Stabilizers of fixed point classes and Nielsen numbers of n-valued maps

Robert F. Brown Junzheng Nan

Abstract

The stabilizer of a fixed point class of a map is the fixed subgroup of the induced fundamental group homomorphism based at a point in the class. A theorem of Jiang, Wang and Zhang is used to prove that if a map of a graph satisfies a strong remnant condition, then the stabilizers of all its fixed point classes are trivial. Consequently, if $\phi_{p,f}$ is the n-valued lift to a covering space p of a map f with strong remnant of a graph, then the Nielsen numbers are related by the equation $N(\phi_{p,f}) = n \cdot N(f)$. Additional information concerning Nielsen numbers is obtained for n-valued lifts of maps of graphs with positive Lefschetz numbers and of maps of spaces with abelian fundamental groups and for extensions of n-valued maps.

1 Introduction

The Nielsen fixed point theory of n-valued maps, that is, upper and lower semicontinuous functions $\phi\colon X \multimap X$ such that $\phi(x)$ is an unordered set of exactly n points of X, was initiated by Schirmer in [14], [15], [16]. She defined a Nielsen number $N(\phi)$ that is a lower bound for the number of fixed points, that is $x \in \phi(x)$, for all n-valued maps that are n-valued homotopic to ϕ. The only examples in those papers were of n-valued maps of the circle. Classes of nontrivial n-valued maps of tori were studied, for instance, in [3] and [4], but there

Received by the editors in July 2016 - In revised form in December 2016.
Communicated by K. Dekimpe, D.L. Gonçalves and P. Wong.
2010 *Mathematics Subject Classification* : 55M20, 54C60.
Key words and phrases : n-valued map, Nielsen number, stabilizer, finite covering space, maps of graphs, remnant.

were no such examples on other spaces prior to [2]. In that paper, a lifting construction defined n-valued maps on graphs, orientable double covers of nonorientable manifolds, handlebodies, free G-spaces and nilmanifolds. The purpose of the present paper is to extend the classes of spaces on which n-valued maps are defined and, especially, to use a recent result of Jiang, Wang and Zhang [12] to refine the computations of the Nielsen numbers of the n-valued maps of graphs obtained in [2].

Let $p\colon \widetilde{X} \to X$ be a finite covering space, of degree n, with X a connected finite polyhedron and $\widetilde{X}$ connected. For $f\colon X \to X$ a map, define $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$, the *$n$-valued lift of f* by $\phi(\tilde{x}) = \{p^{-1}(fp(\tilde{x}))\}$. The relationship between $N(\phi_{p,f})$ and the Nielsen number $N(f)$ of f was established in [2]. It depends on what is called in [12] the "stabilizer" of a fixed point class, which is defined as follows. Let x_1 be a fixed point of f, let $f_{x_1}\colon \pi_1(X, x_1) \to \pi_1(X, x_1)$ be the induced homomorphism and let $Fix(f_{x_1}) = \{\alpha \in \pi_1(X, x_1)\colon f_{x_1}(\alpha) = \alpha\}$ be the fixed subgroup of f_{x_1}. It has long been known, see page 36 of [10], that if $x_2 \in X$ is in the same fixed point class of f as x_1, then $Fix(f_{x_1})$ and $Fix(f_{x_2})$ are isomorphic. If the fixed point classes of f are identified with equivalence classes of the lifts of f to the universal covering space of X under conjugation by deck transformations, then $Fix(f_{x_1})$ can be identified as the stabilizer of the equivalence class corresponding to the fixed point class of x_1 under the action of conjugation by deck transformations on the set of lifts of f. Since the stabilizer is an invariant of the fixed point class $\mathcal{F}_{x_1}$ containing x_1, in [12] the group $Fix(f_{x_1})$ is called the *stabilizer of the fixed point class $\mathcal{F}$* and denoted $Stab(\mathcal{F}_{x_1})$. The symbol $\#(S)$ will mean the cardinality of a finite set S. For x in the fixed point class $\mathcal{F}$ we denote by $\#(p^{-1}(x)/Stab(\mathcal{F}))$ the number of orbits of the restriction to $Stab(\mathcal{F})$ of the monodromy action of $\pi_1(X, x)$ on $p^{-1}(x)$. The relationship between the Nielsen numbers of f and of $\phi_{p,f}$ is the following

Theorem 1.1. ([2]) *Let $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$ be the n-valued lift of a map $f\colon X \to X$ to the covering space $p\colon \widetilde{X} \to X$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{N(f)}$ be the essential fixed point classes of f and let x_j be a point of $\mathcal{F}_j$, then*

$$N(\phi_{p,f}) = \sum_{j=1}^{N(f)} \#(p^{-1}(x_j)/Stab(\mathcal{F}_j)).$$

Thus $N(\phi_{p,f}) \geq N(f)$. If all the stabilizers are trivial, then $N(\phi_{p,f}) = n \cdot N(f)$.

In Section 2 we present a more precise statement of the invariance of the stabilizer, that implies that the stabilizers of all the fixed point classes are isomorphic if the fundamental group of X is abelian. In that case, Theorem 1.1 allows us to calculate the Nielsen number of an n-valued lift of a map $f\colon X \to X$ in terms of the the Nielsen number of f and the homomorphism induced by f on the first integer homology of X. As an application, we calculate the Nielsen numbers of n-valued lifts of maps of tori and of lens spaces.

