Sums of asymptotically midpoint uniformly convex spaces

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Abstract

We study the property of asymptotic midpoint uniform convexity for infinite direct sums of Banach spaces, where the norm of the sum is defined by a Banach space E with a 1-unconditional basis. We show that a sum $(\sum_{n=1}^{\infty} X_n)_E$ is asymptotically midpoint uniformly convex (AMUC) if and only if the spaces X_n are uniformly AMUC and E is uniformly monotone. We also show that $L_p(X)$ is AMUC if and only if X is uniformly convex.

1 Introduction

Convexity properties have long had a central place in Banach space theory. The nicest and simplest of these is uniform convexity. A Banach space $(X, \|\cdot\|)$ is *uniformly convex* (UC) if for all $t \in (0, 2]$, there exists $\delta = \delta(t) > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$ for all $x, y \in S_X$ satisfying $\|x - y\| \geq t$. It was shown early on that every uniformly convex Banach space is reflexive. More generally, a superreflexive space is one which admits an equivalent norm which is uniformly convex. A detailed overview of classical convexity properties can be found in [Meg98].

In 1987 S. Rolewicz [Rol86, Rol87] introduced what he termed property (β) in connection to well-posedness of optimization problems. We say that a Banach space $(X, \|\cdot\|)$ has *property* (β) if, for every $t > 0$, there exists $\delta = \delta(t) > 0$ such

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that for all $x \in X$ satisfying $1 < \|x\| < 1+\delta$, the set $\operatorname{conv}(x \cup B_X) \setminus B_X$ has Kuratowski measure of noncompactness less than t (here $\operatorname{conv}(A)$ is the convex hull of the set A). Recall that the *Kuratowski measure of noncompactness* of a set A is the infimum of all values $t > 0$ such that A can be covered by finitely many sets with diameter at most t.

A third convexity-type property is asymptotic uniform convexity. A Banach space $(X, \|\cdot\|)$ is *asymptotically uniformly convex* (AUC) if, for all $t > 0$, there exists $\delta = \delta(t)$ such that for all $x \in S_X$, there exists a subspace $Y \subset X$ of finite codimension such that

$$\inf_{y \in S_Y} \|x + ty\| > 1 + \delta(t).$$

This property is originally due to V. D. Milman [Mil71], though the current terminology was introduced in [JLPS02]. For a concrete norm, i.e. isometrically, uniform convexity implies property (β), which in turn implies AUC [Rol87]. Spaces which have an equivalent norm with property (β) constitute an isomorphic class that is strictly between superreflexive spaces and those with an equivalent AUC norm, see e.g. [Kut90], [DKLR]. In view of Bourgain's characterization of superreflexivity and [BKL10], it is appropriate to call the (β)-renormable spaces asymptotically superreflexive.

Several equivalent isometric definitions of uniform convexity can be expressed in terms of diameters of given sets tending uniformly to zero, see [Rol86, Laa02, TW05]. In most of these cases, if one replaces the diameter by the Kuratowski measure of noncompactness, one obtains property (β) [Rol86, DKR$^+$13]. However, an exception is a special type of lenses in [Laa02, TW05]. In this case one obtains a new property, namely asymptotic midpoint uniform convexity [DKR$^+$16]. A Banach space $(X, \|\cdot\|)$ is *asymptotically midpoint uniformly convex* (AMUC) if for all $t \in (0, 1)$, there exists $\delta = \delta(t)$ such that for all $x \in S_X$, there exists a subspace $Y \subset X$ of finite codimension such that

$$\inf_{y \in S_Y} \max\{\|x + ty\|, \|x - ty\|\} \geq 1 + \delta(t).$$

Observe that δ can be chosen to be increasing as a function of t. In each of these convexity properties, the function δ is called the *modulus* of the corresponding property.

Since AUC implies AMUC, the new property is isomorphically weaker than property (β). An example was given in [DKR$^+$16] of a norm which is AMUC but not AUC. However, it is not known if AMUC and AUC are isomorphically equivalent (that is, whether any AMUC space admits an equivalent AUC norm). It was shown in [DKR$^+$16] that every AMUC space with an unconditional basis can be renormed to be AUC.

The AMUC property plays a role in the metric geometry of the space. For example, one cannot embed infinitely branching diamond graphs in a space with an equivalent AMUC norm [BCD$^+$].

In the present paper we study the AMUC property in connection with infinite sums of Banach spaces, giving necessary and sufficient conditions for the norm to be AMUC. The norm of the sum is defined using an underlying Banach space admitting a 1-unconditional basis. We give natural conditions which are necessary and sufficient for this norm to be AMUC. For an overview of other results about when geometric properties of a Banach space can be lifted to a larger space, see [Lin04, Sec. 3.4]. Our main result is the following.

