

# EXPOSITORY PAPER

## Hölder's inequality: some recent and unexpected applications

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### Abstract

Hölder's inequality, since its appearance in 1888, has played a fundamental role in Mathematical Analysis and may be considered a milestone in Mathematics. It may seem strange that, nowadays, it keeps resurfacing and bringing new insights to the mathematical community. In this survey we show how a variant of Hölder's inequality (although well-known in PDEs) was essentially overlooked in Functional/Complex Analysis and has had a crucial (and in some sense unexpected) influence in very recent advances in different fields of Mathematics. Some of these recent advances have been appearing since 2012 and include the theory of Dirichlet series, the famous Bohr radius problem, certain classical inequalities (such as Bohnenblust–Hille or Hardy–Littlewood), and Mathematical Physics.

## 1 Introduction

When Leonard James Rogers (1862-1933) and Otto Hölder (1859-1937) discovered, independently, the famous inequality that (nowadays) holds Hölder's name

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(1889, [44]), they could have never imagined that, at that precise moment, they had just started a “revolution” in Functional Analysis (we refer to [47] for a detailed and historical exposition). This tool is a fundamental inequality between integrals and an indispensable tool for the study of, among others,  $L_p$  spaces. Let us recall the classical  $L_p$  version of this inequality.

**Theorem 1.1** (Hölder’s inequality, 1889). *Let  $(\Omega, \Sigma, \mu)$  be a measure space and let  $p, q \in [1, \infty]$  with  $1/p + 1/q = 1$  (Hölder’s conjugates). Then, for all measurable real or complex valued functions  $f$  and  $g$  on  $\Omega$ ,*

$$\int |fg| d\mu \leq \left( \int |f|^p d\mu \right)^{1/p} \left( \int |g|^q d\mu \right)^{1/q}.$$

If one has  $p, q \in (1, \infty)$ ,  $f \in L_p(\mu)$ , and  $g \in L_q(\mu)$ , then this inequality becomes an identity if and only if  $|f|^p$  and  $|g|^q$  are linearly dependent in  $L_1(\mu)$ . When one has  $p = q = 2$  we recover a form of the Cauchy–Schwarz inequality (or Cauchy–Bunyakovsky–Schwarz inequality). Also, Hölder’s inequality is used to prove Minkowski’s inequality (the triangle inequality for  $L_p$  spaces) and to establish that  $L_q(\mu)$  is the dual space of  $L_p(\mu)$  for  $p \in [1, \infty)$ . Of course, we are all familiar with these classical applications of Hölder’s inequality.

As it happens to almost every important result in mathematics, several extensions and generalizations of it have appeared along the time. In the case of Hölder’s inequality, this is not different. One of the extensions is the variant of Hölder’s inequality for mixed  $L_p$  spaces, which appeared in 1961, in the seminal work of A. Benedek and R. Panzone [11]. Later in the 1980’s, R. Blei and J. Fournier re-introduced the inequality for several applications on Lorentz spaces and also on PDEs (see [2, 13, 39]). Mixed  $L_p$  spaces may be seen as a pure exercise of abstraction of the original notion of  $L_p$  spaces, but as a matter of fact we shall show that the theory developed in [11] plays a crucial role in applications to quite different frameworks; it is intriguing that, although widely known (the paper [11] has more than 100 citations, mainly related to PDEs; we refer, for instance to [2, 24, 39]) it was overlooked in important fields of mathematics. This “gap” began to be filled in 2012–2013, when Hölder’s inequality for mixed  $L_p$  spaces was used as an interpolation-type result and we shall show that different fields of Mathematics and even of Physics were positively influenced.

