A closure operator for clopen topologies

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Abstract

A topology τ on a nonempty set X is called a clopen topology provided each member of τ is both open and closed. Given a function f from X to Y, the operator $E \mapsto f^{-1}(f(E))$ is a closure operator on the power set of X whose fixed points are closed subsets corresponding to a clopen topology on X. Conversely, for each clopen topology τ on X, we produce a function f with domain X such that $\tau = \{E \subseteq X : E = f^{-1}(f(E))\}$. We characterize the clopen topologies on X as those that are weak topologies determined by a surjective function with values in some discrete topological space. Paralleling this result, we show that a topology admits a clopen base if and only if it is a weak topology determined by a family of functions with values in discrete spaces.

1 Introduction

In any course on mathematical notation and proof, students consider the operator $E \mapsto f^{-1}(f(E))$ on the power set $\mathcal{P}(X)$ of a nonempty set X where f is a function from X to some other nonempty set Y. Invariably, they are asked to show that (1) $\forall E \subseteq X, E \subseteq f^{-1}(f(E))$, and (2) the operator is the identity map on the power set if and only if f is one-to-one.

Recall that an operator Γ on $\mathcal{P}(X)$ is called a (Kuratowski) *closure operator* (see [4, pp. 38-45] or [7, p. 25]) if it satisfies the following four properties:

- $\Gamma(\emptyset) = \emptyset$;

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- $\forall E \subseteq X,\ E \subseteq \Gamma(E)$;

- $\forall E \subseteq X,\ \Gamma(\Gamma(E)) = \Gamma(E)$;

- $\forall E_1, E_2$ in $\mathcal{P}(X),\ \Gamma(E_1 \cup E_2) = \Gamma(E_1) \cup \Gamma(E_2)$.

Given a topology τ on X, the operator

$$E \mapsto \mathrm{cl}(E) := \{x \in X : x \in V \in \tau \Rightarrow V \cap E \neq \emptyset\}$$

satisfies the above four conditions, and $\{E \subseteq X : \mathrm{cl}(E) = E\}$ is the family of closed sets as determined by τ. Conversely, given a closure operator Γ on $\mathcal{P}(X)$, the family of fixed points of Γ, that is, $\{E \subseteq X : \Gamma(E) = E\}$, is the family of closed sets corresponding to some topology on X.

Our particular operator $E \mapsto f^{-1}(f(E))$ is a closure operator on X, and it is not hard to show that the complement of each set E satisfying $f^{-1}(f(E)) = E$ also satisfies this property. This means that each open subset with respect to the induced topology is also closed. We call a topology τ on X *clopen* provided each member of τ is also τ-closed. Clopen topologies are evidently stable under taking complements of its members, and are also stable under taking arbitrary intersections as well as arbitrary unions. In terms of order (as determined by inclusion), this means that the clopen sets of a clopen topology form a complete Boolean algebra, while the clopen sets as determined by an arbitrary topology just form a Boolean algebra. For the record, we note that each Boolean algebra is isomorphic to the clopen sets of its so-called Stone space [5].

Of course, the familiar discrete and indiscrete topologies on a nonempty set X are clopen topologies; the indiscrete topology is the only connected clopen topology on X. Here is a nontrivial clopen topology on $X = \{a, b, c, d\}$:

$$\{\emptyset, \{a\}, \{d\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

Clopeness of spaces is evidently preserved under topological sums and quotients. Herrlich [2, p. 207] observed that a topological space is clopen if and only if (1) $\langle X, \tau\rangle$ is a topological sum of indiscrete spaces, if and only if (2) the Kolmogorov quotient of $\langle X, \tau\rangle$, that is, the T_0-reflection of the space, is discrete (see Corollary 3.2 below). Because of (1), clopen spaces have been called *indiscretely generated* [2] or *locally indiscrete* [3] in the literature.

The main result of this note shows that each clopen topology on X arises from an operator of the form $E \mapsto f^{-1}(f(E))$ for an appropriate function f with domain X. We also show that each clopen topology on X arises as a weak topology as determined by a surjection from X to an appropriately chosen discrete space. Care is taken to compare and contrast clopen topologies with topologies having a clopen base.

