Parallel Forms, Co-Kähler Manifolds and their Models*

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Abstract

We show how certain topological properties of co-Kähler manifolds derive from those of the Kähler manifolds which construct them. In particular, we show that the existence of parallel forms on a co-Kähler manifold reduces the computation of cohomology from the de Rham complex to certain amenable sub-cdga's defined by geometrically natural operators derived from the co-Kähler structure. This provides a simpler proof of the formality of the foliation minimal model in this context.

1 Introduction

Co-Kähler manifolds may be thought of as odd-dimensional versions of Kähler manifolds and various structure theorems explicitly display how the former are constructed from the latter (see [1, 15]).

In this paper, we take the point of view that topological and geometric properties of co-Kähler manifolds are inherited from those of the Kähler manifolds that construct them. We call this the *hereditary principle* and we shall see this in both topological and geometric contexts. See [2] for further applications of this principle. First, let us recall some basic definitions (see [3] for a detailed introduction).

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Definition 1.1. An **almost contact metric structure** (J, ξ, η, g) on a manifold M^{2n+1} consists of a tensor J of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X,Y) - \eta(X)\eta(Y), \tag{1}$$

for vector fields X and Y, I the identity transformation on TM.

A local J-basis for TM, $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi\}$, may be found with $\eta(X_i) = 0$ for $i = 1, \ldots, n$. The *fundamental 2-form* on M is given by

$$\omega(X,Y) = g(JX, Y),$$

and if $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \eta\}$ is a local 1-form basis dual to the local J-basis, then

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Note that $\imath_\xi \omega = 0$.

Definition 1.2. The geometric structure $(M^{2n+1}, J, \xi, \eta, g)$ is

- **co-symplectic** if $d\omega = 0 = d\eta$;
- **normal** if $[J,J] + 2\, d\eta \otimes \xi = 0$;
- **co-Kähler** if it is co-symplectic and normal; equivalently, if J is parallel with respect to the metric g.

Recently, co-symplectic geometry has attracted a great deal of interest, especially in the context of Poisson geometry, where co-symplectic structures are interpreted as corank 1 Poisson structures (see for instance [5, 9, 12, 14, 16]). Sasakian structures also belong to this family; more precisely, they are normal structures such that $d\eta = \omega$ (see [4, 6, 7]).

Two crucial facts about co-Kähler manifolds are contained in the following lemma. For a direct proof of these facts, see [1].

Lemma 1.3. *On a co-Kähler manifold, the vector field ξ is Killing and parallel. Furthermore, the 1-form η is parallel and harmonic.*

Lemma 1.3 is a key point in Theorem 1.5 below. In fact, in [15] it is shown that we can replace η by a harmonic integral form η_θ with dual parallel vector field ξ_θ and associated metric g_θ, $(1,1)$-tensor J_θ and closed 2-form ω_θ with $i_{\xi_\theta}\omega_\theta = 0$. Then we have the following result of H. Li.

Theorem 1.4 ([15])**.** *With the structure $(M^{2n+1}, J_\theta, \xi_\theta, \eta_\theta, g_\theta)$, there is a compact Kähler manifold (K, h) and a Hermitian isometry $\psi\colon K \to K$ such that M is diffeomorphic to the mapping torus*

$$K_\psi = \frac{K \times [0,1]}{(x,0) \sim (\psi(x),1)}$$

with associated fibre bundle $K \to M = K_\psi \to S^1$.

In [1], the following refinement of Li's result is proved:

Theorem 1.5 ([1], Theorem 3.3)**.** *Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold with integral structure and mapping torus bundle $K \to M \to S^1$. Then M splits as $M \cong S^1 \times_{\mathbb{Z}_m} K$, where $S^1 \times K \to M$ is a finite cover with structure group $\mathbb{Z}_m$ acting diagonally and by translations on the first factor. Moreover, M fibres over the circle $S^1/(\mathbb{Z}_m)$ with finite structure group.*

The first true study of the topological properties of co-Kähler manifolds was made in [8] where the focus was on things such as Betti numbers and a modified Lefschetz property. The two results above allow us to say something about the fundamental group and, moreover, to display the higher homotopy groups as those of the constituent Kähler manifold K (groups which, of course, are generally unknown as well).

