Central configurations, Morse and fixed point indices

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Abstract

We compute the fixed point index of non-degenerate central configurations for the n-body problem in the euclidean space of dimension d, relating it to the Morse index of the gravitational potential function $\bar{U}$ induced on the manifold of all maximal $O(d)$-orbits. In order to do so, we analyze the geometry of maximal orbit type manifolds, and compute Morse indices with respect to the mass-metric bilinear form on configuration spaces.

1 Introduction: central configurations as critical points

Let $E = \mathbb{R}^d$ be the d-dimensional euclidean space, for $d \geq 1$. Fix an integer $n \geq 2$. The *configuration space* of n (colored) points in E is the set of all n-tuples of distinct points in E, and denoted by $\mathbb{F}_n(E)$:

$$\mathbb{F}_n(E) = \{\boldsymbol{q} \in E^n : i \neq j \implies \boldsymbol{q}_i \neq \boldsymbol{q}_j\} = E^n \smallsetminus \Delta,$$

where if $\boldsymbol{q} \in E^n$, its n components are denoted by $\boldsymbol{q}_j$, $j = 1, \ldots, n$; points in $\mathbb{F}_n(E)$ are termed *configurations* of n points in E; its complement in E^n is the set of *collisions*

$$\begin{aligned}\Delta &= \{\boldsymbol{q} \in E^n : \exists (i,j), i \neq j : \boldsymbol{q}_i = \boldsymbol{q}_j\} \\ &= \bigcup_{1 \leq i < j \leq n} \{\boldsymbol{q} \in E^n : \boldsymbol{q}_i = \boldsymbol{q}_j\}.\end{aligned}$$

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For $j = 1, \ldots, n$ let $m_j > 0$ be fixed parameters (that can be interpreted as the mass of the j-th particle in E), under the normalization condition

$$\sum_{j=1}^{n} m_j = 1 \; .$$

If $\boldsymbol{v}, \boldsymbol{w}$ are vectors in (the tangent space of) E^n, then let

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M = \sum_{j=1}^{n} m_j \boldsymbol{v}_j \cdot \boldsymbol{w}_j$$

denote the mass scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$, where $\boldsymbol{v}_j \cdot \boldsymbol{w}_j$ is the standard euclidean scalar product (in E) of the j-th components of $\boldsymbol{v}$ and $\boldsymbol{w}$. The unit sphere in $\mathbb{F}_n(E)$ is termed the *inertia ellipsoid* and denoted by

$$\mathbb{S} = \mathbb{S}_n(E) = \{ \boldsymbol{q} \in \mathbb{F}_n(E) : \|\boldsymbol{q}\|_M^2 = 1 \} \; .$$

It is equal to the unit sphere/ellipsoid in E^n, with collisions removed, $\mathbb{S}_n(E) = S_n(E) \smallsetminus \Delta$. The unit sphere/ellipsoid in E^n is denoted by $S_n(E) = \{ \boldsymbol{q} \in E^n : \|\boldsymbol{q}\|_M^2 = 1 \}$. To simplify notation, if possible we will use the short forms $\mathbb{S}$ and S instead of $\mathbb{S}_n(E)$ and $S_n(E)$.

The *potential function* $U \colon \mathbb{F}_n(E) \to \mathbb{R}$ is simply defined as

$$\sum_{1 \le i < j \le n} \frac{m_i m_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|^\alpha},$$

given a fixed parameter $\alpha > 0$. For $\alpha = 1$, U is the Newtonian gravitational potential. It is invariant under the full group of isometries of E, acting diagonally on $\mathbb{F}_n(E)$.

Let $D = \nabla$ denote the covariant derivative (which is the Levi-Civita connection with respect to the mass-metric) in $\mathbb{F}_n(E)$, which is again the standard derivative. If $F \colon \mathbb{F}_n(E) \to E$ is a smooth function, then $DF = dF$ is the differential of F, which is a section of the cotangent bundle $T^*\mathbb{F}_n(E)$ defined as $DF[\boldsymbol{v}] = D_{\boldsymbol{v}} F$ for each vector field $\boldsymbol{v}$ on $\mathbb{F}_n(E)$. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vector fields on $\mathbb{F}_n(E)$, then $D_{\boldsymbol{v}} \boldsymbol{w}$ is the (Euclidean and covariant) derivative of $\boldsymbol{w}$ in the direction of $\boldsymbol{v}$.