This expository paper is arranged as follows. Section 2 presents some motivation to illustrate the subject of this article. Section 3 is devoted to the aforementioned variant of Hölder’s inequality (Hölder’s inequality for mixed sums), providing a short proof. This result was only written in a proper and organized fashion in 1961 ([11]) but, as it will be left clear along this paper, at least in the topics gathered here (Functional Analysis, Complex Analysis and Quantum Information Theory) it was surely not been taken advantage of before 2012. Our approach is quite different from the one employed in [11] and we shall follow the lines of [10]. Section 4 will recall some useful inequalities that we shall need and Section 5 focuses on recent applications of Hölder’s inequality for mixed sums in Functional Analysis and Quantum Information Theory, culminating with the solution of a classical problem from Complex Analysis: the Bohr radius problem. Applications to the improvement of the constants of the Hardy–Littlewood inequality and separately summing operators are also given.

## 2 Motivation: some interpolative puzzles

As a motivation to the subject treated here, let us suppose that we have the following two inequalities at hand, for certain complex scalar matrix  $(a_{ij})_{i,j=1}^N$ :

$$\sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq C \text{ and } \sum_{j=1}^N \left( \sum_{i=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq C \quad (2.1)$$

for some constant  $C > 0$  and all positive integers  $N$ .

How can one find an optimal exponent  $r$  and a constant  $C_1 > 0$  such that

$$\left( \sum_{i,j=1}^N |a_{ij}|^r \right)^{\frac{1}{r}} \leq C_1$$

for all positive integers  $N$ ? Moreover, how can one obtain a *good* (small) constant  $C_1$ ?

This question (at least concerning the exponent  $r$  can be solved in no less than two ways: interpolation and Hölder's inequality).

First note that, by using a consequence of Minkowski's inequality (see [40]), we know that

$$\left( \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}| \right)^2 \right)^{\frac{1}{2}} \leq \sum_{j=1}^N \left( \sum_{i=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq C. \quad (2.2)$$

If we use Hölder's inequality twice, one can proceed as follows:

$$\begin{aligned} \sum_{i,j=1}^N |a_{ij}|^{\frac{4}{3}} &= \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}|^{\frac{2}{3}} |a_{ij}|^{\frac{2}{3}} \right) \\ &\leq \sum_{i=1}^N \left( \left( \sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{3}} \left( \sum_{j=1}^N |a_{ij}| \right)^{\frac{2}{3}} \right) \\ &\leq \left( \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \left( \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}| \right)^2 \right)^{\frac{1}{3}} \\ &= \left[ \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[ \left( \sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}| \right)^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \leq C^{\frac{4}{3}}. \end{aligned}$$

On the other hand, by means of complex interpolation (see [12]) the solution is shorter; essentially we have two mixed inequalities with exponents  $(1, 2)$  in equation (2.1) and  $(2, 1)$  in equation (2.2). By interpolating these exponents with  $\theta_1 = \theta_2 = 1/2$  we obtain an exponent  $(4/3, 4/3)$  with constant  $C$ . The optimality of the exponent  $4/3$  can be proved using the Kahane–Salem–Zygmund inequality (Theorem 5.1).

The use of Hölder's inequality as above becomes a very arduous work as it increases the number of indexes in the sums. The reader can test the case of three sums using Hölder's inequality. More precisely, as a simple illustration suppose that

$$\sum_{\sigma(i)=1}^N \left( \sum_{\sigma(j)=1}^N \sum_{\sigma(k)=1}^N |a_{ijk}|^2 \right)^{\frac{1}{2}} \leq C$$

for all bijections  $\sigma : \{i, j, k\} \rightarrow \{i, j, k\}$  and all  $N$ . How can we find an optimal exponent  $r$  and a constant  $C_1$  such that

$$\left( \sum_{i,j,k=1}^N |a_{ijk}|^r \right)^{\frac{1}{r}} \leq C_1$$

for every  $N$ ?