Here we use work of Verbitsky [17] and the geometric structure of co-Kähler manifolds to give a completely new decomposition of the cohomology of a co-Kähler manifold in terms of the basic cohomology of the associated transversally Kähler characteristic foliation. This leads to a new, simpler proof of the "Lefschetz" property of [8]. Moreover, we show in Proposition 2.12 how the minimal model of a compact co-Kähler manifold is constructed from the minimal model of the basic cohomology with an extra generator in degree one and this provides an intriguing link between the formality of the co-Kähler manifold and the formality of the basic cohomology model. This can be viewed as either applying the formality result of [10] to prove that of [8, 2] or the reverse!

There is one important thing to note. In [2], we used Theorem 1.5 to derive results about co-Kähler manifolds that were based on taking $G = \mathbb{Z}_m$ invariants of the action on $S^1 \times K$. This included determining the structure of cohomology as well as showing that co-Kähler manifolds satisfy the so-called Toral Rank Conjecture. Here our goal is to go deeper into the geometry of co-Kähler manifolds by examining properties inherent in their differential forms. In this context, we note that the splitting theorem Theorem 1.5 actually uses a modification of the given co-Kähler structure; in fact, the given co-Kähler structure is already lost via the argument of Li to get the mapping torus structure in Theorem 1.4. In some sense, this modification is akin to taking an integral symplectic form near a given symplectic form. While various properties are not affected, it is not quite the original symplectic structure that is being studied. The co-Kähler modification does no harm when, for instance, proving the Toral Rank Conjecture for co-Kähler manifolds since the latter is just a property of the cohomology of the underlying manifold. However, when talking about the co-Kähler version of the Lefschetz Property (see Proposition 2.4, Theorem 2.9), we want to keep track of the given co-Kähler structure because, in fact, the Lefschetz Property refers to it! Then, instead of modifying the co-Kähler structure to one coming from a mapping torus, we work with the characteristic foliation intrinsically associated to the given co-Kähler structure. Thus we prove the Lefschetz Property not for an associated co-Kähler structure (the one called *integral* in [1]), but for the given one. Working with forms has the added benefit of allowing us to relate the minimal models of the co-Kähler manifold and its characteristic foliation as we alluded to above. This relationship cannot be seen from the structure results Theorem 1.4 and Theorem 1.5. We hope this clarifies why we take the approach we do here.

2 Parallel forms and quasi-isomorphisms on co-Kähler manifolds

In [17], Verbitsky shows that, in case a smooth Riemannian manifold has a parallel form, one can define a derivation of the de Rham algebra whose kernel is quasi-isomorphic to the manifold's real cohomology algebra. In this section we will use this construction in the context of co-Kähler manifolds, where the 1-form η is parallel. Once again, we shall see that some topological properties of co-Kähler manifolds may be derived from corresponding properties of Kähler manifolds. This can be interpreted as a geometric incarnation of our hereditary principle.

Let M be a smooth manifold and let $\Omega^*(M;\mathbb{R})$ be the (real) de Rham algebra. A linear map $f \in \mathrm{End}(\Omega^*(M;\mathbb{R}))$ has degree $|f|$ if $f\colon \Omega^k(M;\mathbb{R}) \to \Omega^{k+|f|}(M;\mathbb{R})$. Every linear map $f\colon \Omega^1(M;\mathbb{R}) \to \Omega^{|f|+1}(M;\mathbb{R})$ can be extended to a graded derivation ρ_f of $\Omega^*(M;\mathbb{R})$ by imposing the Leibniz rule, i.e.

$$\begin{aligned} \rho_f\big|_{\Omega^0(M;\mathbb{R})} &= 0 \\ \rho_f\big|_{\Omega^1(M;\mathbb{R})} &= f \\ \rho_f(\alpha \wedge \beta) &= \rho_f(\alpha) \wedge \beta + (-1)^{|\alpha||f|}\alpha \wedge \rho_f(\beta). \end{aligned} \tag{2}$$

where $\alpha, \beta \in \Omega^*(M;\mathbb{R})$ and $|\alpha|$ is the degree of α. (While this apparently well-known fact is used in [17], it is not proved there. See [13, Lemma 4.3] for a proof.) Given two linear operators $f, \tilde{f} \in \mathrm{End}(\Omega^*(M;\mathbb{R}))$, their *supercommutator* is defined as

$$\{f, \tilde{f}\} = f \circ \tilde{f} - (-1)^{|f||\tilde{f}|}\tilde{f} \circ f.$$