Theorem 1. Let $(E, \|\cdot\|_E)$ be a Banach space with a 1-unconditional basis (e_n) and let $(X_n, \|\cdot\|_n)_{n=1}^\infty$ be a sequence of infinite-dimensional Banach spaces. Let $X = (\sum_{n=1}^\infty X_n)_E$, with norm given by $\|x\| = \|(\|x_n\|_n)_{n=1}^\infty\|_E$ for $x = (x_n)_{n=1}^\infty$, $x_n \in X_n$. Then X is AMUC if and only if $(X_n)_{n=1}^\infty$ are uniformly AMUC and E is uniformly monotone.

Recall that a basis (e_j) for a Banach space is *1-unconditional* if it satisfies

$$\left\|\sum_{k=0}^{n} \epsilon_j \alpha_j e_j\right\| \leq \left\|\sum_{k=0}^{n} \alpha_j e_j\right\|$$

for all coefficients $\alpha_j \in \mathbb{R}$ and $\epsilon_j \in \{-1, 1\}$, $j \in \mathbb{N}$. Also, for a Banach space $(E, \|\cdot\|)$ with a 1-unconditional basis (e_n), we say that the norm $\|\cdot\|$ on E is *uniformly monotone* if for every $\epsilon > 0$ there exists $M(\epsilon)$ such that for every $a = \sum_n a_n e_n$ satisfying $a_n \geq 0$, $\|a\| = 1$, and every $b = \sum_n b_n e_n$ satisfying $b_n \geq 0$, $\|b\| \geq \epsilon$, we have $\|a + b\| \geq 1 + M(\epsilon)$.

Theorem 1 remains true with AMUC replaced by AUC. The same proof we provide for Theorem 1 goes through for the AUC case with minor modifications.

A similar characterization for nearly uniform convexity (that is, AUC plus reflexivity) was proved in [KL92]. On the other hand, for the continuous case, the AMUC property of $L_p(X)$ implies uniform convexity for X. This is similar to the result in [Par83] about the uniformly Kadec-Klee property. More precisely, we prove the following theorem.

Theorem 2. Let $(X, \|\cdot\|)$ be a Banach space and let $L_p(X)$ denote the set of p-integrable functions $f : [0, 1] \to X$, where $1 < p < \infty$, with norm $\|f\|_p = (\int_0^1 \|f(x)\|^p \, dx)^{1/p}$. Then $L_p(X)$ is AMUC if and only if X is UC.

2 Proofs of theorems

Proof of Theorem 1. ($\Longrightarrow$) Assume that X is AMUC, and let $\overline{\delta}(\epsilon)$ be its AMUC modulus. Since each X_n can be embedded with the same norm as a subspace of X, it follows immediately that the spaces $(X_n)_{n=1}^\infty$ are uniformly AMUC.

Let $0 < \epsilon < 1$. Consider elements $a = \sum_{n=1}^\infty a_n e_n$ with $a_n \geq 0$ and $\|a\|_E = 1$, and $b = \sum_{n=1}^\infty b_n e_n$ with $b_n \geq 0$, $\|b\|_E \geq \epsilon$. By approximating, we may assume that a and b are supported on $[(e_i)_{i=1}^N]$ for some N.

For every $1 \leq n \leq N$, choose $x_n \in X_n$ so that $\|x_n\|_n = a_n$. By the AMUC property, there exists a subspace $U \subset X$ of finite codimension corresponding to the element in X of finite support $x = (x_n)_{n=1}^N$ and the given value of $\epsilon > 0$. Since the spaces X_n are infinite dimensional, we can choose for each $1 \leq n \leq N$ a vector $y_n \in U$, supported on X_n, so that $\max_\pm \|x_n \pm y_n\|_n = a_n + b_n$. Indeed, take a $v_n \in U$, supported on X_n, so that $\|x_n + v_n\|_n > \|x_n\|_n$ and consider the convex function $h_n(t) = \max_\pm \|x_n \pm t v_n\|_n$. Take $y_n = t v_n$ so that $h_n(t) = a_n + b_n$.