For $f\colon X \to X$ a map of a finite graph, the stabilizer of the fixed point class is a free group that is finitely generated [6]. We denote by $rank(Stab(\mathcal{F}))$ the rank of the stabilizer of a fixed point class $\mathcal{F}$, that is, the smallest number of free generators. We will be making use of the following part of the main result of [12] that relates the rank of the stabilizer of a fixed point class to the fixed point index $ind(\mathcal{F})$ of that class.

Theorem 1.2. (*Jiang, Wang, Zhang* [12]) *Suppose X is a connected finite graph and $f\colon X \to X$ is a map, then*

$$ind(\mathcal{F}) \leq 1 - rank(Stab(\mathcal{F}))$$

for every fixed point class $\mathcal{F}$ of f.

This result implies the index bound $ind(\mathcal{F}) \leq 1$ of [11], [13] and it also implies that if $ind(\mathcal{F}) = +1$, then $Stab(\mathcal{F})$ is the trivial group.

We use Theorem 1.2 in Section 3 to prove that for an n-valued lift $\phi_{p,f}$ of a map f of a graph with Lefschetz number $L(f) > 0$, the bound $N(\phi_{p,f}) \geq N(f)$ can be improved to

$$N(\phi_{p,f}) \geq (n-1)L(f) + N(f).$$

In Section 4 we develop a tool that allows us to describe the stabilizers of fixed point classes in terms of homomorphisms of the fundamental group based at a single point rather than basing the fundamental group at a different fixed point for each fixed point class. We use that tool to obtain a condition on a map of a graph that implies that the stabilizers of all the fixed point classes of f are trivial and thus that $N(\phi_{p,f}) = n \cdot N(f)$ holds for any n-valued lift of f. The required property, called "strong remnant", was introduced by Hart [8] as a stronger version of the remnant condition of Wagner in [17]. In Section 5 we prove that, like the remnant condition, the condition of strong remnant is satisfied by "most" maps of graphs, in a sense that is made precise there.

Finally, in Section 6, we show that we can extend an n-valued map of a finite polyhedron to an n-valued map of a polyhedron that retracts to it, with the Nielsen number unchanged. In particular then, the results regarding n-valued maps of graphs can be extended to n-valued maps of surfaces with boundary and of handlebodies.

We thank Evelyn Hart for sending us several helpful messages in reply to our queries; even finding time to send some of them during her family's European vacation. Conversations with Don Blasius contributed to an example. We thank Michael Crabb for the result credited to him in the paper. Chris Staecker helped with the exposition and brought [1] to our attention. We especially thank the referee for several helpful suggestions for improving this paper.

2 Fixed Subgroups and the Basepoint Isomorphism

Let G be a group and $h\colon G \to G$ an endomorphism. The *fixed subgroup* of h, which we denote by $Fix(h)$ is the group of fixed points of h, that is,

$$Fix(h) = \{g \in G : h(g) = g\}.$$

Let X be a space and $f\colon X \to X$ a map such that $f(x_*) = x_*$. Then f induces the endomorphism $f_{x_*}\colon \pi_1(X, x_*) \to \pi_1(X, x_*)$ of the fundamental group and we call its fixed subgroup the *fixed subgroup of* f_{x_*}, denoted $Fix(f_{x_*})$. Although the group $\pi_1(X, x_*)$ is independent of the basepoint x_*, up to isomorphism, Proposition 2.2 of [2] illustrates the fact that, in general, the isomorphism class of the group $Fix(f_{x_*})$ depends on the choice of the fixed point x_* of f.

Let X be a space, $c\colon [0,1] \to X$ a path with $c(0) = x_0$ and $c(1) = x_1$ and denote by $\beta_c\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ the *basepoint isomorphism* defined by $\beta_c[w] = [c^{-1}wc]$ where $c^{-1}(t) = c(1-t)$.

Proposition 2.1. *Let x_0 and x_1 be fixed points of $f\colon X \to X$ and let $c\colon [0,1] \to X$ be a path such that $c(0) = x_0$ and $c(1) = x_1$. Then $\beta_c(Fix(f_{x_0})) \subseteq Fix(f_{x_1})$ if and only if $[c(fc)^{-1}]$ is in the centralizer of $Fix(f_{x_0})$. Therefore, the restriction of β_c to $Fix(f_{x_0})$ is an isomorphism onto $Fix(f_{x_1})$ if and only if $[c(fc)^{-1}]$ is in the centralizer of $Fix(f_{x_0})$ and $[c^{-1}(fc)]$ is in the centralizer of $Fix(f_{x_1})$.*

Proof. If $[c(fc)^{-1}]$ is in the centralizer of $Fix(f_{x_0})$ and $[w] \in Fix(f_{x_0})$, then

$$\begin{aligned} f_{x_0}(\beta_c[w]) &= [(fc)^{-1}(fw)(fc)] = [(fc)^{-1}w(fc)] \\ &= [c^{-1}(c(fc)^{-1})w(fc)] = [c^{-1}w(c(fc)^{-1})(fc)] \\ &= \beta_c[w]. \end{aligned}$$