Let (M, g) be a smooth Riemannian manifold and let $\eta \in \Omega^k(M;\mathbb{R})$ be a k-form. Define a linear map $\bar{\eta}\colon \Omega^1(M;\mathbb{R}) \to \Omega^{k-1}(M;\mathbb{R})$, with $|\bar{\eta}| = k-2$, by

$$\bar{\eta}(\nu) = \imath_{\nu^\#}\eta\,,$$

where ${}^\#\colon T^*M \to TM$ is the isomorphism given by the metric. Denote by $\rho_\eta\colon \Omega^*(M;\mathbb{R}) \to \Omega^{*+k-2}(M;\mathbb{R})$ the corresponding derivation. Define the linear operator $d_\eta\colon \Omega^*(M;\mathbb{R}) \to \Omega^{*+k-1}(M;\mathbb{R})$ as

$$d_\eta = \{d, \rho_\eta\}\,.$$

Since d_η is the supercommutator of two graded derivations, one sees easily that it is itself a graded derivation of degree $k-1$ and that it supercommutes with d. As a consequence, $\ker(d_\eta) \subset \Omega^*(M;\mathbb{R})$ is a differential subalgebra and has the structure of a cdga. In [17], Verbitsky proves following:

Theorem 2.1. *Let (M, g, η) be a compact Riemannian manifold equipped with a parallel form η. Then the natural embedding*

$$(\ker(d_\eta), d) \hookrightarrow (\Omega^*(M;\mathbb{R}), d)$$

is a quasi-isomorphism.

Let (M, g, η) be a Riemannian manifold equipped with a parallel form η. Theorem 2.1 says that we can recover the cohomology of M by considering the subalgebra of forms ν which are annihilated by d_η, i.e. those for which $d_\eta(\nu) = 0$. This allows one to greatly simplify, in many cases, the computation of the de Rham cohomology of this kind of manifold.

Recall from Lemma 1.3 that the 1-form η is parallel on a co-Kähler manifold. According to Verbitsky's construction, there is a derivation d_η of $(\Omega^*(M;\mathbb{R}), d)$ described explicitly as follows.

Lemma 2.2. *Let (M, J, η, ξ, g) be a co-Kähler manifold. Then $d_\eta = L_\xi$, where L_ξ denotes the Lie derivative in the direction of the vector field ξ.*

Proof. Denote by $\bar{\eta}\colon \Omega^*(M;\mathbb{R}) \to \Omega^*(M;\mathbb{R})$ the operator which acts on 1-forms as $\bar{\eta}(\nu) = \iota_{\nu^\#}\eta$. Since $|\bar{\eta}| = -1$, we have $d_\eta = \{d, \rho_\eta\} = d \circ \rho_\eta + \rho_\eta \circ d$, and $|d_\eta| = 0$. To prove the lemma, by [13], it is enough to consider the action of d_η on 0- and 1-forms. Now, according to the formulas in (2) extending $\bar{\eta}$ to a derivation ρ_η, on 1-forms we have $\rho_\eta = \bar{\eta}$ and

$$\bar{\eta}(\nu) = \iota_{\nu^\#}\eta = \eta(\nu^\#) = g(\xi, \nu^\#) = \nu(\xi) = \iota_\xi \nu.$$

Note that this identifies $\bar{\eta} = \iota_\xi$ which is already a derivation, so $\rho_\eta = \iota_\xi$. Hence, $(d \circ \bar{\eta})(\nu) = d\iota_\xi\nu$ and, on the other hand, $(\bar{\eta} \circ d)(\nu) = \iota_\xi(d\nu)$. By Cartan's magic formula, we obtain

$$d_\eta(\nu) = (d \circ \bar{\eta})(\nu) + (\bar{\eta} \circ d)(\nu) = d\iota_\xi\nu + \iota_\xi(d\nu) = L_\xi(\nu).$$

Thus $d_\eta = L_\xi$ on 1-forms. On a 0-form (i.e. a function) f, we have

$$d_\eta(f) = \rho_\eta(df) = \bar{\eta}(df) = df(\xi) = \xi(f) = L_\xi(f)$$

by the calculation above. Since d_η and L_ξ are graded derivations of the de Rham algebra which agree on 0-forms and 1-forms, the result follows. ■

Let us consider the following graded differential subalgebra $(\Omega^*_\eta(M), d)$ of $(\Omega^*(M;\mathbb{R}), d)$ given by

$$\Omega^*_\eta(M) = \{\nu \in \Omega^*(M;\mathbb{R}) \mid L_\xi(\nu) = 0\}.$$

As a consequence of Theorem 2.1, we obtain the following result.