The search for good constants dominating the respective inequalities is important for applications (see Section 5) and has an extra ingredient when we are using the interpolative approach: the main point is that different interpolations may result in the same exponent, but the constants involved differ. Thus, we must investigate what exponents we shall use to interpolate. More precisely, as we will see in Section 5, the Bohnenblust–Hille inequality for 3-linear forms asserts that there is a constant  $C_3 \geq 1$  such that, for all  $N$  and all 3-linear forms  $T : \ell_\infty^N \times \ell_\infty^N \times \ell_\infty^N \rightarrow \mathbb{K}$ ,

$$\left( \sum_{i_1, i_2, i_3=1}^N |T(e_{i_1}, e_{i_2}, e_{i_3})|^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq C_3 \|T\|,$$

here, as usual,  $\mathbb{K}$  stands for the fields of real or complex numbers,  $e_j$  denotes the canonical vector which entries are 1 at  $j$ -th position and 0 otherwise, and  $\|T\|$  denotes the sup norm.

However, the exponent  $3/2$  can be obtained by a “multiple” interpolation of exponents of inequalities of the form

$$\left( \sum_{i_1=1}^N \left( \sum_{i_2=1}^N \left( \sum_{i_3=1}^N |T(e_{i_1}, e_{i_2}, e_{i_3})|^{q_3} \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|T\|,$$

with

$$(q_1, q_2, q_3) = (1, 2, 2), (2, 1, 2) \text{ and } (2, 2, 1)$$

or

$$(q_1, q_2, q_3) = \left( \frac{4}{3}, \frac{4}{3}, 2 \right), \left( \frac{4}{3}, 2, \frac{4}{3} \right) \text{ and } \left( 2, \frac{4}{3}, \frac{4}{3} \right)$$

and the last procedure provides quite better constants. This is a simple illustration of the core of the new advances that lead to the results presented in this survey paper.

### 3 Hölder's inequality revisited

Essentially, the simplest version of Hölder's inequality asserts that if  $1/p + 1/q = 1$  and  $(a_j) \in \ell_p, (b_j) \in \ell_q$  then  $(a_j b_j) \in \ell_1$ . In this section we present a variation of this result, which may have been seen as a variant of the following general Hölder's inequality presented in the classical work [11] on mixed normed  $L_p$  spaces. We shall now work with  $L_p(\mathbb{N}) = \ell_p$ , since it is the case we are interested in. Let us recall some useful multi-index notation: for a positive integer  $m$  and  $\emptyset \neq D \subset \mathbb{N}$ , we denote the set of multi-indices  $\mathbf{i} = (i_1, \dots, i_m)$ , with each  $i_k \in D$ , by

$$\mathcal{M}(m, D) := \{\mathbf{i} = (i_1, \dots, i_m) \in \mathbb{N}^m; i_k \in D, k = 1, \dots, m\} = D^m.$$

We also denote  $\mathcal{M}(m, n) := \mathcal{M}(m, \{1, 2, \dots, n\})$ . For  $\mathbf{p} = (p_1, \dots, p_m) \in [1, \infty]^m$ , and a Banach space  $X$ , let us consider the space

$$\ell_{\mathbf{p}}(X) := \ell_{p_1}(\ell_{p_2}(\dots(\ell_{p_m}(X))\dots)).$$

Namely, if  $p_1, \dots, p_m < \infty$ , a vector matrix  $(x_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, \mathbb{N})} \in \ell_{\mathbf{p}}(X)$  if, and only if,

$$\left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2=1}^{\infty} \left( \dots \left( \sum_{i_{m-1}=1}^{\infty} \left( \sum_{i_m=1}^{\infty} \|x_{\mathbf{i}}\|_X^{p_m} \right)^{\frac{p_{m-1}}{p_m}} \right)^{\frac{p_{m-2}}{p_{m-1}}} \dots \right)^{\frac{p_2}{p_3}} \right)^{\frac{p_1}{p_2}} \right)^{\frac{1}{p_1}} < \infty.$$

When  $X = \mathbb{K}$ , we just write  $\ell_{\mathbf{p}}$  instead of  $\ell_{\mathbf{p}}(\mathbb{K})$ . Also, we deal with the coordinate product of two scalar matrices  $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}$  and  $\mathbf{b} = (b_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}$ , i.e.,

$$\mathbf{ab} := (a_{\mathbf{i}} b_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}.$$