Conversely, suppose $\beta_c(Fix(f_{x_0})) \subseteq Fix(f_{x_1})$ and let $[w] \in Fix(f_{x_0})$. Then

$$f_{x_0}\beta_c[w] = [(fc)^{-1}(fw)(fc)] = [c^{-1}wc] = \beta_c[w]$$

and consequently,

$$\begin{aligned} [w] &= \beta_{c^{-1}} f_{x_0}\beta_c[w] = [c(fc)^{-1}(fw)(fc)c^{-1}] \\ &= [c(fc)^{-1}w(fc)c^{-1}] = [c(fc)^{-1}][w][(fc)c^{-1}] \\ &= [c(fc)^{-1}][w][c(fc)^{-1}]^{-1} \end{aligned}$$

so

$$[c(fc)^{-1}][w] = [w][c(fc)^{-1}]. \qquad \blacksquare$$

Fixed points x_0 and x_1 are in the same fixed point class if and only if there exists a path c such that $[c(fc)^{-1}] = 1 \in \pi_1(X, x_0)$ so, in that case, the restriction of β_c is an isomorphism between $Fix(f_{x_0})$ and $Fix(f_{x_1})$. We will follow the terminology and notation of [12] from now on and call the fixed subgroup $Fix(f_{x_*})$ at $x_* \in \mathcal{F}$ the stabilizer $Stab(\mathcal{F})$ of its fixed point class $\mathcal{F}$.

If $\pi_1(X, x_*)$ is abelian, then by Proposition 2.1 all the groups $Stab(\mathcal{F})$ are isomorphic. Theorem 1.1 then implies that

$$N(\phi_{p,f}) = \#(p^{-1}(x)/Stab(\mathcal{F})) \cdot N(f)$$

where x is any fixed point of f. If, in addition, $Stab(\mathcal{F}) = 1$ then $N(\phi_{p,f}) = n \cdot N(f)$.

In particular, if $f\colon X \to X$ is a map of the r-torus and A is the r-by-r integer matrix determined by the induced fundamental group homomorphism of f, then

$$N(\phi_{p,f}) = n \cdot |\det(I - A)|.$$

The reason is that if $N(f) = |\det(I - A)| \neq 0$ then $Stab(\mathcal{F}) = 1$ whereas if $\det(I - A) = 0$ then f is homotopic to a fixed point free map and therefore $\phi_{p,f}$ has the same property so $N(\phi_{p,f}) = 0$.

Theorem 2.1. (Michael Crabb) *Let $f\colon X \to X$ be a map of a connected finite polyhedron with abelian fundamental group. The map f induces a homomorphism of the integer homology group $f_*\colon H_1(X) \to H_1(X)$. Let $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$ be the n-valued lift of f to the covering space $p\colon \widetilde{X} \to X$. Denote by $H_1(\widetilde{X}) \subseteq H_1(X)$ the subgroup that is the image of $p_*\colon H_1(\widetilde{X}) \to H_1(X)$, then*

$$N(\phi_{p,f}) = d \cdot N(f)$$

where

$$d = \#(f_* - 1)(H_1(X)/H_1(\widetilde{X})).$$

Proof. If f has no fixed points, then $N(f) = N(\phi_{p,f}) = 0$ so let x be a fixed point of f and let $\tilde{x} \in p^{-1}(x)$. We may identify $p^{-1}(x)$ with $\pi_1(X,x)/\pi_1(\widetilde{X},\tilde{x})$. Let $\mathcal{F}$ be the fixed point class of x, then $Stab(\mathcal{F}) = \{\alpha \in \pi_1(X,x)\colon f_x(\alpha) = \alpha\}$ acts on the left on $p^{-1}(x)$. Since $\pi_1(X,x)$ is abelian, we may replace $\pi_1(X,x)$ and $\pi_1(\widetilde{X},\tilde{x})$ by the integer homology groups $H_1(X)$ and $H_1(\widetilde{X})$ and $Stab(\mathcal{F})$ by

$$K = \ker\{f_* - 1\colon H_1(X) \to H_1(X)\}.$$

Therefore,

$$\begin{aligned} d &= \#(H_1(X)/(K + H_1(\widetilde{X})) \\ &= \#((f_* - 1)(H_1(X))/(f_* - 1)(H_1(\widetilde{X}))) \\ &= \#(f_* - 1)(H_1(X)/H_1(\widetilde{X})) \end{aligned}$$

because the kernel of

$$f_* - 1\colon H_1(X) \to (f_* - 1)(H_1(X))/(f_* - 1)(H_1(\widetilde{X}))$$

is $K + H_1(\widetilde{X})$ and thus

$$Stab(\mathcal{F}) \setminus \pi_1(X,x)/\pi_1(\widetilde{X},\tilde{x}) = H_1(X)/(K + H_1(\widetilde{X})).$$ ∎

To illustrate this result, let $L(m,q)$ denote a lens space, where m and q are relatively prime integers, and let $f\colon L(m,q) \to L(m,q)$ be a map. Then f induces $f_*\colon H_1(L(m,q)) \to H_1(L(m,q))$, an endomorphism of $C[m] = \mathbb{Z}/m\mathbb{Z}$, the cyclic group of order m. Let $f_*(1) = k$. If $k \neq 1$, then by Example 3 on page 34 of [10], the Nielsen number is $N(f) = (k-1,m)$, the greatest common divisor of $k-1$ and m. If r divides m, then there is a covering space $p\colon L(r,q) \to L(m,q)$ of degree m/r. Let $\phi_{p,f}\colon L(r,q) \multimap L(r,q)$ be the m/r-valued lift of f. If $k = 0$, then

$$(f_* - 1)(H_1(X)/H_1(\widetilde{X})) = H_1(X)/H_1(\widetilde{X}) = C[m/r]$$

so $N(\phi_{p,f}) = m/r$.