Corollary 2.3. *On a compact co-Kähler manifold, the natural inclusion*

$$(\Omega^*_\eta(M), d) \hookrightarrow (\Omega^*(M;\mathbb{R}), d)$$

is a cdga quasi-isomorphism and

$$H^*(M;\mathbb{R}) \cong H^*_\eta(M),$$

where $H^*_\eta(M)$ is the cohomology of $(\Omega^*_\eta(M), d)$.

We shall use the cdga $\Omega^*_\eta(M)$ to give an alternative proof of the Lefschetz property and of formality for co-Kähler manifolds in the hereditary framework of the rest of the paper.

Let (M, J, ξ, η, g) be a compact co-Kähler manifold. In [8], the authors defined a Lefschetz map on harmonic forms and proved that it is an isomorphism. This is, of course, different from the Kähler context, where the Lefschetz map can be defined directly on all forms and depends only on the underlying symplectic structure, not on the metric. On forms, the Lefschetz map is $\mathcal{L}^{n-p}\colon \Omega^p(M;\mathbb{R}) \to \Omega^{2n+1-p}(M,\mathbb{R})$, given by

$$\alpha \mapsto \omega^{n-p+1} \wedge \imath_\xi\alpha + \omega^{n-p} \wedge \eta \wedge \alpha \tag{3}$$

One sees immediately that the Lefschetz map does not send closed (resp. exact) forms to closed (resp. exact) forms, as it happens in the Kähler case, hence does not descend to a map on cohomology. However, by restricting the Lefschetz map to the cdga $\Omega^*_\eta(M)$, we *are able to* descend to cohomology.

Proposition 2.4. *The Lefschetz map* (3) *restricts to a map*

$$\mathcal{L}^{n-p}\colon \Omega^p_\eta(M) \to \Omega^{2n+1-p}_\eta(M)$$

for $0 \leqslant p \leqslant n$*, which sends closed (resp. exact) forms to closed (resp. exact) forms. Hence,* $\mathcal{L}$ *descends to the cohomology* $H^*_\eta(M) \cong H^*(M;\mathbb{R})$*.*

Proof. We first show that if $\alpha \in \Omega^p_\eta(M)$, then $\mathcal{L}^{n-p}(\alpha) \in \Omega^{2n+1-p}_\eta(M)$.

$$\begin{aligned} L_\xi(\mathcal{L}^{n-p}(\alpha)) &= L_\xi(\omega^{n-p+1} \wedge \imath_\xi\alpha + \omega^{n-p} \wedge \eta \wedge \alpha) = \omega^{n-p+1} \wedge L_\xi(\imath_\xi\alpha) = \\ &= \omega^{n-p+1} \wedge \imath_\xi d\imath_\xi\alpha = -\omega^{n-p+1} \wedge \imath_\xi\imath_\xi d\alpha = 0, \end{aligned}$$

where we have used the facts that the Lie derivative L_ξ is a derivation, $L_\xi = \imath_\xi d + d\imath_\xi$ (Cartan's Magic formula), $\imath_\xi\imath_\xi = 0$ and $L_\xi\omega = L_\xi\eta = L_\xi\alpha = 0$. For α a closed form in $\Omega^p_\eta(M)$, we have

$$d(\mathcal{L}^{n-p}(\alpha)) = d(\omega^{n-p+1} \wedge \imath_\xi\alpha + \omega^{n-p} \wedge \eta \wedge \alpha) = \omega^{n-p+1} \wedge d\imath_\xi\alpha = 0\,;$$

for $\beta \in \Omega^{p-1}_\eta(M)$,

$$\begin{aligned} \mathcal{L}^{n-p}(d\beta) &= \omega^{n-p+1} \wedge \imath_\xi d\beta + \omega^{n-p} \wedge \eta \wedge d\beta \\ &= -\omega^{n-p+1} \wedge d\imath_\xi\beta - d(\omega^{n-p} \wedge \eta \wedge \beta) \\ &= d(-\omega^{n-p+1} \wedge \imath_\xi\beta - \omega^{n-p} \wedge \eta \wedge \beta). \end{aligned}$$