The following result seems to be first observed by A. Benedek and R. Panzone (see [2, 3, 11]):

**Theorem 3.1** (Hölder's inequality for mixed  $\ell_{\mathbf{p}}$  spaces). *Let  $m, n, N$  be positive integers, and  $\mathbf{r}, \mathbf{q}(1), \dots, \mathbf{q}(N) \in (0, \infty]^m$  be such that*

$$\frac{1}{r_j} = \frac{1}{q_j(1)} + \dots + \frac{1}{q_j(N)}, \quad j \in \{1, 2, \dots, m\},$$

and let  $\mathbf{a}_k := (a_{\mathbf{i}}^k)_{\mathbf{i} \in \mathcal{M}(m, n)}$ ,  $k = 1, \dots, N$ , be scalar matrices. Then

$$\left\| \prod_{k=1}^N \mathbf{a}_k \right\|_{\mathbf{r}} \leq \prod_{k=1}^N \|\mathbf{a}_k\|_{\mathbf{q}(k)}.$$

Recall that, if  $\mathbf{r}, \mathbf{q}(1), \dots, \mathbf{q}(N) \in (0, \infty)^m$ , the previous inequality means the following:

$$\begin{aligned} & \left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_{\mathbf{i}}^1 \cdot a_{\mathbf{i}}^2 \cdot \dots \cdot a_{\mathbf{i}}^N|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \dots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}} \\ & \leq \prod_{k=1}^N \left[ \left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_{\mathbf{i}}^k|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \dots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right]. \end{aligned}$$

Using the above result we are able to recover the interpolative inequality from [4–6, 10] (Theorem 3.2 below), that we can also, in some sense, call Hölder’s inequality for multiple exponents. We shall illustrate along the paper several applications (in different fields) of this result. Just before that, for a positive real number  $\theta$ , let us define  $\mathbf{a}^\theta := (a_i^\theta)_{i \in \mathcal{M}(m,n)}$ . It is straightforward to see that

$$\|\mathbf{a}^\theta\|_{\mathbf{q}/\theta} = \|\mathbf{a}\|_{\mathbf{q}}^\theta,$$

where  $\mathbf{q}/\theta := (q_1/\theta, \dots, q_m/\theta)$ .

**Theorem 3.2** (Hölder’s inequality for multiple exponents -interpolative approach).

Let  $m, n, N$  be positive integers and  $\mathbf{q}, \mathbf{q}(1), \dots, \mathbf{q}(N) \in [1, \infty)^m$  be such that  $\left(\frac{1}{q_1}, \dots, \frac{1}{q_m}\right)$  belongs to the convex hull of  $\left(\frac{1}{q_1(k)}, \dots, \frac{1}{q_m(k)}\right)$ ,  $k = 1, \dots, N$ . Then for all scalar matrix  $\mathbf{a} = (a_i)_{i \in \mathcal{M}(m,n)}$ ,

$$\|\mathbf{a}\|_{\mathbf{q}} \leq \prod_{k=1}^N \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k},$$

i.e.,

$$\begin{aligned} & \left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_i|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \\ & \leq \prod_{k=1}^N \left[ \left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_i|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \dots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right]^{\theta_k}, \end{aligned}$$

where  $\theta_k$  are the coordinates of  $\left(\frac{1}{q_1(k)}, \dots, \frac{1}{q_m(k)}\right)$  on the convex hull.