2 Preliminaries

In the sequel we denote the positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, and the real numbers by $\mathbb{R}$. By a *partition* $\{A_j : j \in J\}$ of a set X, we mean a cover of X consisting of nonempty subsets such that $\forall j_1, j_2 \in J,\ A_{j_1} \cap A_{j_2} \neq \emptyset \Rightarrow A_{j_1} = A_{j_2}$. We write $\chi_A : X \to \{0,1\}$ for the characteristic function for a subset A of a nonempty set X.

We denote a nonempty set X equipped with a topology τ by the notation $\langle X, \tau \rangle$. We call a topological space $\langle X, \tau \rangle$ *discrete* if τ is the discrete topology.

Suppose X is a nonempty set, and $\{\langle Y_j, \sigma_j \rangle : j \in J\}$ is a family of topological spaces, and $\forall j \in J,\ f_j : X \to Y_j$. Then the topology generated by the family $\{f_j^{-1}(V) : V \in \sigma_j, j \in J\}$ is the weakest topology τ on X for which each $f_j : \langle X, \tau \rangle \to \langle Y, \sigma_j \rangle$ is continuous and is called the *weak topology* on X determined by f [7, p. 55]. We denote the weak topology by $\tau_{\{f_j : j \in J\}}$.

For a general topological space $\langle X, \tau \rangle$, if $x \in X$, we put

$$[x] := \cap \{V \in \tau : x \in V\}.$$

That $[x] = \{x\}$ for all $x \in X$ is equivalent to $\langle X, \tau \rangle$ satisfying the T_1-separation axiom. When $\langle X, \tau \rangle$ is a clopen space, then $\{[x] : x \in X\}$ forms a base for the topology, for if $x \in V \in \tau$, then $x \in [x] \subseteq V$. In this setting, we call $\{[x] : x \in X\}$ the *essential base* for the topology, as each base must contain these sets. Clearly, a clopen topology is T_0 if and only if it is discrete. Notice that the essential base for the clopen topology on $\{a, b, c, d\}$ of the Introduction partitions the underlying set, and this is no accident.

Each subspace of a clopen space is a clopen space; in fact, the essential base for the relative topology is the trace of the essential base for the underlying space on the subspace. However, a product of clopen spaces need not be one: if we equip $\{0,1\}$ with the discrete topology, then $\{0,1\}^{\mathbb{N}}$ being homeomorphic to the Cantor set as a subspace of the line [7, p. 217] is not a clopen space. We leave the easy proof of the next proposition that we will use in the sequel to the reader.

Proposition 2.1. *Let X be a nonempty set and let $\langle Y, \sigma \rangle$ be a clopen space. If $f : X \to Y$ is any function, then $\{f^{-1}(V) : V \in \sigma\}$, the weak topology on X determined by $\{f\}$, is a clopen topology on X.*

It can be shown that the essential base for the weak topology in Proposition 2.1 consists of $\{f^{-1}([y]) : y \in f(X)\}$ (see Theorem 3.5 below). We next prove a partial converse to Proposition 2.1.

Proposition 2.2. *Let $\langle Y, \sigma \rangle$ be a topological space and let $f : X \to Y$ be onto. Suppose that the weak topology $\tau_{\{f\}}$ on X is clopen. Then $\langle Y, \sigma \rangle$ is a clopen space.*

Proof. Since $V \in \sigma$ if and only if $f^{-1}(V) \in \tau_{\{f\}}$, by [1, Prop 2.4.3], $\langle Y, \sigma \rangle$ is homeomorphic to a quotient of a clopen space. ∎

A topological space with a clopen base is called *zero-dimensional* [7, p. 210]. Such spaces perhaps originate in a paper of Vedenissoff [6], who proved that a

Hausdorff space has a clopen base if and only if it is homeomorphic to a subspace of a product of 2-point discrete spaces. That a topological space $\langle X, \tau \rangle$ has a base consisting of clopen subsets is not enough to conclude that $\langle X, \tau \rangle$ is a clopen space: $\mathbb{Q}$ as a subspace of $\mathbb{R}$ has a clopen base consisting of all sets of the form $(\alpha, \beta) \cap \mathbb{Q}$ where $\alpha < \beta$ are irrationals. As another example, the standard base for the Sorgenfrey line [7, p. 34] is also a clopen base.

Our next result directly connects topological spaces with a clopen base with clopen topologies.