■

Consider the following two subalgebras of $\Omega^*_\eta(M)$:

$$\Omega^p_1(M) = \{\alpha \in \Omega^p_\eta(M) \mid \imath_\xi\alpha = 0\},\ \Omega^p_2(M) = \mathbb{Q} \oplus \{\alpha \in \Omega^p_\eta(M) \mid \eta \wedge \alpha = 0\}.$$

Lemma 2.5. *$\Omega^p_\eta(M) = \Omega^p_1(M) \oplus \Omega^p_2(M)$ for all $p > 0$ and $\Omega^*_i(M)$ is a differential subalgebra of $\Omega^*_\eta(M)$, $i = 1, 2$.*

Proof. Given any $\alpha \in \Omega^p_\eta(M)$, we can write tautologically

$$\alpha = (\alpha - \eta \wedge \imath_\xi \alpha) + \eta \wedge \imath_\xi \alpha =: \alpha_1 + \alpha_2. \tag{4}$$

Since $\eta(\xi) = 1$, we see immediately that $\imath_\xi \alpha_1 = 0$, so $\alpha_1 \in \Omega^p_1(M)$. Clearly $\alpha_2 \in \Omega^p_2(M)$. Now suppose that $\alpha \in \Omega^p_1(M) \cap \Omega^p_2(M)$. Then $\eta \wedge \alpha = 0$ and hence, by applying $\imath_\xi$, we get $0 = \alpha - \eta \wedge \imath_\xi \alpha = \alpha$, which gives $\alpha = 0$.

Now, if $\alpha \in \Omega^p_\eta(M)$, then $L_\xi \alpha = d\imath_\xi \alpha + \imath_\xi d\alpha = 0$, so $\imath_\xi d\alpha = -d\imath_\xi \alpha$. If $\alpha \in \Omega^p_1(M)$, then we also have $\imath_\xi(d\alpha) = -d\imath_\xi \alpha = 0$ since $\alpha \in \Omega^p_1(M)$. Hence $d\colon \Omega^p_1(M) \to \Omega^{p+1}_1(M)$.

Finally, suppose $\alpha \in \Omega^p_2(M)$. Then, since η is closed, we have $\eta \wedge d\alpha = -d(\eta \wedge \alpha) = 0$. Hence $d\colon \Omega^p_2(M) \to \Omega^{p+1}_2(M)$. ∎

As a consequence, the cohomology $H^p_\eta(M)$ of the cdga $\Omega^*_\eta(M)$ can be written as

$$H^p_\eta(M) \cong H^p_1(M) \oplus H^p_2(M)\,,$$

where $H^p_i(M) = H^p(\Omega^*_i(M))$, $i = 1, 2$. Now consider a form $\alpha \in \Omega^p_2(M)$. Applying the derivation $\imath_\xi$ to the equation $\eta \wedge \alpha = 0$, we obtain $\alpha = \eta \wedge \imath_\xi \alpha$, where clearly $\imath_\xi \alpha \in \Omega^{p-1}_1(M)$. This tells us that $\Omega^p_2(M) = \eta \wedge \Omega^{p-1}_1(M)$ and, since $d\eta = 0$, we have a differential splitting

$$\Omega^p_\eta(M) = \Omega^p_1(M) \oplus \eta \wedge \Omega^{p-1}_1(M)\,.$$

From this, we immediately deduce

Corollary 2.6. *The cohomology $H^p_\eta(M)$ of $\Omega^*_\eta(M)$ splits as*

$$H^p_\eta(M) = H^p_1(M) \oplus [\eta] \wedge H^{p-1}_1(M)$$

This corollary shows that the cohomology of $\Omega^*_\eta(M)$ only depends on the cohomology of the cdga $\Omega^*_1(M)$.

Let us now consider the *characteristic foliation* $\mathcal{F}_\xi$ on a compact co-Kähler manifold (M, J, ξ, η, g) given by $(\mathcal{F}_\xi)_x = \langle \xi_x \rangle$ for every $x \in M$. Such a foliation is Riemannian and transversally Kähler. Indeed, at every point $x \in M$, the orthogonal space to ξ is endowed with a Kähler structure given by (J, g, ω), and all these data vary smoothly with x.

Recall that, given a foliation $\mathcal{F}$ on a compact manifold M, the *basic cohomology* is defined as the cohomology of the complex $\Omega^*(M, \mathcal{F})$, where

$$\Omega^p(M, \mathcal{F}) = \{\alpha \in \Omega^p(M) \mid \imath_X \alpha = \imath_X d\alpha = 0\ \forall X \in \mathfrak{X}(\mathcal{F})\}$$

and $\mathfrak{X}(\mathcal{F})$ denotes the subalgebra of vector fields tangent to $\mathcal{F}$. In our case, we have the following.