*Proof.* For  $j = 1, \dots, m$  we have

$$\frac{1}{q_j} = \frac{\theta_1}{q_j(1)} + \dots + \frac{\theta_N}{q_j(N)} = \frac{1}{q_j(1)/\theta_1} + \dots + \frac{1}{q_j(N)/\theta_N}.$$

Since  $\|\mathbf{a}^{\theta_k}\|_{\mathbf{q}(k)/\theta_k} = \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k}$ , by the Hölder inequality for mixed  $\ell_p$  spaces we conclude that

$$\|\mathbf{a}\|_{\mathbf{q}} = \|\mathbf{a}^{\theta_1 + \dots + \theta_N}\|_{\mathbf{q}} = \left\| \prod_{k=1}^N \mathbf{a}^{\theta_k} \right\|_{\mathbf{q}} \leq \prod_{k=1}^N \|\mathbf{a}^{\theta_k}\|_{\mathbf{q}(k)/\theta_k} = \prod_{k=1}^N \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k}. \quad \blacksquare$$

For the sake of completeness of this article, we would also like to present the following proof, which is based on interpolation.

*Proof.* (Interpolative Approach) We shall follow the lines of [4, Proposition 2.1]. We shall proceed by induction on  $N$  and we also employ the fact that, for any Banach space  $X$  and  $\theta \in [0, 1]$ ,

$$\ell_r(X) = [\ell_p(X), \ell_q(X)]_\theta,$$

with  $\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$ , for  $i = 1, \dots, m$  (see [12]). If

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \dots + \frac{\theta_N}{q_i(N)},$$

with  $\sum_{k=1}^N \theta_k = 1$  and each  $\theta_k \in [0, 1]$ , then we also have

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \frac{1-\theta_1}{p_i},$$

setting

$$\frac{1}{p_i} = \frac{\alpha_2}{q_i(2)} + \dots + \frac{\alpha_N}{q_i(N)}, \quad \text{and } \alpha_j = \frac{\theta_j}{1-\theta_1},$$

for  $i = 1, \dots, m$  and  $j = 2, \dots, N$ . So  $\alpha_j \in [0, 1]$  and  $\sum_{j=2}^N \alpha_j = 1$ . Therefore, by the induction hypothesis, we conclude that

$$\|\mathbf{a}\|_{\mathbf{q}} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \|\mathbf{a}\|_{\mathbf{p}}^{1-\theta_1} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \left[ \prod_{j=2}^N \|\mathbf{a}\|_{\mathbf{q}(j)}^{\alpha_j} \right]^{1-\theta_1} = \prod_{k=1}^N \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k}. \quad \blacksquare$$

Combining the previous result with Minkowski's inequality we have a very useful inequality (see [10, Remark 2.2]):

**Corollary 3.3.** *Let  $m, n$  be positive integers,  $1 \leq k \leq m$  and  $1 \leq s \leq q$ . Then for all scalar matrix  $(a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ ,*

$$\left( \sum_{\mathbf{i} \in \mathcal{M}(m,n)} |a_{\mathbf{i}}|^{\rho} \right)^{\frac{1}{\rho}} \leq \prod_{S \in \mathcal{P}_k(m)} \left( \sum_{\mathbf{i}_S} \left( \sum_{\mathbf{i}_{\bar{S}}} |a_{\mathbf{i}}|^q \right)^{\frac{s}{q}} \right)^{\frac{1}{s} \cdot \frac{1}{\binom{m}{k}}},$$

where

$$\rho := \frac{msq}{kq + (m-k)s}$$

and  $\mathcal{P}_k(m)$  stands for the set of subsets  $S \subseteq \{1, \dots, m\}$  with  $\text{card}(S) = k$ .

The above corollary shows that Blei's inequality (see Corollary 3.4 below) is just a very particular case of a huge family of similar inequalities. For our purposes, the crucial point is that the use of Blei's inequality is far from being a good option to obtain good estimates for the constants of the Bohnenblust–Hille and related inequalities. Just to illustrate the strength of Theorem 3.2 and Corollary 3.3, we present here quite a simple proof (see [10]) of Blei's inequality.