If $k \geq 2$, then by Theorem 2.1,

$$\#(f_* - 1)(H_1(X)/H_1(\widetilde{X})) = \#(f_* - 1)(C[m/r]) = \frac{m/r}{(k-1,m/r)}$$

and therefore

$$N(\phi_{p,f}) = \frac{m \cdot (k-1,m)}{r \cdot (k-1,m/r)}.$$

3 Maps with Positive Lefschetz Numbers

Let $f\colon X \to X$ be a map of a connected (finite) graph. Since the Nielsen theory of single-valued maps is invariant of homotopy type, we take X to be a wedge $X = a_1 \vee a_2 \vee \cdots \vee a_m$ of circles at a vertex that we will always denote by x_0. The circles are oriented so they generate the free group $\pi_1(X, x_0)$. We homotope f so that it maps a neighborhood of x_0 to x_0. Then, using the simplicial approximation theorem, we further homotope f so that each circle a_i is a union of arcs on each of which the restriction of f takes the endpoints to x_0 and the interior homeomorphically onto either some $a_j - \{x_0\}$ or $a_j^{-1} - \{x_0\}$. The map f is then said to be in *standard form*.

Proposition 3.1. *Let $f\colon X \to X$ be a map of a graph. For $p\colon \widetilde{X} \to X$ a covering space of degree n, let $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$ be the n-valued lift of f. If the Lefschetz number $L(f) > 0$, then*

$$N(\phi_{p,f}) \geq (n-1)L(f) + N(f).$$

Proof. We assume that f is in standard form and let $\mathcal{F}_1, \ldots, \mathcal{F}_{N(f)}$ be the essential fixed point classes. By Theorem 1.2, $ind(\mathcal{F}_i) \leq 1$. Therefore, since $\sum_{i=1}^{N(f)} ind(\mathcal{F}_i) = L(f)$, at least $L(f)$ of the fixed point classes are of index $+1$. By Theorem 1.2, if $ind(\mathcal{F}_i) = 1$, then $Stab(\mathcal{F}_i)$ is the trivial group. Therefore $\#(p^{-1}(x_i)/Stab(\mathcal{F}_i)) = n$ and consequently Theorem 1.1 implies that

$$N(\phi_{p,f}) \geq n \cdot L(f) + (N(f) - L(f)) = (n-1)L(f) + N(f). \qquad \blacksquare$$

4 Maps with Strong Remnant

Let X be a wedge of oriented circles based at x_0 and $f\colon X \to X$ a map in standard form fixing x_0 and with one fixed point x_j for each appearance a_{i_j} of a generator a_i or its inverse in $f(a_i)$. Then f induces $f_{x_0}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$. As in [17], write $f_{x_0}(a_i) = V_j a_{i_j}^{\epsilon_j} \overline{V}_j$, where $\epsilon_j \in \{+1, -1\}$ and define the *Wagner tails* corresponding to a_{i_j} as follows: $W_j = V_j$ if $\epsilon_j = +1$, $W_j = V_j a_{i_j}^{-1}$ if $\epsilon_j = -1$, $\overline{W}_j = \overline{V}_j^{-1}$ if $\epsilon_j = +1$ and $\overline{W}_j = \overline{V}_j^{-1} a_{i_j}$ if $\epsilon_j = -1$.

The following proposition allows us to study the stabilizers of the fixed point classes of maps of X as fixed subgroups of endomorphisms of the fundamental group at the basepoint x_0. In this way, it becomes entirely an algebraic problem.

Proposition 4.1. *For a fixed point x_j of f we use the corresponding Wagner tails to define $\theta_{x_j}, \bar{\theta}_{x_j}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ by $\theta_{x_j}[\omega] = W_j^{-1}[f(\omega)]W_j$ and $\bar{\theta}_{x_j}[\omega] = \overline{W}_j^{-1}[f(\omega)]\overline{W}_j$. The stabilizer of $\mathcal{F}_{x_j}$, the fixed point class of f that contains x_j, is isomorphic to the fixed subgroups of θ_{x_j} and $\bar{\theta}_{x_j}$.*

Proof. Without loss of generality, we let $x_j = x_1 \in a_1$. Let γ_+ be the arc in the circle a_1 containing x_1 from x_1 to x_0 in the positive direction and γ_- in the negative. Let $\beta_+\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ be the basepoint isomorphism defined by $\beta_+[w] =$

$[\gamma_+^{-1}w\gamma_+]$. Lemma 1.2 of [17] implies that $f(\gamma_+) = W_1\gamma_+$, meaning that they are homotopic relative to the endpoints, so

$$\begin{aligned} f_{x_1}\beta_+[w] &= [f_{x_1}(\gamma_+^{-1})f(w)f_{x_1}(\gamma_+)] \\ &= [\gamma_+^{-1}W_1^{-1}f(w)W_1\gamma_+] \\ &= \beta_+\theta_{x_1}[w]. \end{aligned}$$

If $[w] \in Fix(\theta_{x_1})$, then

$$f_{x_1}\beta_+[w] = \beta_+\theta_{x_1}[w] = \beta_+[w]$$

so $\beta_+(Fix(\theta_{x_1})) \subseteq Fix(f_{x_1})$. Similarly, $\theta_{x_1}\beta_+^{-1} = \beta_+^{-1}f_{x_1}$ and therefore $\beta_+^{-1}(Fix(f_{x_1})) \subseteq Fix(\theta_{x_1})$ so these groups are isomorphic. Replacing γ_+ with γ_-, it is proved in the same way that the stabilizer of $\mathcal{F}_{x_j}$ is isomorphic to the fixed subgroup of $\bar{\theta}_{x_j}$. ∎