Theorem 2.3. *Let $\langle X, \tau \rangle$ be a topological space. The following conditions are equivalent:*

1. *τ has a clopen base $\mathcal{B}$;*

2. *τ is the weak topology determined by a family of functions each with domain X and with values in $\{0, 1\}$ equipped with the discrete topology;*

3. *τ is the weak topology determined by a family of functions each with domain X and with values in a clopen space (that can vary with the function);*

4. *τ is the supremum of a family of clopen topologies on X (with respect to the lattice of topologies on X).*

Proof. (1) $\Rightarrow$ (2). The desired family of functions is $\{\chi_B : B \in \mathcal{B}\}$. Since $\mathcal{B}$ consists of clopen subsets of X, the generating sets for the weak topology consists only of elements of τ, and furthermore, $\mathcal{B} = \{\chi_B^{-1}(\{1\}) : B \in \mathcal{B}\}$ generates the topology.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4). If the family of functions in (3) is $\{f_j : X \to \langle Y_j, \sigma_j \rangle : j \in J\}$, then by Proposition 2.1, the weak topology generated by each f_j, namely

$$\{f_j^{-1}(V) : V \in \sigma_j\}$$

is a clopen topology, and τ is the smallest topology containing the individual weak topologies.

(4) $\Rightarrow$ (1). Suppose τ is the the supremum of $\{\tau_j : j \in J\}$ where each τ_j is clopen; then $\cup_{j \in J} \tau_j$ is a family of subsets of X generating τ and each member is clopen with respect to τ. Since a finite intersection of τ-clopen sets is τ-clopen, finite intersections of members of $\cup_{j \in J} \tau_j$ form a clopen base for τ. ∎

We also could have listed in Theorem 2.3 as an additional characteristic property for zero-dimensionality this known one but chose not to: τ is the weak topology determined by a family of functions each with domain X with values in a zero-dimensional space (that can vary with the function). For the benefit of the reader, we include a short direct argument for (1) $\Rightarrow$ (4) of Theorem 2.3: $\tau = \bigvee_{B \in \mathcal{B}} \tau_B$ where for each $B \in \mathcal{B}$, $\tau_B := \{\emptyset, B, B^c, X\}$.

The proof of Theorem 2.3 shows that if a topological space has a clopen base, then it is the weak topology determined by the characteristic functions for the members of that base. In view of [7, Theorem 8.10], this yields the following attractive corollary.

Corollary 2.4. *Let $\langle X, \tau \rangle$ be a topological space with clopen base $\mathcal{B}$, and let $\langle Y, \sigma \rangle$ be a second topological space. Then $f : Y \to X$ is continuous if and only if $\forall B \in \mathcal{B}$, $\chi_B \circ f$ is continuous.*

3 Results

We first show that in a clopen space $\langle X, \tau \rangle$, the essential base partitions X. With this in mind, we will write $x_1 \equiv x_2$ in X if $[x_1] = [x_2]$.

Proposition 3.1. *Let $\langle X, \tau \rangle$ be a clopen space. Then $\{[x] : x \in X\}$ partitions X. Conversely, if a topological space has the property that $\{[x] : x \in X\}$ is a locally finite family of closed sets, then the space is already clopen.*

Proof. Without the clopenness assumption, these sets cover X. It remains to show that if $x_1 \in [x_2]$, then $[x_1] = [x_2]$. By definition, each open set containing x_2 contains x_1 and so $[x_1] \subseteq [x_2]$. For the reverse inclusion, suppose $[x_1]$ fails to contain $[x_2]$. Take $p \in [x_2] \backslash [x_1]$; there exists some open neighborhood V of x_1 with $p \notin V$. Now $x_2 \notin V$ either, else $p \in [x_2] \Rightarrow p \in V$. Then V^c would be a neighborhood of x_2 that fails to contain x_1, contradicting $x_1 \in [x_2]$. We conclude that $[x_2] \subseteq [x_1]$, so that $[x_2] = [x_1]$.

For the converse, simply note that if $V \in \tau$, then by definition, $V = \cup_{x \in V} [x]$, and the union of a locally finite family of closed sets is closed [7, Lemma 20.5]. ∎

The condition that $\{[x] : x \in X\}$ be a partition of X by closed sets is not enough to guarantee that the space is clopen, for in any T_1 space $\langle X, \tau \rangle$, $\{[x] : x \in X\}$ consists of the singleton subsets of X.