**Corollary 3.4** (Blei's inequality - approach by Defant, Popa, and Schwarting, [30]). *Let  $A$  and  $B$  be two finite non-void index sets. Let  $(a_{ij})_{(i,j) \in A \times B}$  be a scalar matrix with positive entries, and denote its columns by  $\alpha_j = (a_{ij})_{i \in A}$  and its rows by  $\beta_i = (a_{ij})_{j \in B}$ . Then, for  $q, s_1, s_2 \geq 1$  with  $q > \max(s_1, s_2)$  we have*

$$\left( \sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left( \sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left( \sum_{j \in B} \|\alpha_j\|_q^{s_2} \right)^{\frac{f(s_2, s_1)}{s_2}},$$

with

$$w : [1, q]^2 \rightarrow [0, \infty), \quad w(x, y) := \frac{q^2(x + y) - 2qxy}{q^2 - xy},$$

$$f : [1, q]^2 \rightarrow [0, \infty), \quad f(x, y) := \frac{q^2x - qxy}{q^2(x + y) - 2qxy}.$$

*Proof.* Let us consider the exponents

$$(q, s_2), (s_1, q)$$

and

$$(\theta_1, \theta_2) = (f(s_2, s_1), f(s_1, s_2)).$$

Note that  $(w(s_1, s_2), w(s_1, s_2))$  is obtained by interpolating  $(q, s_2)$  and  $(s_1, q)$  with  $\theta_1, \theta_2$ , respectively. Then, from Theorem 3.2, we have

$$\left( \sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left( \sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left( \sum_{i \in A} \|\beta_i\|_{s_2}^q \right)^{\frac{f(s_2, s_1)}{q}}.$$

Now, since  $q > s_2$  we just need to use Proposition 4.6 to change the order of the last sum.  $\blacksquare$

We invite the interested reader to compare the above proof with the proof presented in [30, pages 226-227], in which the classical Hölder's inequality is needed several times.

## 4 Some useful inequalities

The main recent advances presented here are direct or indirect consequence of the improvements obtained in the polynomial and multilinear Bohnenblust–Hille inequalities, which were obtained by using the theory of mixed  $L_p$  spaces, more specifically the variant of Hölder's inequality (Theorem 3.2). Three other important ingredients are also needed: the Khinchine inequality (and its version for multiple sums), Kahane–Salem–Zygmund's inequality in its polynomial and multilinear versions and a variant of Minkowski's inequality. Before that, let us provide a brief account on polynomials and multilinear operators, that shall be needed in the remaining sections of this survey.

Polynomials in Banach spaces (at least for complex scalars) are of fundamental importance in the theory of Infinite Dimensional Holomorphy (see [35, 50]). In general the theory of polynomials and multilinear operators between normed spaces has its importance in different areas of Mathematics, from Number Theory, or Dirichlet series, to Functional Analysis.

In this section we recall the concepts of polynomials and multilinear operators between Banach spaces and some results that many authors would call "folklore", and that will be needed here. Let  $E, E_1, \dots, E_m$ , and  $F$  be Banach spaces. A  $m$ -linear operator  $T : E_1 \times \dots \times E_m \rightarrow F$  is a map that is linear in each coordinate separately. When  $E_1 = \dots = E_m = E$  we say that  $T$  is symmetric if



$T(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = T(x_1, \dots, x_m)$  for all bijections  $\sigma : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ . A  $m$ -homogeneous polynomial is a map  $P : E \rightarrow F$  such that

$$P(x) = T(x, \dots, x)$$

for some  $m$ -linear operator  $T : E \times \dots \times E \rightarrow F$ . Continuity is defined in the obvious fashion. The spaces of continuous  $m$ -homogeneous polynomials from  $E$  to  $F$  are represented by  $\mathcal{P}(^m E; F)$  and the space of continuous multilinear operators from  $E_1 \times \dots \times E_m$  to  $F$  is denoted by  $\mathcal{L}(E_1, \dots, E_m; F)$ . Both vector spaces are Banach spaces when endowed with the sup norm in the unit ball  $B_E$  or in the product of the the unit balls  $B_{E_1} \times \dots \times B_{E_m}$ .