Let G be the free group on generators $a_1, \ldots, a_n$, let $h\colon G \to G$ be a homomorphism and set $h(a_i) = A_i$. The homomorphism *h has remnant* if each A_i can be written in the form $A_i = P_i\overline{A}_iS_i$ where P_i is the longest initial subword of A_i that can be cancelled by $A_j^{\epsilon_j}$, for $\epsilon_j \in \{+1, -1\}$, except for A_i^{-1}, the subword S_i is the longest such terminal subword and $\overline{A}_i \neq 1$. Then $\overline{A}_i$ is called the *remnant* of A_i. The homomorphism *h has strong remnant* if h has remnant and $\overline{A}_i \neq a_i$ for all i.

A map $f\colon (X, x_0) \to (X, x_0)$ of a graph, in standard form, *has remnant* [17] if the induced homomorphism $f_{x_0}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ has remnant and it *has strong remnant* [8] if f_{x_0} has strong remnant.

Let $|Q|$ denote the length of the word Q in a free group, that is, the minimum number of generators and their inverses needed to write it.

Lemma 4.1. *If a homomorphism $h\colon G \to G$ has strong remnant, then $Fix(h) = 1$.*

Proof. Suppose $h(Q) = Q \neq 1$. Write $Q = \prod_{j=1}^{m} a_{\delta_j}^{u_j}$ in reduced form for some m, then $Q = h(Q) = \prod_{j=1}^{m} A_{\delta_j}^{u_j}$. The homomorphism h has remnant so Q contains $\sum_{k \in C_i} |u_k|$ appearances of the remnant $\overline{A}_i$, where $C_i = \{j | \delta_j = i\}$. Since $|Q| = |h(Q)|$, then $Q = \prod_{j=1}^{m} \overline{A}_{\delta_j}^{u_j}$ in reduced form. By the uniqueness of the representation of freely reduced words, $\overline{A}_i = a_i$ for some $1 \leq i \leq n$. However, this contradicts the hypothesis that h has strong remnant so $Fix(h) = 1$. ∎

Let $f\colon X \to X$ be a map that has remnant and let x_i be a fixed point of f corresponding to a_i. Then, x_i is called *front-special* if x_i corresponds to the first letter of the remnant $\overline{A}_i$ and the fixed point index $ind(f, x_i) = -1$. The fixed point x_i is called *back-special* if x_i corresponds to the last letter of $\overline{A}_i$ and $ind(f, x_i) = -1$.

Lemma 4.2. *Suppose $f\colon X \to X$ has strong remnant. Let $x_j \in a_i$ be a fixed point that corresponds to a_i or a_i^{-1} and is not a back-special fixed point of f. If $|W_j| < |P_i\overline{A}_i|$, then the fixed subgroup of θ_{x_j} is trivial and therefore the fixed point class of f containing x_j has trivial stabilizer.*

Proof. Without loss of generality, set $i = 1$. By Proposition 4.1 and Lemma 4.1, it is sufficient to prove that θ_{x_j} has strong remnant. For $1 \leq k \leq n$, let $\theta_{x_j}(a_k) = W_j^{-1}(f_{x_0}(a_k))W_j = Y_k$, then since f has remnant, we may write

$$Y_k = W_j^{-1} A_k W_j = W_j^{-1} P_k \overline{A}_k S_k W_j.$$

Since $|W_j| < |P_1\overline{A}_1|$, then $\overline{A}_1$ cannot be cancelled completely when we reduce $W_j^{-1}A_1$. Let Z be the subword of $\overline{A}_1$ that is not cancelled, then Z is a subword of Y_1 because $P_1\overline{A}_1$ is not cancelled in A_1W_j and $|W_j| < |P_1\overline{A}_1|$. Since $\overline{A}_1$ is the remnant for A_1, then Z is not cancelled when we reduce $Y_1Y_1 = W_j^{-1}A_1A_1W_j$. Similarly, Z is not cancelled when we reduce $Y_1Y_k, Y_kY_1, Y_1^{-1}Y_k$ and $Y_1Y_k^{-1}$ for $k = 2, \ldots, n$. Thus the remnant $\overline{Y}_1$ of Y_1 contains $Z \neq 1$. However, $\overline{Y}_1 = Z = a_1$ would imply that x_j is a back-special fixed point of f, contrary to the hypotheses. For $k = 2, \ldots, n$, since $Y_k = W_j^{-1}A_kW_j$ where $W_j^{-1}A_k = V_j^{-1}A_k$ or $a_1^{-1}V_j^{-1}A_k$ whereas $A_kW_j = A_kV_j$ or $A_kV_ja_1^{-1}$ and V_j is a subword of A_j, then the remnant $\overline{A}_k$ is a subword of $\overline{Y}_k$ and, since f has strong remnant, then $\overline{Y}_k \neq a_k$. Therefore, θ_{x_j} has strong remnant. ∎

The corresponding argument establishes

Lemma 4.3. *Suppose $f\colon X \to X$ has strong remnant, a fixed point x_j in the circle a_i for some $1 \leq i \leq n$, is not a front-special fixed point of f that corresponds to a_i or a_i^{-1} and $|\overline{W}_j| < |\overline{A}_iS_i|$, then the fixed subgroup of $\bar{\theta}_{x_j}$ is trivial, where $\bar{\theta}_{x_j}[\omega] = \overline{W}_j^{-1}[f(\omega)]\overline{W}_j$, and therefore the fixed point class $\mathcal{F}_{x_j}$ of f containing x_j has trivial stabilizer.*