Let ∇^S denote the covariant derivative (Levi-Civita connection) on S, induced by the mass-metric of $\mathbb{F}_n(E)$ restricted to S, i.e. the restriction to S of the Riemannian structure of $\mathbb{F}_n(E)$. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vector fields defined in a neighborhood of S, then the covariant derivative $\nabla^S_{\boldsymbol{v}} \boldsymbol{w}$ is equal, at $x \in S$, to the orthogonal projection of $D_{\boldsymbol{v}} \boldsymbol{w}$, projected orthogonally to the tangent space $T_x S$ (cf. proposition 3.1 at page 11 of [7], or proposition 1.2 at page 371 of [8]). The same holds with $\mathbb{S} \subset S$ instead of S. If Π denote the projection $T\mathbb{F}_n(E) \mapsto T\mathbb{S}$, then $\nabla^S_{\boldsymbol{v}} \boldsymbol{w} = \Pi D_{\boldsymbol{v}} \boldsymbol{w}$.

If $F \colon \mathbb{F}_n(E) \to \mathbb{R}$ is a smooth function, and $f = F|_{\mathbb{S}}$ is its restriction to $\mathbb{S}$, then $\nabla^S f = df$ is the restriction of dF to the tangent bundle $T\mathbb{S}$. Let $\operatorname{grad}(f) = df^\sharp$ and $\operatorname{grad}(F) = dF^\sharp$ denote the gradients of f and F respectively (i.e., the images of the differentials under the musical isomorphisms induced by the mass-metric). For each $x \in \mathbb{S}$, $df^\sharp(x) \in T_x \mathbb{S}$ and $dF^\sharp(x) \in T_x \mathbb{F}_n(E)$ satisfy the equations

$$\langle df^\sharp, \boldsymbol{v} \rangle_M = df[\boldsymbol{v}] = \langle dF^\sharp, \boldsymbol{v} \rangle_M = dF[\boldsymbol{v}]$$

for any $v \in T_x\mathbb{S}$, and hence $\mathrm{grad}(f) = df^\sharp$ is the projection of $\mathrm{grad}(F) = dF^\sharp$ on the tangent space $T_x\mathbb{S}$. A *critical point* of f is a point $x \in \mathbb{S}$ such that $df = 0 \iff \mathrm{grad}(f) = 0$, which is equivalent to say that $\mathrm{grad}(F)$ is orthogonal to $T_x\mathbb{S}$.

The *Hessian* of the function f, at a critical point x of f in $\mathbb{S}$, is (cf. page 343 of [8]) equal to the bilinear form $\mathrm{Hess}(f)[\boldsymbol{v},\boldsymbol{w}]$, defined on the tangent space $T_x\mathbb{S}$ as

$$\mathrm{Hess}(f)[\boldsymbol{v},\boldsymbol{w}](x) = (\nabla^S_{\boldsymbol{v}}\nabla^S_{\boldsymbol{w}}f - \nabla^S_{\nabla^S_{\boldsymbol{v}}\boldsymbol{w}}f)(x) = (\nabla^S_{\boldsymbol{v}}\nabla^S_{\boldsymbol{w}}f)(x)$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vector fields defined in a neighborhood of x.

The Hessian of F is simply the symmetric matrix of all the second derivatives D^2F:

$$\begin{aligned}\mathrm{Hess}(F)[\boldsymbol{v},\boldsymbol{w}](x) &= (D_{\boldsymbol{v}}D_{\boldsymbol{w}}F)(x) = D^2F(x)[\boldsymbol{v},\boldsymbol{w}]\\ &= \sum_{\substack{i=1\dots n\\ \beta=1,\dots,d}}\ \sum_{\substack{j=1,\dots,n\\ \gamma=1,\dots,d}} \frac{\partial^2 F}{\partial \boldsymbol{q}_{i\beta}\partial \boldsymbol{q}_{j\gamma}}\boldsymbol{v}_{i\beta}\boldsymbol{w}_{j\gamma}\end{aligned}$$

where $\boldsymbol{q}_{i\beta}, \boldsymbol{v}_{i\beta}$ and $\boldsymbol{w}_{j\gamma}$ are the d cartesian components in E ($\mathbb{R}^d$ as the tangent space of E) of $\boldsymbol{q}_i$, $\boldsymbol{v}_i$ and $\boldsymbol{w}_j$ respectively.