Lemma 2.7. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold and let $\mathcal{F}_\xi$ be the characteristic foliation. Then $\Omega^*_1(M) = \Omega^*(M, \mathcal{F}_\xi)$.*

Proof. This is clear, since

$$\alpha \in \Omega^*_1(M) \Leftrightarrow L_\xi \alpha = \imath_\xi \alpha = 0 \Leftrightarrow \imath_\xi d\alpha = \imath_\xi \alpha = 0 \Leftrightarrow \alpha \in \Omega^p(M, \mathcal{F}_\xi).$$

■

Corollary 2.8. *On a compact co-Kähler manifold M, $H^*_1(M) \cong H^*(M, \mathcal{F}_\xi)$ and*

$$H^*(M; \mathbb{R}) \cong H^*_\eta(M) = H^*(M, \mathcal{F}_\xi) \oplus [\eta] \wedge H^{*-1}(M, \mathcal{F}_\xi)\,.$$

Theorem 2.9. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold. Then the Lefschetz map*

$$\begin{aligned} \mathcal{L}^{n-p} \colon H^p(M; \mathbb{R}) \cong H^p_\eta(M) \to H^{2n+1-p}_\eta(M) \cong H^{2n+1-p}(M; \mathbb{R}), \\ \alpha \mapsto \omega^{n-p+1} \wedge \imath_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha \end{aligned}$$

is an isomorphism for $0 \leqslant p \leqslant n$.

Proof. First note that, by Poincaré duality, it is sufficient to show that $\mathcal{L}^{n-p}$ has zero kernel. Now, by Corollary 2.3, on a compact co-Kähler manifold we have an isomorphism $H^p_\eta(M) \cong \mathcal{H}^p(M)$. In particular, Corollary 2.8 tells us that the (harmonic) cohomology of M can be computed as a cylinder on the basic cohomology of the characteristic foliation. Since the latter is transversally Kähler, in view of [11], the map $H^p(M, \mathcal{F}_\xi) \to H^{2n-p}(M, \mathcal{F}_\xi)$ given by multiplication with the Kähler form ω^{n-p} is an isomorphism for $p \leqslant n$. Again by Corollary 2.8, the corresponding map $H^p_1(M) \to H^{2n-p}_1(M)$ is also an isomorphism.

Now consider the Lefschetz map $\mathcal{L}^{n-p} \colon H^p_\eta(M) \to H^{2n+1-p}_\eta(M)$ given by (3). Decompose any $\alpha \in H^p_\eta(M)$ as $\alpha = \alpha_1 + \alpha_2$ according to (4) so that $\imath_\xi \alpha_1 = 0$ and $\alpha_2 = \eta \wedge \imath_\xi \alpha$. We shall show that the Lefschetz map is non-zero on both α_1 and α_2 with $\mathcal{L}^{n-p}(\alpha_1) \in \eta \wedge H^{2n-p}_1(M)$ and $\mathcal{L}^{n-p}(\alpha_2) \in H^{2n+1-p}_1(M)$. Then, because these sub-algebras are complementary, we will have $\mathcal{L}^{n-p}(\alpha) \neq 0$ for all $\alpha \neq 0$.

For $\alpha_1 \in H^p_1(M) \cong H^p(M, \mathcal{F}_\xi)$, because $\imath_\xi \alpha_1 = 0$, the first term in the Lefschetz map definition applied to α_1 vanishes. Hence, we get that $\omega^{n-p} \wedge \alpha_1 \neq 0$ in $H^{2n-p}(M, \mathcal{F}_\xi)$ and, in view of Corollary 2.8, this implies that $\omega^{n-p} \wedge \eta \wedge \alpha_1$ is non-zero in $\eta \wedge H^{2n-p}_1(M) \subseteq H^{2n+1-p}_\eta(M)$.