Then we also have $\|y_n\|_n \geq b_n$. Let $\theta = \pm 1$ be such that $\max\limits_{\theta=\pm} \left\| \sum_{n=1}^{N}(x_n + \theta y_n) \right\|$ is achieved. The finitely supported element $y = \sum_{i=1}^{N} y_n$ clearly belongs to U and $\|y\| \geq \|b\|_E \geq \epsilon$. Then by AMUC, for either $\theta = 1$ or $\theta = -1$ we have $\left\| \sum_{n=1}^{N}(x_n + \theta y_n) \right\| \geq 1 + \overline{\delta}(\epsilon)$. Let θ_n be such that $\|x_n + \theta_n y_n\|_n = \max\limits_{\pm} \|x_n \pm y_n\|_n$. By 1-unconditionality,

$$\begin{aligned}\|a + b\|_E &= \left\| \sum_{n=1}^{N} \|x_n + \theta_n y_n\|_n e_n \right\|_E \\ &\geq \left\| \sum_{n=1}^{N} \|x_n + \theta y_n\|_n e_n \right\|_E \\ &= \left\| \sum_{n=1}^{N} (x_n + \theta y_n) \right\| \geq 1 + \overline{\delta}(\epsilon).\end{aligned}$$

Hence E is uniformly monotone.

($\Longleftarrow$) Assume that $(X_n, \|\cdot\|_n)_{n=1}^{\infty}$ are uniformly AMUC and E is uniformly monotone.

Consider $x = (x_n)_{n=1}^{N}$ of finite support in X, where $N \geq 1$, with $\|x\| = 1$. Let $0 < \epsilon < 1$. Choose subspaces $U_n \subset X_n$, $1 \leq n \leq N$, of finite codimension satisfying the following two conditions:

(1) If $y_n \in U_n$, then $\|x_n + y_n\|_n \geq \|x_n\|_n$.

(2) If $y_n \in U_n$ with $\|y_n\|_n \geq \frac{\epsilon}{3}\|x_n\|_n$, then $\max\limits_{\pm}\|x_n \pm y_n\|_n \geq \|x_n\|_n(1 + \overline{\delta}(\frac{\epsilon}{3}))$.

The first condition can be achieved by assuming that U_n is intersected with $V_n = \{y_n \in X_n : f_n(y_n) = 0\}$, where f_n is chosen as $\|f_n\|_{X_n^*} = 1$ and $f_n(x_n) = \|x_n\|_n$. Then

$$\|x_n + y_n\|_n \geq f_n(x_n + y_n) = f_n(x_n) = \|x_n\|_n.$$

Let $U = \{y = (y_n)_{n=1}^{\infty} : y_n \in U_n, 1 \leq n \leq N\}$. Then U is of finite codimension in X. Suppose $y = (y_n)_{n=1}^{\infty} \in U$ with $\|y\| \geq \epsilon$. Let $A = \{1 \leq n \leq N : \|y_n\|_n < \frac{\epsilon}{3}\|x_n\|_n\}$ and $B = \{1 \leq n \leq N : \|y_n\|_n \geq \frac{\epsilon}{3}\|x_n\|_n\}$. Then $\|\sum_{n\in A} y_n\| \leq \frac{\epsilon}{3} \|\sum_{n\in A} x_n\| \leq \frac{\epsilon}{3}$.

We have $\|\sum_{n\in A} y_n\| + \|\sum_{n\in B} y_n\| + \left\|\sum_{n=N+1}^{\infty} y_n\right\| \geq \|y\| \geq \epsilon$. So either (i) $\left\|\sum_{n=N+1}^{\infty} y_n\right\| \geq \frac{\epsilon}{3}$ or (ii) $\|\sum_{n\in B} y_n\| \geq \frac{\epsilon}{3}$.

Suppose that (i) occurs. It follows from condition (1) that for both choices of signs, $\|x_n \pm y_n\|_n \geq \|x_n\|_n$, $1 \leq n \leq N$. Then by 1-unconditionality of (e_n) and uniform monotonicity, $\|x \pm y\| \geq \left\|x \pm \sum_{n=N+1}^{\infty} y_n\right\| \geq 1 + M(\epsilon/3)$.

Suppose next that (ii) occurs.

Case 1. In the first case, assume that $\|\sum_{n\in B} x_n\| < \min\left(\frac{1}{2}, \frac{M(\epsilon/6)}{4}\right)$. Note that $\max\limits_{\pm}\|x_n \pm y_n\|_n \geq \|y_n\|_n$ for all n. Indeed, $g_n(t) = \|tx_n + y_n\|_n$, $t \in \mathbb{R}$, is a convex function, thus $g_n(0) \leq \max(g_n(-1), g_n(1))$. Hence, if we choose $\theta_n = \pm 1$ so that $\|x_n + \theta_n y_n\|_n = \max \|x_n \pm y_n\|_n$, we obtain by 1-unconditionality $\|\sum_{n\in B} x_n + \theta_n y_n\| \geq \|\sum_{n\in B} y_n\|$.