Using the mass-metric, if $\boldsymbol{N}$ denotes the unit vector field normal to $T_x\mathbb{S}$ in $T_x\mathbb{F}_n(E)$, the projection of $\nabla^S_{\boldsymbol{v}}\boldsymbol{u}$ of any vector field $\boldsymbol{u}$ on $T_x\mathbb{S}$ is

$$\nabla^S_{\boldsymbol{v}}\boldsymbol{u} = D_{\boldsymbol{v}}\boldsymbol{u} - \langle D_{\boldsymbol{v}}\boldsymbol{u}, \boldsymbol{N}\rangle_M \boldsymbol{N},$$

and

$$df^\sharp = dF^\sharp - \langle dF^\sharp, \boldsymbol{N}\rangle_M \boldsymbol{N}\ .$$

The Hessian can be written also as (cf. page 344 of [8]) $\mathrm{Hess}(f)[\boldsymbol{v},\boldsymbol{w}](x) = \langle \nabla^S_{\boldsymbol{v}} df^\sharp, \boldsymbol{w}\rangle_M$ and $\mathrm{Hess}(F)[\boldsymbol{v},\boldsymbol{w}](x) = \langle D_{\boldsymbol{v}} dF^\sharp, \boldsymbol{w}\rangle_M$. It follows therefore that

$$\begin{aligned}\mathrm{Hess}(f)[\boldsymbol{v},\boldsymbol{w}](x) &= \langle \nabla^S_{\boldsymbol{v}} df^\sharp, \boldsymbol{w}\rangle_M\\ &= \langle \nabla^S_{\boldsymbol{v}}\left(dF^\sharp - \langle dF^\sharp, \boldsymbol{N}\rangle_M \boldsymbol{N}\right), \boldsymbol{w}\rangle_M\\ &= \langle \nabla^S_{\boldsymbol{v}}\left(dF^\sharp\right), \boldsymbol{w}\rangle_M - \langle \nabla^S_{\boldsymbol{v}}\left(\langle dF^\sharp, \boldsymbol{N}\rangle_M \boldsymbol{N}\right), \boldsymbol{w}\rangle_M.\end{aligned}$$

Because of the product rule for each function φ and each vector field $\boldsymbol{u}$

$$\nabla^S_{\boldsymbol{v}}(\varphi\boldsymbol{u}) = \varphi\nabla^S_{\boldsymbol{v}}\boldsymbol{u} + (d\varphi[\boldsymbol{v}])\boldsymbol{u}$$

$$\implies \nabla^S_{\boldsymbol{v}}\left(\langle dF^\sharp, \boldsymbol{N}\rangle_M \boldsymbol{N}\right) = \langle dF^\sharp, \boldsymbol{N}\rangle_M \nabla^S_{\boldsymbol{v}}\boldsymbol{N} + d\left(\langle dF^\sharp, \boldsymbol{N}\rangle_M\right)[\boldsymbol{v}]\boldsymbol{N}$$

which implies that

$$\langle \nabla^S_{\boldsymbol{v}}\left(\langle dF^\sharp, \boldsymbol{N}\rangle_M \boldsymbol{N}\right), \boldsymbol{w}\rangle_M = \langle dF^\sharp, \boldsymbol{N}\rangle_M \langle \nabla^S_{\boldsymbol{v}}\boldsymbol{N}, \boldsymbol{w}\rangle_M$$

since $\boldsymbol{N}$ is orthogonal to $\boldsymbol{w}$. The same argument can be applied to show that for any vector field $\boldsymbol{u}$ (not necessarily tangent to $\mathbb{S}$)

$$\langle \nabla^S_{\boldsymbol{v}}\boldsymbol{u}, \boldsymbol{w}\rangle_M = \langle D_{\boldsymbol{v}}\boldsymbol{u}, \boldsymbol{w}\rangle_M,$$

and therefore that, evaluated at the critical point x,

$$\begin{aligned}\operatorname{Hess}(f)[\boldsymbol{v},\boldsymbol{w}] &= \langle D_{\boldsymbol{v}}\left(dF^\sharp\right),\boldsymbol{w}\rangle_M - \langle dF^\sharp,\boldsymbol{N}\rangle_M\langle D_{\boldsymbol{v}}\boldsymbol{N},\boldsymbol{w}\rangle_M \\ &= D^2F[\boldsymbol{v},\boldsymbol{w}] - \langle dF^\sharp,\boldsymbol{N}\rangle_M\langle D_{\boldsymbol{v}}\boldsymbol{N},\boldsymbol{w}\rangle_M.\end{aligned}$$

