(v) \otimes \eta^i\Big) = \sum_{i=0}^{n-1} q(f_i(v)) \eta^{2i}.\end{aligned}$$

It follows that $q(f_i(v)) = \alpha_i q(v)$ for all $i$, hence $f_i \in \mathrm{Sim}(V, q)$ and

$$\mu(V, q)(f_i) = \alpha_i \in F. \tag{2}$$

This proves the existence of the desired $K$-algebra isomorphism

$$\mathrm{Sim}(V_K, q_K) \simeq \mathrm{Sim}(V, q) \otimes_F K. \tag{3}$$

Note that (3) is also an isomorphism of right vector spaces over $D_K$. Moreover, (2) shows that

$$\mu(V_K, q_K)(f) = \lambda = \sum_{i=0}^{n-1} \mu(V, q)(f_i) \eta^{2i},$$

proving that $(\mathrm{Sim}(V_K, q_K), \mu(V_K, q_K)) \simeq (\mathrm{Sim}(V, q), \mu(V, q))_K$. ∎

By Theorem 3.5 the anisotropy of $q$ over $(D, \theta)$ reduces to the anisotropy of $\mu$ over $(F, \mathrm{id})$. We know that an anisotropic totally singular quadratic space over $(F, \mathrm{id})$ remains anisotropic over every separable extension $K$ of $F$ (this is immediate from the corresponding result for symmetric bilinear forms [8, (10.2.1)]). Hence, using the isometry $\mu(V_K, q_K) \simeq (\mu(V, q))_K$ in Proposition 4.1 we obtain the following result.

**Corollary 4.2.** *Let $K/F$ be a separable field extension such that $D_K$ is a division ring. If $q$ is anisotropic then $q_K$ is also anisotropic.*

**Remark 4.3.** It is easy to see that for any quadratic space $(V, q)$ over $(D, \theta)$ there exists a hermitian form $(V, h)$ over $(D, \theta)$ such that

$$q(x) = h(x, x) + \mathrm{Alt}(D, \theta) \in D / \mathrm{Alt}(D, \theta).$$

This correspondence clearly respects field extensions, and $(V, q)$ is anisotropic if and only if $(V, h)$ is *direct*, i.e., the condition $h(v, v) \in \mathrm{Alt}(D, \theta)$ for $v \in V$ implies that $v = 0$. In view of these facts, Corollary 4.2 also follows from the main result of [5], that direct hermitian forms remain direct over separable extensions.

**Corollary 4.4.** *Let $K/F$ be a separable field extension such that $D_K$ is a division ring. Then*

$$(\overline{\mathrm{Sim}}(V_K, q_K), \bar{\mu}(V_K, q_K)) \simeq (\overline{\mathrm{Sim}}(V, q), \bar{\mu}(V, q))_K.$$

*Proof.* Proposition 4.1 and Corollary 4.2 imply that $(\mu(V,q)_{\mathrm{an}})_K \simeq \mu(V_K, q_K)_{\mathrm{an}}$. The result therefore follows from Lemma 3.4. ∎

## 5 Applications to orthogonal involutions

Let $(A, \sigma)$ be a central simple algebra with orthogonal involution over $F$. By [10, (2.6)] we have $F \cap \mathrm{Alt}(A, \sigma) = \{0\}$. Let

$$S(A, \sigma) = \{x \in A \mid \sigma(x)x \in F \oplus \mathrm{Alt}(A, \sigma)\}.$$

Hence, for every $x \in S(A, \sigma)$ there exists a unique element $\alpha \in F$ such that $\sigma(x)x + \alpha \in \mathrm{Alt}(A, \sigma)$. As in [11] we denote the element $\alpha$ by $q_\sigma(x)$. According to [11, (3.2)], $S(A, \sigma)$ is an $F$-subalgebra of $A$ and the pair $(S(A, \sigma), q_\sigma)$ is a totally singular quadratic space over $(F, \mathrm{id})$ satisfying $q_\sigma(xy) = q_\sigma(x)q_\sigma(y)$ for every $x, y \in S(A, \sigma)$. We denote the kernel of the map $q_\sigma : S(A, \sigma) \to F$ by $\mathfrak{m}(A, \sigma)$.