Corollary 3.2. *Let $\langle X, \tau \rangle$ be a clopen space. Then each nonempty $V \in \tau$ can be written uniquely as a union of sets from the essential base: namely, V can only be expressed as the union of all the essential basic open sets that it contains.*

Corollary 3.3. *Let $\langle X, \tau \rangle$ be a clopen space. The following are equivalent:*

1. *$\langle X, \tau \rangle$ is compact;*

2. *the essential base for the topology is finite;*

3. *the topology τ consists of a finite family of sets.*

Corollary 3.4. *Each clopen space $\langle X, \tau \rangle$ is pseudo-metrizable.*

Proof. A compatible pseudometric d is given by $d(x_1, x_2) = 0$ if $x_1 \equiv x_2$ and $d(x_1, x_2) = 1$ otherwise. The open balls of radius at most one as determined by d consist of members of the essential base, while balls of larger radii give X. ∎

Theorem 3.5. *Let $f\colon X \to \langle Y, \sigma\rangle$ be onto and equip X with the weak topology $\tau_{\{f\}}$. Suppose $\langle Y, \sigma\rangle$ is clopen so that $\langle X, \tau_{\{f\}}\rangle$ is as well. Then $[y] \mapsto f^{-1}([y])$ is a bijection from the essential base of $\langle Y, \sigma\rangle$ to the essential base of $\langle X, \tau_{\{f\}}\rangle$.*

Proof. Let $p \in f^{-1}([y])$. We claim $[p] = f^{-1}([y])$. First, since $[p] \in \tau_{\{f\}}$, $\exists V \in \sigma$ with $[p] = f^{-1}(V)$. Now $V = (V - [y]) \cup (V \cap [y])$. If $V - [y]$ were nonempty, then by surjectivity $f^{-1}(V - [y]) \cap [p] \neq \emptyset$ and this contradicts $f^{-1}(V - [y]) \cap f^{-1}([y]) = \emptyset$ because $[p] \subseteq f^{-1}([y]) \in \tau_{\{f\}}$. This establishes the claim.

Obviously the map $[y] \mapsto f^{-1}([y])$ is one-to-one as the preimage operator preserves disjointness. To see it is onto, fix $x \in X$ and consider $[x]$. Since $[x] \in \tau_{\{f\}}$, $\exists V \in \sigma$ with $f^{-1}(V) = [x]$. Now if $V \neq [y]$ for some $y \in Y$, then by Proposition 3.1, $\exists y_1, y_2 \in V$ with $[y_1] \cap [y_2] = \emptyset$ and $[y_1] \cup [y_2] \subseteq V$. Then $f^{-1}([y_1])$ and $f^{-1}([y_2])$ are disjoint nonempty open subsets of $[x]$ which is impossible. ∎

We next verify that if $f : X \to Y$ is a function between nonempty sets X and Y, then $E \mapsto f^{-1}(f(E))$ a is closure operator.

Proposition 3.6. *Let $f : X \to Y$ where X, Y are nonempty sets; then $\Gamma(E) = f^{-1}(f(E))$ is a closure operator on $\mathcal{P}(X)$.*

Proof. The properties $\Gamma(\emptyset) = \emptyset$ and $E \subseteq \Gamma(E)$ are obvious from the definition of Γ. That Γ preserves unions of size two follows from the fact that both the direct image and preimage operators preserve arbitrary unions. Finally, idempotency follows from the formula

$$f(f^{-1}(f(E)) = f(E), \quad E \subseteq X$$

which is valid in general. ∎

We denote the topology on X associated with our closure operator by τ_f, that is,

$$\tau_f := \{E^c : f^{-1}(f(E)) = E\}.$$

We now prove that each member of τ_f is closed as well.

Proposition 3.7. *Let X and Y be nonempty sets and let $f : X \to Y$. Suppose for some $E \subseteq X$, we have $f^{-1}(f(E)) = E$. Then $f^{-1}(f(E^c)) = E^c$.*

Proof. We need only show that $f^{-1}(f(E^c)) \subseteq E^c$. To this end, let $x \in f^{-1}(f(E^c))$. As $f(x) \in f(E^c)$, there exists $p \in E^c$ with $f(p) = f(x)$. Now if $x \in E$ were true, then $p \in f^{-1}(f(E)) = E$ which is impossible. We conclude that $x \in E^c$ as required. ∎

We now come to the main result of this note.