The inertia ellipsoid S is defined by the equation $\|\boldsymbol{q}\|_M^2 = 1$, or equivalently $h(\boldsymbol{q}) = \frac{1}{2}$ where $h(\boldsymbol{q}) = \dfrac{1}{2}\|\boldsymbol{q}\|_M^2$. The normal unit vector $\boldsymbol{N}$ is equal to $dh^\sharp = \boldsymbol{q}$, and thus

$$\begin{aligned}\operatorname{Hess}(f)[\boldsymbol{v},\boldsymbol{w}] &= D^2F[\boldsymbol{v},\boldsymbol{w}] - \langle dF^\sharp,\boldsymbol{q}\rangle_M\langle D_{\boldsymbol{v}}\boldsymbol{q},\boldsymbol{w}\rangle_M \\ &= D^2F[\boldsymbol{v},\boldsymbol{w}] - \langle dF^\sharp,\boldsymbol{q}\rangle_M\langle \boldsymbol{v},\boldsymbol{w}\rangle_M.\end{aligned}$$

If $F = U$, then U is homogeneous of degree $-\alpha$, and therefore $\langle dU^\sharp,\boldsymbol{q}\rangle_M = dU(\boldsymbol{q})[\boldsymbol{q}] = -\alpha U(\boldsymbol{q})$. The following equation follows, at any critical point x of the restriction of U to $\mathbb{S}$.

$$\operatorname{Hess}(U|_{\mathbb{S}})[\boldsymbol{v},\boldsymbol{w}] = D^2U(x)[\boldsymbol{v},\boldsymbol{w}] + \alpha U(x)\langle \boldsymbol{v},\boldsymbol{w}\rangle_M. \tag{1.1}$$

A *central configuration* is a configuration $\boldsymbol{q} \in \mathbb{F}_n(E)$ with the property that there exists a multiplier $\lambda \in \mathbb{R}$ such that

$$dU^\sharp(\boldsymbol{q}) = \lambda\boldsymbol{q}, \tag{1.2}$$

where $dU^\sharp$ is the gradient in E^n of the potential function U, with respect to the mass-metric. Equation (1.2) implies that $\lambda = -\alpha\dfrac{U(\boldsymbol{q})}{\|\boldsymbol{q}\|_M^2}$ (for more on central configurations see e.g. [17] (§369–§382bis at pp. 284–306), [15], [10], [12], [18], [1], [6], [2], [11], [5]). An equivalent definition for a normalized (i.e. $\boldsymbol{q} \in \mathbb{S}$) central configuration is the following:

(1.3) *$\boldsymbol{q} \in \mathbb{S}_n(E)$ is a central configuration if and only if it is a critical point for the restriction $U|_{\mathbb{S}}$ of the potential function to $\mathbb{S} = \mathbb{S}_n(E)$.*

Let $c\colon E^n \to E^n$ be the isometry defined as $c(\boldsymbol{q}) = \boldsymbol{q}'$, with

$$\boldsymbol{q}'_j = \boldsymbol{q}_j - 2\boldsymbol{q}_0 \tag{1.4}$$

for each $j = 1,\ldots,n$, and with $\boldsymbol{q}_0 = \sum_{j=1}^n m_j\boldsymbol{q}_j$. It is an isometry, since $\|\boldsymbol{q}'\|_M^2 = \sum_{j=1}^n m_j|\boldsymbol{q}_j - 2\boldsymbol{q}_0|^2 = \sum_{j=1}^n m_j(|\boldsymbol{q}_j|^2 + 4|\boldsymbol{q}_0|^2 - 4\boldsymbol{q}_j\cdot\boldsymbol{q}_0) = \sum_{j=1}^n m_j|\boldsymbol{q}_j|^2 + 4(\sum_{j=1}^n m_j)|\boldsymbol{q}_0|^2 - 4|\boldsymbol{q}_0|^2 = \|\boldsymbol{q}\|_M^2$. It is the orthogonal reflection around the space of all configurations with center of mass $\boldsymbol{q}_0$ equal to zero: $c\boldsymbol{q} = \boldsymbol{q} \iff \boldsymbol{q}_0 = 0$. It is easy to see that if $\boldsymbol{q}$ is a central configuration then $c\boldsymbol{q} = \boldsymbol{q}$, and hence $\boldsymbol{q}$ has center of mass $\boldsymbol{q}_0$ in 0. Let Y be defined as $Y = \{\boldsymbol{q} \in E^n : \boldsymbol{q}_0 = \boldsymbol{0}\}$, and $\mathbb{S}^c = \mathbb{S} \cap Y$, $S^c = S \cap Y$. In other words, elements of $\mathbb{S}^c$ are normalized configurations with center of mass in 0. Since the potential function is invariant up to translations, $U(c\boldsymbol{q}) = U(\boldsymbol{q})$, and any critical point of the restriction $U|_{\mathbb{S}^c}$ is a critical point of $U|_{\mathbb{S}}$ (for example, by Palais Principle of Symmetric Criticality [13]). Thus it is equivalent to define central configurations as critical points of $U|_{\mathbb{S}^c}$ or as critical points of $U|_{\mathbb{S}}$.