Let $D$ be a division algebra, Brauer-equivalent to $A$ and let $\theta : D \to D$ be an involution of the first kind on $D$. By [10, (4.2)] there exists a unique, up to a scalar factor in $F^\times$, nonsingular hermitian space $(V, h)$ over $(D, \theta)$ such that

$$(A, \sigma) \simeq (\mathrm{End}_D(V), \sigma_h), \tag{4}$$

where $\sigma_h$ is the adjoint involution of $\mathrm{End}_D(V)$ with respect to $h$. Moreover, since $\sigma$ is orthogonal, the form $h$ is *non-alternating*, i.e., $h(v, v) \notin \mathrm{Alt}(D, \theta)$ for some $v \in V$. Define a map $\varphi_\sigma : V \to D/\,\mathrm{Alt}(D, \theta)$ via

$$\varphi_\sigma(v) = h(v, v) + \mathrm{Alt}(D, \theta).$$

It is readily seen that $(V, \varphi_\sigma)$ is a totally singular quadratic space over $(D, \theta)$. Note that $\varphi_\sigma$ is uniquely determined by $\sigma$, up to a scalar factor in $F^\times$.

**Theorem 5.1.** *Considering the isomorphism* (4) *as an identification we have*

*(i)* $S(A, \sigma) = \mathrm{Sim}(V, \varphi_\sigma)$.

*(ii)* $q_\sigma = \mu(V, \varphi_\sigma)$.

*(iii)* $\mathfrak{m}(A, \sigma) = \mathcal{I}(V, \varphi_\sigma)$.

*Proof.* Let $f \in S(A, \sigma)$ and set $\lambda = q_\sigma(f) \in F$. Then there exists $g \in A$ such that $\sigma(f)f + \lambda = g - \sigma(g)$. Hence, for every $v \in V$ we have

$$\begin{aligned}
h(f(v), f(v)) - \lambda h(v, v) &= h(\sigma(f)f(v), v) - \lambda h(v, v) \\
&= h((\sigma(f)f - \lambda)(v), v) \\
&= h((\sigma(g) - g)(v), v) \\
&= h(\sigma(g)(v), v) - h(g(v), v) \\
&= h(v, g(v)) - h(g(v), v) \\
&= h(v, g(v)) - \theta(h(v, g(v))) \in \mathrm{Alt}(D, \theta).
\end{aligned}$$

It follows that $\varphi_\sigma(f(v)) = \lambda\varphi_\sigma(v)$ for all $v \in V$, hence $f \in \mathrm{Sim}(V, \varphi_\sigma)$ and $\mu(V, \varphi_\sigma)(f) = \lambda = q_\sigma(f)$. This proves that $S(A, \sigma) \subseteq \mathrm{Sim}(V, \varphi_\sigma)$.

Let $f \in \mathrm{Sim}(V, \varphi_\sigma)$ and set $\lambda = \mu(V, \varphi_\sigma)(f)$. Since

$$\begin{aligned} h((\sigma(f)f + \lambda)(v), v) &= h(\sigma(f)f(v), v) + \lambda h(v, v) \\ &= h(f(v), f(v)) + \lambda h(v, v), \end{aligned}$$

the equality $\varphi_\sigma(f(v)) = \lambda\varphi_\sigma(v)$ implies that

$$h((\sigma(f)f + \lambda)(v), v) \in \mathrm{Alt}(D, \theta) \quad \text{for all } v \in V. \tag{5}$$