Theorem 4.1. *If $f\colon X \to X$ has strong remnant, then $Stab(\mathcal{F}) = 1$ for any fixed point class $\mathcal{F}$.*

Proof. Let x_j be a fixed point of index -1 which, without loss of generality, we may assume is in the circle a_1. Since f has strong remnant, a fixed point with index -1 cannot be both front-special and back-special because that would imply that $\overline{A}_1 = a_1$. Since $ind(f, x_j) = -1$, then x_j is represented by a_1 so $W_j = V_j$ and $\overline{W}_j = \overline{V}_j^{-1}$. Therefore at least one of $|\overline{W}_j^{-1}| < |\overline{A}_1S_1|$ or $|W_j| < |P_1\overline{A}_1|$ must be true. Thus, by Lemma 4.2 or 4.3, the stabilizer of the fixed point class containing any points of index -1 is trivial. Since $ind(\mathcal{F}) \leq 1$ for any fixed point class $\mathcal{F}$ by Theorem 1.2, a fixed point class containing only points of index $+1$ consists of a single point. The index of that class is $+1$ so its stabilizer is trivial by Theorem 1.2. ∎

The example of $f_{x_0}(a) = a$ on the circle demonstrates that the conclusion of Theorem 4.1 fails if f does not have strong remnant.

Let $f\colon X \to X$ be a map where X is a connected graph. For $p\colon \widetilde{X} \to X$ a covering space of degree n, let $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$ be the n-valued lift of f. As a consequence of Theorem 4.1, if f has strong remnant then

$$N(\phi_{p,f}) = n \cdot N(f).$$

In this case, it is easy to calculate $N(f)$ because, by part 2 of Theorem 3.3 of [7], if f has strong remnant then any two fixed points in the same fixed point class are directly related in the sense of [17] so the Nielsen number can be determined by comparing Wagner tails.

5 Generic Properties of Maps of Graphs

Let $f\colon Y \to Y$ be a map of a graph that is of the homotopy type of a wedge of n circles then, up to homotopy, f is characterized by an ordered n-tuple $\mathbb{X} = (X_1, \ldots, X_n)$ of words in the free group F_n on n generators. Let $B^n(M)$ denote the set of n-tuples of words in F_n all of length less than or equal to M and $B^n(m, M) \subseteq B^n(M)$ the n-tuples for which all the words are of length at least m. For a property possessed by a subset of n-tuples, we identify the property with the set itself. Specializing a concept due to Gromov [5], we define a property $\mathcal{S}$ of n-tuples of words in F_n to be *generic* if

$$\lim_{k\to\infty} \frac{\#(\mathcal{S} \cap B^n(k))}{\#(B^n(k))} = 1.$$

We denote by $\mathcal{S}(r)$, for $r \geq 1$, the set of n-tuples $\mathbb{X} = (X_1, \ldots, X_n)$ that have *minimum remnant length* r, that is, $|\overline{X}_i| \geq r$ for all i, where $X_i = P_i\overline{X}_iS_i$ for $\overline{X}_i$ the remnant of X_i in $\mathbb{X}$, compare [7]. Thus $\mathcal{S}(1)$ is the set of n-tuples with remnant in the sense of [17]. The following result is a consequence of [1]. However, we take this opportunity to present a self-contained, elementary proof, modelled on that of Theorem 3.7 of [17], in order to add some details and to correct some minor errors in the published proof of that theorem.

Theorem 5.1. *The minimum remnant length property $\mathcal{S}(r)$ is generic for all $r \geq 1$.*

Proof. Suppose given $\epsilon > 0$. We write

$$\frac{\#(\mathcal{S}(r) \cap B^n(k))}{\#(B^n(k))} = \frac{\#(\mathcal{S}(r) \cap B^n(k))}{\#(B^n(m_0, k))} \cdot \frac{\#(B^n(m_0, k))}{\#(B^n(k))}.$$

We will prove that there exists m_0 such that there is $M > m_0$ with the property that if $k \geq M$ then each factor of the product is greater than $\sqrt{1-\epsilon}$.

Without loss of generality, we assume that m has the same parity as r so that $(m-r)/2$ is an integer. If $\mathbb{X} \in B^n(m, M)$ does not have minimum remnant length r, then there is at least one X_i such that at least one of $|P_i| > (m-r)/2$ or $|S_i| > (m-r)/2$ is true. We observe that

$$\begin{aligned} \frac{\#(X = P_iY; X \in B^1(m, M))}{\#(B^1(m, M))} &\leq \frac{1}{(2n)(2n-1)^{(m-r)/2-1}} \\ &< \frac{1}{(2n-1)^{(m-r)/2}} \end{aligned}$$

and that this inequality does not depend on the length of the word X_i and so it holds for any value of $M > m$. Since X_i must be tested against all $X_j^{\pm 1} \in \mathbb{X}$ except

X_i^{-1} at both the start and the end of the word,

$$\frac{\#((X_i \notin \mathcal{S}(r)) \cap B^1(m,M))}{\#(B^1(m,M))} \le \frac{2(2n-1)}{(2n-1)^{(m-r)/2}}$$

and therefore

$$\frac{\#((\mathbb{X} \notin \mathcal{S}(r)) \cap B^n(m,M))}{\#(B^n(m,M))} \le \frac{2n(2n-1)}{(2n-1)^{(m-r)/2}} < \frac{4n^2}{(2n-1)^{(m-r)/2}}.$$