Because $\alpha_2 = \eta \wedge \imath_\xi \alpha$, we see that the second term in the Lefschetz map definition applied to α_2 vanishes. Now, $\imath_\xi \alpha_2 \in H^{p-1}_1(M) \cong H^{p-1}(M, \mathcal{F}_\xi)$, so $\omega^{n-p+1} \wedge \imath_\xi \alpha_2 \neq 0$ in $H^{2n-p+1}(M, \mathcal{F}_\xi) \cong H^{2n-p+1}_1(M)$. Therefore, when $p \geqslant 1$,

$$\begin{aligned} \mathcal{L}^{n-p}(\alpha) &= \omega^{n-p+1} \wedge \imath_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha \\ &= \omega^{n-p+1} \wedge \imath_\xi \alpha_2 + \omega^{n-p} \wedge \eta \wedge \alpha_1 \\ &\neq 0, \end{aligned}$$

so $\mathcal{L}^{n-p}$ has zero kernel and is thus an isomorphism on cohomology. Furthermore, when $p = 0$, we get

$$\mathcal{L}^n(1) = \omega^n \wedge \eta \neq 0\,,$$

since $\omega^n \wedge \eta$ is a volume form by assumption and, hence, cannot be exact. ∎

Since $H^p_\eta(M) \cong \mathcal{H}^p(M)$ (harmonic forms) on a compact co-Kähler manifold, we obtain

Corollary 2.10. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold. Then the Lefschetz map $\mathcal{L}^{n-p}\colon \mathcal{H}^p(M) \to \mathcal{H}^{2n+1-p}(M)$ is an isomorphism for $0 \leqslant p \leqslant n$.*

In [10] the authors prove that the minimal model $\mathcal{M}_{M,\mathcal{F}}$ of the basic forms $\Omega^*(M, \mathcal{F})$ of a transversally Kähler foliation $\mathcal{F}$ on a compact manifold is formal. We would like to use our characterization (in a slightly different form) of the cohomology of a compact co-Kähler manifold to give an alternative proof of this formality in the context of co-Kähler geometry as well as a new description of the minimal model of a co-Kähler manifold. Note that Corollary 2.8 may be phrased as the following.

Corollary 2.11. *Let M be a compact co-Kähler manifold; then $H^*_1(M) \cong H^*(M, \mathcal{F}_\xi)$ and*

$$H^*(M;\mathbb{R}) \cong H^*_\eta(M) = H^*(M, \mathcal{F}_\xi) \otimes \wedge([\eta])\,.$$

Furthermore, the splitting $\Omega^p_\eta(M) = \Omega^p_1(M) \oplus \eta \wedge \Omega^{p-1}_1(M)$ (for each p) may be written as

$$\Omega^*_\eta(M) = \Omega^*_1(M) \otimes \wedge(\eta)\,.$$

Using this description, we can now see the transversally Kähler structure reflected in the minimal model of M.

Proposition 2.12. *Let M be a compact co-Kähler manifold. Then $\mathcal{M}_{M,\mathcal{F}}$ is formal in the sense of Sullivan and the minimal model of M splits as a tensor product of cdga's*

$$\mathcal{M}_M \cong \mathcal{M}_{M,\mathcal{F}} \otimes \wedge(\eta, d = 0).$$

Proof. We use two facts: first, by [8, 2], we know that M is formal; secondly, we know that, in a cdga decomposition $A \cong B \otimes C$, A is formal if and only if both B and C are formal. ∎

Remark 2.13. The proof above is much simpler than the original in [10], but is only for transversally Kähler foliations arising from co-Kähler structures. Of course, if we, on the other hand, assume the formality of $\mathcal{M}_{M,\mathcal{F}}$ (by [10]), then $\wedge(\eta, d = 0)$ and the identification $\Omega^*(M, \mathcal{F}) \cong \Omega^*_1(M)$ allow us to obtain the following diagram.

$$\begin{array}{ccc}
H^*_\eta(M) & \xrightarrow{\cong} & H^*(M, \mathcal{F}_\xi) \otimes \wedge([\eta]) \\
\uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle \simeq} \\
\mathcal{M}_M & \overset{\rho}{\dashrightarrow} & \mathcal{M}_{M,\mathcal{F}} \otimes \wedge(\eta, d = 0) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\Omega^*_\eta(M) & = & \Omega^*(M, \mathcal{F}) \otimes \wedge(\eta)
\end{array}$$

Here, the quasi-isomorphism ρ is obtained from a standard lifting lemma in minimal model theory applied to the bottom part of the diagram. By composition, we then obtain θ and we see it is a quasi-isomorphism. Hence, M is formal and, again by the lifting lemma, the quasi-isomorphism ρ is an isomorphism.

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