Theorem 3.8. *Let $\langle X, \tau\rangle$ be a topological space. Then the space is clopen if and only if there exists a function f with domain X such that $\tau = \tau_f$.*

Proof. Sufficiency has been established by Proposition 3.7. For necessity, let $Y = \{[x] : x \in X\}$ and define $f : X \to Y$ by $f(x) = [x]$ $(x \in X)$. We intend to show that $\tau = \{E \subseteq X : f^{-1}(f(E)) = E\}$. Now it is clear that $f^{-1}(f(E)) = \cup_{e \in E} [e]$. Furthermore, since the space is clopen, a nonempty subset A of X is open if and only if $A = \cup_{a \in A}[a]$. Thus, E is a fixed point of the operator if and only $E \in \tau$, that is, $\tau = \tau_f$. ∎

Returning to our clopen topology on $\{a, b, c, d\}$ of the Introduction, we see that an appropriate function f relative to Theorem 3.8 is given by $f(a) = \{a\}$, $f(d) = \{d\}$, and $f(b) = f(c) = \{b, c\}$.

Given a nonempty set X, a topological space $\langle Y, \sigma \rangle$, and a function $f : X \to Y$, we now provide necessary and sufficient conditions for the coincidence of τ_f and $\tau_{\{f\}}$.

Theorem 3.9. *Let X be a nonempty set, let $\langle Y, \sigma \rangle$ be a topological space, and let f be a function from X to Y. The following conditions are equivalent:*

1. $\tau_f = \tau_{\{f\}}$;

2. *the relative topology on $f(X)$ determined by σ is discrete.*

Proof. (1) $\Rightarrow$ (2). Suppose condition (2) fails. Then there exists $y_0 \in f(X)$ such that $\forall V \in \sigma$, $V \cap f(X) \neq \{y_0\}$. Now $f^{-1}(\{y_0\})$ is a fixed point of the operator $f^{-1}(f(\cdot))$, that is, $f^{-1}(\{y_0\}) \in \tau_f$. But $f^{-1}(\{y_0\}) \notin \tau_{\{f\}}$, because if $y_0 \in V \in \sigma$, then $V \cap f(X)$ contains $\{y_0\}$ properly so that $f^{-1}(V)$ contains $f^{-1}(\{y_0\})$ properly. Thus, condition (1) fails.

(2) $\Rightarrow$ (1). Let $A \in \tau_{\{f\}}$ be nonempty. By definition, there exists $V \in \sigma$ with $f^{-1}(V) = A$. It follows that

$$f^{-1}(f(A)) \subseteq f^{-1}(V) = A,$$

so that $f^{-1}(f(A)) = A$, i.e., A belongs to τ_f. For the reverse inclusion, let $A \in \tau_f$ be nonempty. By the discreteness of the relative topology on $f(X)$, there exists $V \in \sigma$ with $f(A) = V \cap f(X)$ so that

$$f^{-1}(V) = f^{-1}(V \cap f(X)) = f^{-1}(f(A)) = A.$$

We have shown that $A \in \tau_{\{f\}}$ and this completes the proof. ∎

Our final result of this section, which complements Theorem 2.3, explains exactly how clopen topologies arise as weak topologies.

Theorem 3.10. *A topology τ on a nonempty set X is a clopen topology if and only if it is the weak topology on X induced by a surjective function from X to a discrete space.*

Proof. Sufficiency is a consequence of Proposition 2.1. For necessity, from Theorem 3.8, $\tau = \tau_f$ where f is a function with domain X, and without loss of generality, we can replace the target space by $f(X)$ without affecting the validity of the equality. From Theorem 3.9, τ_f is the weak topology determined by the corestriction of f to $f(X)$, equipped with the discrete topology. ∎

As a courtesy to the reader, we include a less abstract argument for necessity in our last result. Suppose $\langle X, \tau \rangle$ is a clopen space. Equip $\mathcal{Y} = \{[x] : x \in X\}$ with the discrete topology (note that this is the Kolmogorov quotient of the original space), and define $f : X \to \mathcal{Y}$ by $f(x) = [x]$. If $\mathcal{E}$ is a subset of $\mathcal{Y}$ then $f^{-1}(\mathcal{E}) = \cup\{[x] : [x] \in \mathcal{E}\}$ which is a union of members of τ. Conversely, if $V \in \tau$, then

$$V = \cup_{x \in V}[x] = f^{-1}(\{[x] : x \in V\})$$

and $\{[x] : x \in V\}$ is open in the discrete target space.