Set $B_+ = \{n \in B : \theta_n = 1\}$ and $B_- = \{n \in B : \theta_n = -1\}$. Then $\|\sum_{n\in B}(x_n + \theta_n y_n)\| \leq \left\|\sum_{n\in B_+}(x_n + y_n)\right\| + \left\|\sum_{n\in B_-}(x_n - y_n)\right\|$. So either for $\theta = 1$ or $\theta = -1$ we have by 1-unconditionality $\|\sum_{n\in B} x_n + \theta y_n\| \geq \frac{1}{2}\|\sum_{n\in B} y_n\| \geq \frac{\epsilon}{6}$.

In case 1, $\|\sum_{n\in A} x_n\| \geq \frac{1}{2}$, so by 1-unconditionality and scaling in uniform monotonicity,

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty}(x_n + \theta y_n)\right\| &\geq \left\|\sum_{n\in A}(x_n + \theta y_n) + \sum_{n\in B}(x_n + \theta y_n)\right\| \\
&\geq \left\|\sum_{n\in A} x_n + \sum_{n\in B}(x_n + \theta y_n)\right\| \\
&= \left\|\sum_{n\in A} x_n\right\| \left\|\left(\frac{\sum_{n\in A} x_n}{\|\sum_{n\in A} x_n\|} + \frac{\sum_{n\in B}(x_n + \theta y_n)}{\|\sum_{n\in A} x_n\|}\right)\right\| \\
&\geq \left\|\sum_{n\in A} x_n\right\| + \frac{1}{2}M\left(\frac{\epsilon}{6}\right) \geq \left\|\sum_{n=1}^{N} x_n\right\| + \frac{1}{2}M\left(\frac{\epsilon}{6}\right) - \frac{1}{4}M\left(\frac{\epsilon}{6}\right) \\
&= 1 + \frac{1}{4}M\left(\frac{\epsilon}{6}\right).
\end{aligned}$$

Case 2. $\|\sum_{n\in B} x_n\| \geq \min\left(\frac{1}{2}, \frac{1}{4}M\left(\frac{\epsilon}{6}\right)\right)$.

For $n \in B$, $\|y_n\|_n \geq \frac{\epsilon}{3}\|x_n\|_n$, so by uniform AMUC, $\|x_n + \theta_n y_n\|_n = \max_{\pm}\|x_n \pm y_n\|_n \geq \|x_n\|_n\left(1 + \overline{\delta}\left(\frac{\epsilon}{3}\right)\right)$.

So for $n \in B_+$, $\|x_n + y_n\|_n - \|x_n\|_n \geq \|x_n\|_n\overline{\delta}\left(\frac{\epsilon}{3}\right)$, while for $1 \leq n \leq N$, $n \notin B_+$, $\|x_n + \theta_n y_n\|_n - \|x_n\|_n \geq 0$. Then by 1-unconditionality and uniform monotonicity

$$\begin{aligned}
\left\|\sum_{n=1}^{N}(x_n + y_n)\right\| &= \left\|\sum_{n=1}^{N}\|x_n\|e_n + \sum_{n=1}^{N}\left(\|x_n + y_n\|_n - \|x_n\|_n\right)e_n\right\|_E \\
&\geq \left\|\sum_{n=1}^{N}\|x_n\|e_n + \sum_{n\in B_+}\|x_n\|_n\overline{\delta}\left(\frac{\epsilon}{3}\right)e_n\right\|_E \\
&\geq 1 + M\left(\overline{\delta}\left(\frac{\epsilon}{3}\right)\left\|\sum_{n\in B_+}\|x_n\|_n e_n\right\|_E\right) \\
&= 1 + M\left(\overline{\delta}\left(\frac{\epsilon}{3}\right)\left\|\sum_{n\in B_+} x_n\right\|\right).
\end{aligned}$$

Similarly, $\left\|\sum_{n=1}^{N}(x_n - y_n)\right\| \geq 1 + M\left(\overline{\delta}\left(\frac{\epsilon}{3}\right)\left\|\sum_{n\in B_-} x_n\right\|\right)$.