2 Fixed points, $SO(d)$-orbits and projective configuration spaces

Following [3, 4], consider the function $F \colon \mathbb{S}_n(E) \to S_n(E)$ defined as

$$F(\boldsymbol{q}) = -\frac{dU^\sharp(\boldsymbol{q})}{\|dU^\sharp(\boldsymbol{q})\|_M} \tag{2.1}$$

where $dU^\sharp$ is the gradient of U, with respect to the mass-metric.

First, consider the isometry c defined above in (1.4). Since $F(c\boldsymbol{q}) = cF(\boldsymbol{q})$, $F(\mathbb{S}^c) \subset S^c$. Moreover, as the image of F is in S^c, if F^c denotes the restriction $F^c \colon \mathbb{S}^c \to S^c$,

$$\mathrm{Fix}(F^c) = \mathrm{Fix}(F), \tag{2.2}$$

and the fixed point indexes are exactly the same.

Let $O(d)$ be the special orthogonal group, acting diagonally on E^n, and $SO(d)$ the special orthogonal subgroup. The inertia ellipsoid $\mathbb{S}$, S and Y are $O(d)$-invariant in E^n, and so are $\mathbb{S}^c$ and S^c. Let $\pi \colon S \to S/G$ denote the quotient map onto the space of G-orbits, for $G = SO(d)$ or $G = -O(d)$.

Since U is a G-invariant function, F is a G-equivariant map, and hence it induces a map on the quotient spaces:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{F} & S \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{S}/G & \xrightarrow[f]{} & S/G \end{array} \tag{2.3}$$

A fixed point of F is a normalized configuration $\boldsymbol{q}$ such that $F(\boldsymbol{q}) = \boldsymbol{q}$. A fixed point of f is a conjugacy class $[\boldsymbol{q}]$ of configurations such that $f([\boldsymbol{q}]) = [\boldsymbol{q}]$, i.e. it is a conjugacy class $[\boldsymbol{q}]$ such that $F(\boldsymbol{q}) = g\boldsymbol{q}$ for some $g \in G$. It follows from Theorem (2.5) of [4] that if $G = SO(d)$, then $F(\boldsymbol{q}) = g\boldsymbol{q} \iff F(\boldsymbol{q}) = \boldsymbol{q}$, or equivalently that

$$G = SO(d) \implies \pi(\mathrm{Fix}(F)) = \mathrm{Fix}(f), \tag{2.4}$$

and hence also that $\pi(\mathrm{Fix}(F^c)) = \mathrm{Fix}(f^c)$.

(2.5) Remark. Elements in $\mathbb{S}/G$ are called projective configurations: for $d = 2$ and $G = SO(2)$, S/G is the $(n-1)$-dimensional complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$, and S^c is a hyperplane in it, hence a $(n-2)$-dimensional complex projective space $\mathbb{P}^{n-2}(\mathbb{C})$ For $n = 3$, it is the Riemann sphere. Projective configurations are projective classes of elements $[\boldsymbol{q}_1 : \boldsymbol{q}_2 : \boldsymbol{q}_3]$ in $\mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ such that $m_1\boldsymbol{q}_1 + m_2\boldsymbol{q}_2 + m_3\boldsymbol{q}_3 = 0$, $\boldsymbol{q}_j \in \mathbb{C}$, and $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$, $\boldsymbol{q}_1 \neq \boldsymbol{q}_3$, $\boldsymbol{q}_2 \neq \boldsymbol{q}_3$.

For $d = 1$, projective configurations are equivalence classes under the action of the orthogonal group $G = O(1) = \mathbb{Z}_2$.

The following Corollary of (2.4) shows that the difference is minor.

(2.6) Corollary. *If $\boldsymbol{q} \in \mathbb{S}$ is a central configuration such that $F(\boldsymbol{q}) = g\boldsymbol{q}$, with $g \in O(d)$ (acting diagonally on E^n), then $g = 1$.*

Proof. Let $E' = E \oplus \mathbb{R}$ be the euclidean space of dimension $d+1$, and $E \subset E'$ one of its d-dimensional subspaces. If $\boldsymbol{q} \in \mathbb{S} \subset \mathbb{F}_n(E)$, then $\boldsymbol{q} \in \mathbb{S} \subset \mathbb{F}_n(E) \subset \mathbb{F}_n(E')$, and there exists $g' \in SO(d+1)$ such that $g'E = E$ and the restriction of g' to E is equal to g: it follows that $F(\boldsymbol{q}) = g'\boldsymbol{q}$, in $\mathbb{F}_n(E')$, and therefore $g' = 1$, from which it follows that $g = 1$. ∎