4 The Sorgenfrey Line

As we observed earlier, each topology with a clopen base is trivially a supremum of four-element clopen topologies. However, a supremum of clopen topologies can sometimes be achieved in a more illuminating way. We provide a construction of the Sorgenfrey line [7, p. 34] as the supremum of a family of clopen topologies using our closure operator Γ. We will denote the Sorgenfrey topology on $\mathbb{R}$, having as a base all half-open intervals of the form $[a, b)$ with $a < b$, by τ_S .

Consider the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Q}$. Let $x/\sim := \{y \in \mathbb{R} : x \sim y\}$ and $X/\sim := \{x/\sim : x \in \mathbb{R}\}$. Let J be a set of representatives for the blocks in $X/\sim$. For each $\alpha \in J$ and $n \in \mathbb{N}$, define $f_{\alpha,n} : \mathbb{R} \to \mathbb{R}$ by $f_{\alpha,n}(x) := \lfloor n(x - \alpha) \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function, also known as the greatest integer function. For $\alpha \in J$, let $F_\alpha := \{f_{\alpha,n} : n \in \mathbb{N}\}$. For $f_{\alpha,n} \in F_\alpha$, put $\Gamma_{\alpha,n} := f_{\alpha,n}^{-1}(f_{\alpha,n}(\cdot))$. We define the topology $\rho_{\alpha,n}$ on $\mathbb{R}$ to be the clopen topology determined by the closure operator $\Gamma_{\alpha,n}$ on $\mathcal{P}(\mathbb{R})$.

Fix $\alpha \in J$ and $n \in \mathbb{N}$, and let $x \in \mathbb{R}$. Then x may be expressed as $\alpha + y$ for some $y \in \mathbb{R}$ and y in turn may be expressed as $y = \frac{m}{n} + r$ with $m \in \mathbb{Z}$ and $r \in [0, \frac{1}{n})$. Therefore,

$$f_{\alpha,n}(x) = \lfloor n(x - \alpha) \rfloor = \lfloor m + nr \rfloor = m$$

because $nr \in [0, 1)$.

Thus $\Gamma_{\alpha,n}(\{x\}) = [\alpha + \frac{m}{n}, \alpha + \frac{m+1}{n})$ and so the essential basic open sets of $\rho_{\alpha,n}$ are $B_{\alpha,n} := \{[\alpha + \frac{m}{n}, \alpha + \frac{m+1}{n}) : m \in \mathbb{Z}\}$.

For $\alpha \in J$, let ρ_α be the supremum of the topologies $\{\rho_{\alpha,n} : n \in \mathbb{N}\}$. Then, we write ρ for the supremum of the topologies $\{\rho_\alpha : \alpha \in J\}$

Proposition 4.1. *Let $\alpha \in J$.*

1. *ρ_α contains all open intervals and half-open intervals of the form* $\{[x, r) : x \in \alpha/\sim,\ r \in \mathbb{R}, r > x\}$;

2. $\rho = \tau_S$.

Proof. (1) Let $a, b \in \mathbb{R}$ with $a < b$ and let $x \in (a, b)$. Then $\exists n \in \mathbb{N}$ such that $a < x - \frac{1}{n}$ and $x + \frac{1}{n} < b$. As before, we may express x as $x = \alpha + \frac{m}{n} + r$ for some $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $r \in [0, \frac{1}{n})$. Since $\rho_{\alpha,n} \subseteq \rho_\alpha$, $x \in [\alpha + \frac{m}{n}, \alpha + \frac{m+1}{n}) \in \rho_\alpha$ and $[\alpha + \frac{m}{n}, \alpha + \frac{m+1}{n}) \subseteq (a, b)$. This shows that ρ_α contains the usual topology of the real line.

Next let $x \in \alpha/\sim$ and $r \in \mathbb{R}, r > x$. As we have just seen, $(x, r) \in \rho_\alpha$. Let $n \in \mathbb{N}$ satisfy $\frac{1}{n} < r - x$. Then $[x, x + \frac{1}{n}) \in \rho_\alpha$ and so $[x, r)$ contains a ρ_α-neighborhood of each of its points.