The following characterizations of continuous polynomials are elementary (analogous results hold for multilinear operators):

**Proposition 4.1.** *Let  $P \in \mathcal{P}(^m E; F)$ . The following assertions are equivalent:*

- (i)  $P \in \mathcal{P}(^m E; F)$ ;
- (ii)  $P$  is continuous at zero;
- (iii) There is a constant  $M > 0$  such that  $\|P(x)\| \leq M \|x\|^m$ , for all  $x \in E$ .

The Polarization Formula relates polynomials and symmetric multilinear operators in a very useful way. Its proof is a kind of consequence of the Leibniz formula and some combinatorial tricks (see [35, 50]).

**Theorem 4.2** (Polarization Formula). *If  $T \in \mathcal{L}(^m E; F)$  is symmetric then*

$$T(x_1, \dots, x_m) = \frac{1}{m!2^m} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m T(x_0 + \varepsilon_1 x_1 + \dots + \varepsilon_m x_m)^m,$$

for all  $x_0, x_1, x_2, \dots, x_m \in E$ .

The following result is an immediate consequence of the Polarization Formula:

**Corollary 4.3.** *For each  $m$ -homogeneous polynomial there is a unique  $m$ -linear operator associated to it. In other words, if  $P$  is a  $m$ -homogeneous polynomial, then there exists only one symmetric  $m$ -linear operator  $T$  (sometimes called polar of  $P$ ) such that*

$$P(x) = T(x, \dots, x)$$

for all  $x$ .

In general, if  $T$  is the symmetric  $m$ -linear operator associated to a  $m$ -homogeneous polynomial  $P$  we have

$$\|P\| \leq \frac{m^m}{m!} \|T\|, \tag{4.1}$$

where  $\|P\| = \sup_{\|z\|=1} |P(z)|$ . The constant  $\frac{m^m}{m!}$  is usually called polarization constant.

If  $P$  is a homogeneous polynomial of degree  $m$  on  $\mathbb{K}^n$  given by

$$P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

and  $L$  is the polar of  $P$ , then

$$L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n}) = \frac{a_\alpha}{\binom{m}{\alpha}}, \quad (4.2)$$

where  $\{e_1, \dots, e_n\}$  is the canonical basis of  $\mathbb{K}^n$  and  $e_k^{\alpha_k}$  stands for  $e_k$  repeated  $\alpha_k$  times, the  $\alpha_j$ 's are non negative integers with  $|\alpha| := \sum_{j=1}^n \alpha_j = m$ , and  $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  for  $\alpha = (\alpha_1, \dots, \alpha_n)$ .

#### 4.1 Khinchine's inequality

The Khinchine inequality in its modern presentation has its origins in [56]. Let  $(\varepsilon_i)_{i \geq 1}$  be a sequence of independent Rademacher variables. For any  $p \in (0, \infty)$ , there exists a constant  $A_{\mathbb{R}, p}$  such that, given any sequence  $(a_i)$  of real numbers with finite support,

$$\left( \sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} \leq A_{\mathbb{R}, p}^{-1} \left( \int_{[0,1]^m} \left| \sum_{i=1}^{\infty} a_i \varepsilon_i(\omega) \right|^p d\omega \right)^{1/p}.$$

For complex scalars it is more useful (since it gives better constants) to use the following version of Khinchine's inequality (called Khinchine's inequality with Steinhaus variables): for any  $p \in (0, \infty)$ , there exists a constant  $A_{\mathbb{C}, p}$  such that, for any sequence  $(a_i)$  of complex numbers with finite support

$$\left( \sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} \leq A_{\mathbb{C}, p}^{-1} \left( \int_{\mathbb{T}^\infty} \left| \sum_{i=1}^{\infty} a_i z_i \right|^p dz \right)^{1/p},$$

with  $\mathbb{T}^\infty$  denoting the infinite polycircle, i.e.,

$$\mathbb{T}^\infty = \left\{ z = (z_i)_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |z_i| = 1 \text{ for all } i \in \mathbb{N} \right\},$$

and  $dz$  denoting the standard Lebesgue probability measure on  $\mathbb{T}^\infty$ . The best constants  $A_{\mathbb{R}, p}$  and  $A_{\mathbb{C}, p}$  were obtained by Haagerup and König, respectively (see [41] and [46]). More precisely,