We choose $m = m_0$ so that

$$\frac{4n^2}{(2n-1)^{(m_0-r)/2}} < 1 - \sqrt{1-\epsilon}.$$

Denoting the negation of a property $\mathcal{S}$ by the symbol $\sim \mathcal{S}$, we have proved that

$$\frac{\#(\sim \mathcal{S}(r) \cap B^n(m_0,M))}{\#(B^n(m_0,M))} < \frac{4n^2}{(2n-1)^{(m_0-r)/2}}$$

and therefore that

$$\frac{\#(\mathcal{S}(r) \cap B^n(m_0,M))}{\#(B^n(m_0,M))} > \sqrt{1-\epsilon}.$$

Now, choose M so that

$$\frac{1-(2n-1)^{m_0}}{1-(2n-1)^M} < 1 - \sqrt{1-\epsilon}.$$

then for $k \ge M$ we have

$$\begin{aligned}
\frac{\#(B^n(m_0,k))}{\#(B^n(k))} &= \frac{\sum_{j=m_0}^{k} 2n(2n-1)^{j-1}}{1+\sum_{j=1}^{k} 2n(2n-1)^{j-1}} \\
&> \frac{\sum_{j=1}^{k} (2n-1)^{j-1} - \sum_{j=1}^{m_0-1} (2n-1)^{j-1}}{\sum_{j=1}^{k} (2n-1)^{j-1}} \\
&= 1 - \frac{\sum_{j=1}^{m_0-1} (2n-1)^{j-1}}{\sum_{j=1}^{k} (2n-1)^{j-1}} \\
&= 1 - \frac{1-(2n-1)^{m_0}}{1-(2n-1)^k} > \sqrt{1-\epsilon}.
\end{aligned}$$

Therefore

$$\frac{\#(\mathcal{S}(r) \cap B^n(k))}{\#(B^n(k))} > 1-\epsilon$$

and we have proved that

$$\lim_{k\to\infty} \frac{\#(\mathcal{S}(r) \cap B^n(k))}{\#(B^n(k))} = 1. \qquad \blacksquare$$

We denote the set of n-tuples that have strong remnant by $\mathcal{S}(s)$ then, since $\mathcal{S}(2) \subset \mathcal{S}(s)$, we have

Corollary 5.1. *The strong remnant property $\mathcal{S}(s)$ is generic.*

In [9], a map f is called *essentially fix trivial* if $Stab(\mathcal{F}) = 1$ for all essential fixed point classes $\mathcal{F}$ of f. We extend the definition by calling f *totally fix trivial* if $Stab(\mathcal{F}) = 1$ for *all* its fixed point classes, essential or not. Thus Theorem 6.1 implies that "most" maps of wedges of circles have only trivial stabilizers of their fixed point classes in the following sense:

Corollary 5.2. *For maps of graphs, the totally fix trivial property is generic.*

Therefore, for "most" n-valued lifts $\phi_{p,f}$ of maps f of graphs, the Nielsen number is $N(\phi_{p,f}) = n \cdot N(f)$.

6 Extensions of n-Valued Maps

Let X be a finite polyhedron, Y a subpolyhedron of X, $\phi\colon Y \multimap Y$ an n-valued map, and $r\colon X \to Y$ a retraction to the subpolyhedron. Then the n-valued map $\widehat{\phi} = \iota \circ \phi \circ r\colon X \multimap X$, where $\iota\colon Y \to X$ is inclusion, is well-defined. We call $\widehat{\phi}$ the *extension* of ϕ with respect to the retraction r.

Theorem 6.1. *The Nielsen number of an n-valued map is the same as that of any extension of it, that is, $N(\widehat{\phi}) = N(\phi)$.*

Proof. By Theorem 6 of [14], we can homotope ϕ so that the fixed point set $Fix(\phi)$ is finite. Note that $Fix(\widehat{\phi}) = Fix(\widehat{\phi}|Y) = Fix(\phi)$. We claim that the fixed point classes of ϕ and $\widehat{\phi}$ are identical. Suppose x and y are in the same fixed point class of ϕ in the sense of [15]. That means that there exists a path $c\colon [0,1] \to Y$ from x to y such that g_j, for some j, is homotopic to c relative to the endpoints, where $\{g_i\}_{1\le i\le n}$ is the splitting of $\phi c\colon I \multimap Y$. Since the path c is in Y, then $\widehat{\phi} \circ c = \phi \circ c$. Thus, they have the same splitting and g_i is a map in the splitting of $\widehat{\phi} \circ c$. Therefore, x and y are in the same fixed point class of $\widehat{\phi}$. Conversely, suppose x and y are in the same fixed point class of $\widehat{\phi}$. Then, there exists a path $c\colon [0,1] \to X$ from x to y such that $\widehat{g}_j$ is homotopic to c, for some integer j, where $\{\widehat{g}_i\}_{1\le i\le n}$ is the splitting of $\widehat{\phi}c$. Therefore, $r\widehat{g}_j$ and rc are homotopic relative to the endpoints, which are in Y. Since the image of $\widehat{\phi}$ is in Y, the map $r \circ \widehat{g}_j = \widehat{g}_j$ is a member of the splitting of $\phi \circ (r \circ c)$. Thus, $r \circ c$ is a path from x to y such that $r \circ c$ is homotopic to $\widehat{g}_j$ relative to the endpoints. This shows that x and y are in the same fixed point class of ϕ and this establishes the claim that the fixed point classes of ϕ and of $\widehat{\phi}$ are identical.