Homological calculations on configurations spaces for the sake of central configurations have been done by Palmore [14], Pacella [12] and McCord [9]. We can arrange all the spaces inertia ellipsoids and the corresponding projective quotients as in diagram (2.7).

$$\begin{array}{ccccccc}
\mathbb{S}_n^c(\mathbb{R}) & \xrightarrow{\iota_1} & \mathbb{S}_n^c(\mathbb{R}^2) & \xrightarrow{\iota_2} & \mathbb{S}_n^c(\mathbb{R}^3) & \xrightarrow{\iota_3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{S}_n^c(\mathbb{R})/SO(1) & \xrightarrow{\bar\iota_1} & \mathbb{S}_n^c(\mathbb{R}^2)/SO(2) & \xrightarrow{\bar\iota_2} & \mathbb{S}_n^c(\mathbb{R}^3)/SO(3) & \xrightarrow{\bar\iota_3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{S}_n^c(\mathbb{R})/O(1) & \xrightarrow{\bar{\bar\iota}_1} & \mathbb{S}_n^c(\mathbb{R}^2)/O(2) & \xrightarrow{\bar{\bar\iota}_2} & \mathbb{S}_n^c(\mathbb{R}^3)/O(3) & \xrightarrow{\bar{\bar\iota}_3} & \cdots
\end{array} \tag{2.7}$$

For each d, $\mathbb{S}_n^c(\mathbb{R}^d)$ is a deformation retract of $\mathbb{F}_n^c(\mathbb{R}^d)$, which in turn is a deformation retraction of $\mathbb{F}_n(\mathbb{R}^d)$ (where $\mathbb{F}_n^c(E)$ denotes the space of all configurations with center of mass in 0). The Poincaré polynomial for the cohomology of the configuration space $\mathbb{F}_n(\mathbb{R}^d)$ is equal to

$$P(t) = \prod_{k=1}^{n-1}(1 + kt^{d-1}),$$

as shown e.g. in Theorem 3.2 of [16] (see also Proposition 2.11.2 of [11]).

Now, note that in the sequence of projections

$$\mathbb{S}_n^c(\mathbb{R}^d) \to \mathbb{S}_n^c(\mathbb{R}^d)/SO(d) \to \mathbb{S}_n^c(\mathbb{R}^d)/O(d)$$

the second map corresponds to the projection given by the action of the quotient group $\mathbb{Z}_2 = O(d)/SO(d)$ on the quotient space $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$ ($SO(d)$ is normal in $O(d)$). For $d \geq 2$, let h be the orthogonal reflection of $\mathbb{R}^d$ around $\mathbb{R}^{d-1} \subset \mathbb{R}^d$: its coset $hSO(d)$ is the generator of $O(d)/SO(d)$, and hence the image $\operatorname{Im}(\bar\iota_{d-1})$ in $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$ is fixed by $O(d)/SO(d)$. Actually, it is equal to the fixed point subset of $O(d)/SO(d)$ in $\mathbb{S}_n^c(\mathbb{R}^d)/SO(d)$. Outside the image of $\bar\iota_{d-1}$, therefore the $\mathbb{Z}_2$ action is free: let $\mathbb{M}_n(\mathbb{R}^d)$ denote the manifold

$$\mathbb{M}_n(\mathbb{R}^d) = \left(\mathbb{S}_n^c(\mathbb{R}^d)/SO(d) \smallsetminus \operatorname{Im}(\bar\iota_{d-1})\right)/\mathbb{Z}_2 = \mathbb{S}_n^c(\mathbb{R}^d)/O(d) \smallsetminus \operatorname{Im}(\bar{\bar\iota}_{d-1}), \tag{2.8}$$

where the last equality holds since $\bar\iota_{d-1}$ factors through $\mathbb{S}_n^c(\mathbb{R}^{d-1})$.

The next proposition follows from the dimension of $SO(d)$ and the previous remarks.

(2.9) *The subspace of all points in* $\mathbb{S}^c_n(\mathbb{R}^d)/O(d)$ *with maximal orbit type is the open subspace* $\mathbb{M}_n(\mathbb{R}^d)$ *defined in* (2.8)*, and it is is a manifold of dimension*

$$\dim \mathbb{M}_n(\mathbb{R}^d) = d(n-1) - 1 - d(d-1)/2.$$

For $d = 1$, it is the projective space $\mathbb{P}^{n-2}(\mathbb{R})$ minus collisions. For $d = 2$, it is a $(2n-4)$ dimensional manifold (where $\mathbb{P}^{n-2}(\mathbb{C})$ minus collinear and minus collisions is its double cover).