(2) For each $\alpha \in J$, $\rho_\alpha \subseteq \tau_S$ and ρ_α is generated by the half-open intervals that include their minimum in $\alpha/\sim$. Since $X = \cup_{\alpha \in J} \alpha/\sim$, the supremum ρ of the topologies ρ_α, $\alpha \in J$, contains all half-open intervals of the form $[a, b)$ where $a < b$ are otherwise arbitrary real numbers. ■

5 On the operator $E \mapsto f(f^{-1}(E))$

Recall that an operator Θ on $\mathcal{P}(Y)$ is called an *interior operator* if it satisfies these four properties [7, p. 27]

- $\Theta(Y) = Y$;
- $\forall E \subseteq Y,\ E \supseteq \Theta(E)$;
- $\forall E \subseteq Y,\ \Theta(\Theta(E)) = \Theta(E)$;
- $\forall E_1, E_2$ in $\mathcal{P}(Y)$, $\Theta(E_1 \cap E_2) = \Theta(E_1) \cap \Theta(E_2)$.

If $f\colon X \to Y$, we are led to dually look at $E \mapsto f(f^{-1}(E))$ on $\mathcal{P}(Y)$ to see if it is an interior operator. The first bullet point fails unless f is surjective, so we look more generally at the modified operator Θ on $\mathcal{P}(Y)$ defined by

$$\Theta(E) = \begin{cases} Y & \text{if } E = Y \\ f(f^{-1}(E)) & \text{otherwise.} \end{cases}$$

Clearly, $\Theta = f(f^{-1}(\cdot))$ if and only if f is surjective.

Theorem 5.1. *Let $f\colon X \to Y$; then Θ as just defined is an interior operator on $\mathcal{P}(Y)$.*

Proof. The first two bullet points are obviously satisfied. The third follows from the general fact that

$$f^{-1}(f(f^{-1}(E))) = f^{-1}(E) \quad \text{for } E \subseteq Y.$$

For the fourth, we must show $\forall E_1, E_2 \subseteq Y$,

$$f(f^{-1}(E_1)) \cap f(f^{-1}(E_2)) = f(f^{-1}(E_1 \cap E_2)),$$

or equivalently,

$$f(f^{-1}(E_1)) \cap f(f^{-1}(E_2)) = f(f^{-1}(E_1) \cap f^{-1}(E_2)).$$

From basic set theory, we need only show

$$f(f^{-1}(E_1)) \cap f(f^{-1}(E_2)) \subseteq f(f^{-1}(E_1) \cap f^{-1}(E_2)).$$

Let $y \in f(f^{-1}(E_1)) \cap f(f^{-1}(E_2))$ be arbitrary. $\exists x_j \in f^{-1}(E_j)$ with $f(x_j) = y$, $j = 1, 2$. Since $y \in f(f^{-1}(E_1)) \subseteq E_1$ and $f(x_2) = y$ we see $x_2 \in f^{-1}(E_1)$ as well. This shows $y \in f(f^{-1}(E_1) \cap f^{-1}(E_2))$ as required. ∎

We are now led to ask: when is the topology on Y associated with Θ a clopen topology? This topology consists of all $E \subseteq Y$ where $\Theta(E) = E$.

Theorem 5.2. *Let $f\colon X \to Y$. Then $\{E \subseteq Y\colon \Theta(E) = E\}$ is a clopen topology if and only if f is surjective. In this case, the topology is discrete.*

Proof. Suppose f is not surjective and put $E = f(X)$. Then $f(f^{-1}(E)) = f(X) = E$, so E is open in the associated topology. On the other hand, $E^c \neq \emptyset$ while $f(f^{-1}(E^c)) = f(\emptyset) = \emptyset$, so E^c is not open. Conversely, if f is surjective, then from elementary set theory $\forall E \subseteq Y, f(f^{-1}(E)) = E$, that is, each subset of Y is open so that the topology is discrete (and clopen). ∎

In general, the open subsets induced from a possibly non-surjective $f\colon X \to Y$ by the interior operator Θ consist of all subsets of the range of f plus the codomain Y. Thus, from Theorem 3.9, $\tau_f = \tau_{\{f\}}$ with respect to the topology on Y determined by our interior operator.

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