Now $\left\|\sum_{n\in B_+} x_n\right\| + \left\|\sum_{n\in B_-} x_n\right\| \geq \|\sum_{n\in B} x_n\| \geq \min\left(\frac{1}{2}, \frac{1}{4}M\left(\frac{\epsilon}{6}\right)\right)$. This gives

$$\max_{\pm}\left\|\sum_{n\in B_{\pm}} x_n\right\| \geq \min\left(\frac{1}{4}, \frac{1}{8}M\left(\frac{\epsilon}{6}\right)\right).$$

Thus, by 1-unconditionality,

$$\begin{aligned}\max_{\theta=\pm 1}\left\|\sum_{n=1}^{\infty}(x_n+\theta y_n)\right\| &\geq \max_{\theta=\pm 1}\left\|\sum_{n=1}^{N}(x_n+\theta y_n)\right\| \\ &\geq 1+M\left(\overline{\delta}\left(\frac{\epsilon}{3}\right)\max_{\pm}\left\|\sum_{n\in B_{\pm}}x_n\right\|\right) \\ &\geq 1+M\left(\overline{\delta}\left(\frac{\epsilon}{3}\right)\min\left(\frac{1}{4},\frac{1}{8}M\left(\frac{\epsilon}{6}\right)\right)\right).\end{aligned}$$

For a general $x \in S_X$, we can approximate x by $(x_n)_{n=1}^{\infty}$ for some N. Since our estimates for $\|x+\theta y\|$ do not depend on N, we get the result. ∎

Proof of Theorem 2. This proof is based on arguments of Partington [Par83, Thm. 2] and Smith and Turett [ST80, Thm. 8]. First suppose that $L_p(X)$ is AMUC. To show X is UC, fix $\epsilon \in (0,1)$ and let $x, y \in S_X$ such that $\|x-y\| = \epsilon$. If $\|x+y\| \leq 1$, there is nothing to prove, so assume without loss of generality that $\|x+y\| > 1$. Let $r_n : [0,1] \to \{-1,1\}$ be the n-th Rademacher function, and let $f_n = (x+y)\chi_{[0,1]}/2 + (x-y)r_n/2 \in L_p(X)$. Observe that $1 = \|f_n\|_p$, and that $f_n = \|(x+y)/2\|(\widetilde{x}+t\widetilde{y}_n)$ for $\widetilde{x} = (x+y)\chi_{[0,1]}/\|x+y\|$, $t = \|x-y\|/\|x+y\|$, and $\widetilde{y}_n = (x-y)r_n/\|x-y\|$. Observe that $\frac{\epsilon}{2} \leq t < \epsilon$ and that $\widetilde{y}_n$ converges weakly to zero. Let $Y \subset L_p(X)$ be a subspace of finite codimension satisfying the AMUC property for $\widetilde{x}$. It is well-known that we can find a sequence $(w_n) \subset Y$ approximating $(\widetilde{y}_n)$. That is, given $\epsilon' > 0$, for all sufficiently large n it holds that $\|\widetilde{y}_n - w_n\|_p < \epsilon'$. Hence for sufficiently large n we have

$$1 = \left\|\frac{x+y}{2}\right\| \|\widetilde{x}+t\widetilde{y}_n\|_p \geq \left\|\frac{x+y}{2}\right\| \left(\|\widetilde{x}+tw_n\|_p - t\|w_n-\widetilde{y}_n\|_p\right). \tag{1}$$

Observe that $\|\widetilde{x}+t\widetilde{y}_n\|_p = \|\widetilde{x}-t\widetilde{y}_n\|_p$, which implies in particular that

$$\left|\|\widetilde{x}+tw_n\|_p - \|\widetilde{x}-tw_n\|_p\right| \leq 2\epsilon'\epsilon.$$

Moreover $\|w_n\| \geq \|\widetilde{y}_n\| - \epsilon' \geq \frac{1}{2}$. Thus $\max_{\pm}\|\widetilde{x} \pm tw_n\|_p \geq 1+\delta(\frac{t}{2})$ by the AMUC property for $L_p(X)$. From (1) we obtain

$$\begin{aligned}1 &\geq \left\|\frac{x+y}{2}\right\| \left(\max_{\pm}\|\widetilde{x}\pm tw_n\|_p - 2\epsilon'\epsilon - t\|\widetilde{y}_n - w_n\|_p\right) \\ &\geq \left\|\frac{x+y}{2}\right\| \left(1+\delta\left(\frac{t}{2}\right) - 3\epsilon'\epsilon\right) \geq \left\|\frac{x+y}{2}\right\| \left(1+\delta\left(\frac{\epsilon}{4}\right) - 3\epsilon'\epsilon\right).\end{aligned}$$

Letting $\epsilon' \to 0$ we obtain

$$\left\|\frac{x+y}{2}\right\| \leq \frac{1}{1+\delta(\frac{\epsilon}{4})} \leq \min\left\{\frac{1}{2}, 1-\frac{1}{2}\delta\left(\frac{\epsilon}{4}\right)\right\}.$$

For the converse, a result of Day [Day41] states that $L_p(X)$ is UC (and hence AMUC) whenever X is UC. ∎

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