Since Y is locally contractible, there is a contractible (open) neighborhood U_0 of a fixed point y_0 of ϕ. Therefore, there exist maps $\{f_i\colon U_0 \to Y\}_{1\le i\le n}$ splitting ϕ such that $f_j(y_0) = y_0$ for some j. If $x \in r^{-1}(U_0) = U$, then

$$\widehat{\phi}(x) = \phi(r(x)) = \{f_i(r(x))\}_{1\le i\le n}$$

and $y_0 = \widehat{f_j}(y_0) = f_j(r(y_0))$. Let $V_0 \subset U_0$ be a neighborhood of y_0 such that $f_j(V_0) \subset U_0$ and let $V = r^{-1}(V_0)$ so $\widehat{f_j}(V) \subset U_0 \subset U$. Consider

$$\widehat{f_j}|V\colon \left(V \xrightarrow{r} V_0 \xrightarrow{f} U_0\right) \hookrightarrow U$$

and

$$f_j|V_0\colon V_0 \hookrightarrow \left(V \xrightarrow{r} V_0 \xrightarrow{f} U_0\right).$$

By the commutativity property of the fixed point index [10] , we have $ind(\widehat{f_j}|V, V) = ind(f_j|V_0, V_0)$. The excision property implies that $ind(\widehat{f_j}, U) = ind(f_j, U_0)$. Thus, according to the definition in [15], $ind(\widehat{\phi}, x_0) = ind(\phi, x_0)$ and therefore $\mathcal{F}$ is essential as a fixed point class of $\widehat{\phi}$ if and only if it is essential as a fixed point class of ϕ and we conclude that $N(\widehat{\phi}) = N(\phi)$. ∎

We may construct a class of multiply fixed n-valued maps of surfaces with boundary as follows. Suppose a graph $\widetilde{X}$ is a finite covering of degree n of a graph X by a covering map $p\colon \widetilde{X} \to X$. Suppose Y is a surface with boundary containing $\widetilde{X}$ and there is a retraction $r\colon Y \to \widetilde{X}$. Let $f\colon X \to X$ be a map and extend the lift $\phi_{p,f}\colon \widetilde{X} \multimap \widetilde{X}$ to $\widehat{\phi}_{p,f} = \iota \circ \phi_{p,f} \circ r\colon Y \multimap Y$. If f has strong remnant, then

$$N(\widehat{\phi}_{p,f}) = n \cdot N(f)$$

by Theorem 5.1

References

[1] Arzhantseva, G. and Ol'shanskii, A, *The class of groups all of whose subgroups with lesser number of generators are free is generic*, Math. Notes **59**, 350 - 355 (1996).

[2] Brown, R., *Construction of multiply fixed n-valued maps*, Topology Appl. **196**, 249 - 259 (2015).

[3] Brown, R. and Lin, J., *Coincidences of projections and linear n-valued maps of tori*, Topology Appl. **157**, 1990 - 1998 (2010).

[4] Crabb, M., *Lefschetz indices for n-valued maps*, J. Fixed Point Theory Appl. **17**, 153 - 186 (2015).

[5] Gromov, M., *Hyperbolic groups*, Math Sci. Res. Inst. Publ. **8**, Springer, 95 - 263 (1987).

[6] Goldstein, R. and Turner, E., *Fixed subgroups of homomorphisms of free groups*, Bull. London Math. Soc. **18:5**, 468 - 470 (1986).

[7] Hart, E., *Reidemeister conjugacy for finitely generated free fundamental groups*, Fund. Math. **199**, 93 - 118 (2008).

[8] Hart, E., *Notes on Wagner's algorithm*, unpublished.

[9] Heath, P., Keppelmann, E. and Wong, P., *Addition formulae for Nielsen numbers and Nielsen type numbers of fibre preserving maps*, Topology Appl. **67**, 133 - 157 (1995).

[10] Jiang, B., *Lectures on Nielsen Fixed Point Theory*, Contemp. Math. **14** (1983).

[11] Jiang, B., *Bounds for fixed points on surfaces*. Math. Ann. **311**, 467 - 479 (1998).

[12] Jiang, B., Wang, S. and Zhang, Q., *Bounds for fixed points and fixed subgroups on surfaces and graphs*, Alg. Geom. Top. **11**, 2297 - 2318 (2011).

[13] Kelly, M., *Bounds on the fixed point indices of self-maps of certain simplicial complexes*, Topology Appl. **108**, 179 - 196 (2000).

[14] Schirmer, H., *Fix-finite approximation of n-valued multifunctions*, Fund, Math. **121**, 73-80 (1984).

[15] Schirmer, H., *An index and Nielsen number for n-valued multifunctions*, Fund. Math. **124**, 207 - 219 (1984).

[16] Schirmer, H., *A minimum theorem for n-valued multifunctions*, Fund. Math. **126**, 83 - 92 (1985).

[17] Wagner, J., *An algorithm for calculating the Nielsen number of surfaces with boundary*, Trans. Amer. Math. Soc. **351**, 41 - 62 (1999).

Department of Mathematics
University of California
Los Angeles, CA 90095-1555
e-mail: rfb@math.ucla.edu

Department of Mathematics and Statistics
Boston University
Boston, MA 02215
e-mail: jnan@bu.edu