As already observed, $h$ is non-alternating. Hence, by [9, Ch. I, (6.2.4)], $(V, h)$ has an orthogonal basis, i.e., a basis $\{v_1, \cdots, v_n\}$ satisfying $h(v_i, v_j) = 0$ for $i \neq j$. Note that $h(v_i, v_i) \neq 0$ for all $i$, because $h$ is nonsingular. For $j = 1, \cdots, n$, write

$$(\sigma(f)f + \lambda)(v_j) = \sum_{i=1}^{n} v_i\alpha_{ij}, \tag{6}$$

where $\alpha_{ij} \in D$. By (5) there exists $d_j \in D$, $j = 1, \cdots, n$, such that

$$h((\sigma(f)f + \lambda)(v_j), v_j) = d_j - \theta(d_j). \tag{7}$$

For $j = 1, \cdots, n$, set $w_j = v_j(h(v_j, v_j)^{-1}\theta(d_j)) \in V$, so that $h(w_j, v_j) = d_j$ and $h(w_j, v_i) = 0$ for $i \neq j$. Let $g \in \mathrm{End}_D(V)$ be the map induced by

$$g(v_j) = w_j + \sum_{i=1}^{j-1} v_i\alpha_{ij}, \quad \text{for } j = 1, \cdots, n.$$

Then for every $i, j$ we have

$$h((g - \sigma(g))(v_j), v_i) = h(g(v_j), v_i) - h(v_j, g(v_i)) = \begin{cases} h(v_i\alpha_{ij}, v_i) & i < j \\ d_j - \theta(d_j) & i = j \\ -h(v_j, v_j\alpha_{ji}) & i > j \end{cases} \tag{8}$$

On the other hand, using (6) we obtain $h(v_i\alpha_{ij}, v_i) = h((\sigma(f)f + \lambda)(v_j), v_i)$ and

$$h(v_j, v_j\alpha_{ji}) = h(v_j, (\sigma(f)f + \lambda)(v_i)) = h((\sigma(f)f + \lambda)(v_j), v_i).$$

Hence, (8) and (7) imply that

$$h((g - \sigma(g))(v_j), v_i) = h((\sigma(f)f + \lambda)(v_j), v_i),$$

for every $i, j$. Since $h$ is nonsingular, we get $(g - \sigma(g))(v_j) = (\sigma(f)f + \lambda)(v_j)$ for $j = 1, \cdots, n$, i.e.,

$$\sigma(f)f + \lambda = g - \sigma(g) \in \mathrm{Alt}(A, \sigma).$$

Hence, $f \in S(A, \sigma)$ and $\lambda = q_\sigma(f) \in F$. This proves parts $(i)$ and $(ii)$. The third part follows from the second. ∎

We recall from [5] that an involution $\sigma$ on a central simple algebra $A$ is said to be *direct* if $\sigma(a)a \in \mathrm{Alt}(A,\sigma)$ for $a \in A$ implies that $a = 0$. The following result complements the characterization [11, (3.8)] of direct involutions.

**Corollary 5.2.** *For a central simple algebra with orthogonal involution $(A,\sigma)$ over $F$ the following statements are equivalent.* (1) *$\sigma$ is direct.* (2) *$q_\sigma$ is anisotropic.* (3) *$S(A,\sigma)$ is a field.* (4) $\mathfrak{m}(A,\sigma) = \{0\}$. (5) *$\varphi_\sigma$ is anisotropic.* (6) *$\mu(V,\varphi_\sigma)$ is anisotropic.* (7) $\mathcal{I}(V,\varphi_\sigma) = \{0\}$. (8) $\mu(V,\varphi_\sigma) \simeq \bar{\mu}(V,\varphi_\sigma)$. (9) $\mathrm{Sim}(V,\varphi_\sigma)$ *is a field.*

*Proof.* The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) were proved in [11, (3.8)] and (2) $\Leftrightarrow$ (4) is evident. The other equivalences follow from Theorem 3.5 and Theorem 5.1. ∎

**Remark 5.3.** As already mentioned in Remark 4.3, every totally singular generalized quadratic form $q$ is derived from a hermitian form. Hence, in view of Theorem 5.1, the functoriality of the pair $(\mathrm{Sim}(V,q), \mu(V,q))$ under separable extensions also follows from the functoriality of $S(A,\sigma)$ established in [12, (3.5)]. Moreover, it is worth pointing out that this functoriality does not hold in general. This can be easily seen using Theorem 5.1 and the corresponding example of non-functoriality of the form $q_\sigma$ given in [11, (3.18)].

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