(2.10) $\mathbb{S}^c_n(\mathbb{R}^2)/SO(2)$ *has the same homotopy type of* $\mathbb{F}_{n-2}(\mathbb{R}^2 \smallsetminus \{p,q\})$*, where* p, q *are two arbitrary distinct points of* $\mathbb{R}^2$*.*

Proof. It is Lemma 4.1 of [9]. ∎

It follows that the Poincaré polynomial (where β_j are Betti numbers) of $\mathbb{S}^c_n(\mathbb{R}^2)/SO(2)$ is

$$p(t) = \prod_{k=2}^{n-1}(1+kt) = \sum_{j=0}^{n-2} \beta_j t^j. \tag{2.11}$$

(see also Proposition 2.11.3 of [11]). McCord in [9] proved also that

$$\dim H^k(\mathbb{M}_n(\mathbb{R}^2)) = \begin{cases} \sum_{j=0}^{k} \beta j & \text{if } k \leq n-3 \\ 0 & \text{otherwise,} \end{cases}$$

while Pacella in (2.4) of [12] computed the $SO(3)$-equivariant homology (using Borel homology) Poincaré series of $\mathbb{S}^c_n(\mathbb{R}^3) \sim \mathbb{F}_n(\mathbb{R}^3)$ as

$$P^{SO(3)}(t) = \frac{\prod_{k=2}^{n-1}(1+kt^2)}{1-t^2}.$$

(2.12) Remark. The projective quotient $\mathbb{S}^c_n(\mathbb{R}^2)/SO(2)$ is a manifold (it is the projective space $\mathbb{P}^{n-2}(\mathbb{C})$ with collisions removed). It contains $\mathbb{S}^c_n(\mathbb{R})/O(1)$ as a submanifold (the collinear configurations). For $d \geq 3$ the isotropy groups of the action start being non-trivial, and the filtration of subspaces of constant orbits type in $\mathbb{S}^c_n(\mathbb{R}^d)/SO(d)$ is given by the horizontal arrows $\bar{\iota}_j$ in diagram (2.7).

3 Fixed points and Morse indices

Let $\boldsymbol{q} \in \mathbb{S}^c_n(\mathbb{R}^d)$ a central configuration, and hence a fixed point of the map F defined above in (2.1), such that its $O(d)$-orbits lies in the maximal orbit type submanifold $\mathbb{M}_n(\mathbb{R}^d) \subset \mathbb{S}^c(\mathbb{R}^d)/O(d)$.

(3.1) *If* $DF\colon T_{\boldsymbol{q}}\mathbb{S} \to T_{\boldsymbol{q}}\mathbb{S}$ *denotes the differential of* F *at the central configuration* $\boldsymbol{q}$*, then for any* $\boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{q}}\mathbb{S}$ *the following equation holds:*

$$D^2U(\boldsymbol{q})[\boldsymbol{v},\boldsymbol{w}] = -\alpha U(\boldsymbol{q})\langle DF[\boldsymbol{v}], \boldsymbol{w}\rangle_M.$$

Proof. As we have seen in the introduction, $\langle D_v dU^\sharp, \boldsymbol{w}\rangle_M = D^2U[\boldsymbol{v},\boldsymbol{w}]$, and if $\boldsymbol{q}$ is a normalized central configuration then by (1.2) $dU^\sharp(\boldsymbol{q}) = \lambda\boldsymbol{q}$ with $\lambda = -\alpha\frac{U(\boldsymbol{q})}{\|\boldsymbol{q}\|_M^2} = -\alpha U(\boldsymbol{q})$. It follows that $\langle dU^\sharp, \boldsymbol{w}\rangle_M = 0$, being $\boldsymbol{w}$ tangent to $\mathbb{S}$, and $\|dU^\sharp\|_M = -\lambda = \alpha U(\boldsymbol{q})$. Also,

$$\begin{aligned}
\langle DF[\boldsymbol{v}], \boldsymbol{w}\rangle_M &= \langle D_{\boldsymbol{v}}\left(-\frac{dU^\sharp}{\|dU^\sharp\|_M}\right), \boldsymbol{w}\rangle_M \\
&= -\langle\left(\frac{D_{\boldsymbol{v}} dU^\sharp}{\|dU^\sharp\|_M}\right), \boldsymbol{w}\rangle_M - \langle D_{\boldsymbol{v}}\left(\frac{1}{\|dU^\sharp\|_M}\right) dU^\sharp, \boldsymbol{w}\rangle_M \\
&= -\frac{1}{\|dU^\sharp\|_M}\langle D_{\boldsymbol{v}} dU^\sharp, \boldsymbol{w}\rangle_M - 0 \\
&= -\frac{1}{\alpha U(\boldsymbol{q})} D^2U(\boldsymbol{q})[\boldsymbol{v},\boldsymbol{w}].
\end{aligned}$$

■

Combining (3.1) with equation (1.1) the following corollary holds.

(3.2) Corollary. *If $\boldsymbol{q}$ is as above, then for each $\boldsymbol{v},\boldsymbol{w} \in T_{\boldsymbol{q}}\mathbb{S}$*

$$\operatorname{Hess}(U|_{\mathbb{S}})[\boldsymbol{v},\boldsymbol{w}] = \alpha U(\boldsymbol{q})\left(\langle \boldsymbol{v},\boldsymbol{w}\rangle_M - \langle DF[\boldsymbol{v}], \boldsymbol{w}\rangle_M\right).$$

Finally, consider again the group $O(d)$ acting on $\mathbb{S}_n^c(\mathbb{R}^d)$. Let F and $\boldsymbol{q}$ be the map and the central configuration defined above. Recall that $f\colon \mathbb{S}/O(d) \to S/O(d)$ denotes the map defined on the quotient. Let $[\boldsymbol{q}] \in \mathbb{M}_n(\mathbb{R}^d)\mathbb{S}/O(d)$ denote the projective class (i.e. the $O(d)$-orbit of $\boldsymbol{q}$) of $\boldsymbol{q}$, which is a fixed point of f, and is a critical point of the map $\bar{U}\colon \mathbb{M}_n(\mathbb{R}^d) \to \mathbb{R}$ induced on $\mathbb{M}_n$ by U, defined simply as $\bar{U}([x]) = U(x)$ for each $x \in \mathbb{S}_n^c(\mathbb{R}^d)$.

(3.3) Theorem. *The point $[\boldsymbol{q}]$ is a non-degenerate critical point of $\bar{U}$ if and only if it is a non-degenerate fixed point of f. If $\operatorname{ind}([\boldsymbol{q}], f)$ denotes the fixed point index of $[\boldsymbol{q}]$ for f, and $\mu([\boldsymbol{q}])$ the Morse index of $[\boldsymbol{q}]$, then the following equation holds:*

$$\operatorname{ind}([\boldsymbol{q}], f) = (-1)^{\mu([\boldsymbol{q}])}.$$

Proof. The point $[\boldsymbol{q}]$ is a non-degenerate critical point if and only if the dimension of the kernel of the Hessian $\operatorname{Hess}(U|_{\mathbb{S}})(\boldsymbol{q})$ is equal to the dimension of $SO(d)$, i.e. $d(d-1)/2$. By (3.2), the kernel is equal to the eigenspace of $DF(\boldsymbol{q})$ corresponding to the eigenvalue 1, which has dimension $d(d-1)/2$ if and only if the fixed point $[\boldsymbol{q}]$ is non-degenerate. Now, if this holds then the index $\operatorname{ind}([\boldsymbol{q}], f)$ is equal to the number $(-1)^e$. where e is the number of negative eigenvalues $1 - f'$, which is the same as the number of negative eigenvalues of $1 - F'$. Again by (3.2) and since $U > 0$, e is equal to the number of negative eigenvalues of $\operatorname{Hess}(U|_{\mathbb{S}})$, which is by definition the Morse index $\mu([\boldsymbol{q}])$. ■

(3.4) Remark. Unfortunately, a former version of this statement had a wrong formula for $\operatorname{ind}(\boldsymbol{q})$. In fact, in (3.5) of [4] one should put $\epsilon = 0$, and not $\epsilon = d(n-1) - 1 - d(d-1)/2 = \dim \mathbb{M}_n(\mathbb{R}^d)$. The error occurred because I used the wrong sign of U in (3.1) ($V = -U$ instead of U).

(3.5) Example. For $d = 1$ and any n, all critical points are local minima of U, and hence $\mu = 0$, and fixed points have index 1. The map induced on the quotient can be regularized on binary collisions (see [4, 3]), hence the map on the quotient can be extended to a self-map $f\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ with three fixed points of index 1. Therefore the Lefschetz number of f is 3, and f has degree -2.

For $d = 2$ and $n = 3$, the three Euler configurations have $\mu = 1$, while the two Lagrange points have $\mu = 1$, hence the map f induced on the quotient $\mathbb{P}^1(\mathbb{C})$ (again, by regularizing the binary collisions) has Lefschetz number equal to $L(f) = 2 - 3 = -1$. Therefore the degree of f is equal to -2.

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