- $A_{\mathbb{R}, p} = \frac{1}{\sqrt{2}} \left( \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p}$  if  $1.8474 \approx p_0 \leq p < 2$ ;
- $A_{\mathbb{R}, p} = 2^{\frac{1}{2} - \frac{1}{p}}$  if  $0 < p < p_0$ ;
- $A_{\mathbb{C}, p} = \Gamma\left(\frac{p+2}{2}\right)^{1/p}$  if  $p \in [1, 2]$ ;

- $A_{\mathbb{K},p} = 1$  if  $p \geq 2$  and  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

The (apparently) *strange* value  $p_0 \approx 1.8474$  is, to be precise, the unique number  $p_0 \in (1, 2)$  with

$$\Gamma\left(\frac{p_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

The notation  $A_{\mathbb{K},p}$  will be kept along this paper.

Using Fubini's theorem and Minkowski's inequality (see, for instance, [30, Lemma 2.2] for the real case and [51, Theorem 2.2] for the complex case), these inequalities have a multilinear version: for any  $n, m \geq 1$ , for any family  $(a_i)_{i \in \mathbb{N}^m}$  of real (resp. complex) numbers with finite support,

$$\left(\sum_{i \in \mathbb{N}^m} |a_i|^2\right)^{1/2} \leq A_{\mathbb{R},p}^{-m} \left(\int_{[0,1]^m} \left|\sum_{i \in \mathbb{N}^m} a_i \varepsilon_{i_1}^{(1)}(\omega_1) \dots \varepsilon_{i_m}^{(m)}(\omega_m)\right|^p d\omega_1 \dots d\omega_m\right)^{1/p}$$

where  $(\varepsilon_i^{(1)}), \dots, (\varepsilon_i^{(m)})$  are sequences of independent Rademacher variables (resp.

$$\left(\sum_{i \in \mathbb{N}^m} |a_i|^2\right)^{1/2} \leq A_{\mathbb{C},p}^{-m} \left(\int_{(\mathbb{T}^\infty)^m} \left|\sum_{i \in \mathbb{N}^m} a_i z_{i_1}^{(1)} \dots z_{i_m}^{(m)}\right|^p dz^{(1)} \dots dz^{(m)}\right)^{1/p},$$

in the complex case).

### 4.2 Kahane–Salem–Zygmund's inequality: suitable random polynomials

The essence of the Kahane–Salem–Zygmund inequalities, as we describe below, probably appeared for the first time in [45], but our approach follows the lines of Boas' paper [15]. Paraphrasing Boas, the Kahane–Salem–Zygmund inequalities use probabilistic methods to construct a homogeneous polynomial (or multilinear operator) with a relatively small supremum norm but relatively large majorant function. Both the multilinear and polynomial versions are needed for our goals.

**Theorem 4.4** (Kahane–Salem–Zygmund's inequality - Multilinear version, [15]). *Let  $m, n$  be positive integers. There exists a  $m$ -linear map  $T_{m,n} : \ell_\infty^n \times \dots \times \ell_\infty^n \rightarrow \mathbb{K}$  of the form*

$$T_{m,n}(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^n \pm z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$$

such that

$$\|T_{m,n}\| \leq \sqrt{32 \log(6m)} \times n^{\frac{m+1}{2}} \times \sqrt{m!}.$$

The original version of the Kahane–Salem–Zygmund inequality appears in the framework of complex scalars but it is simple to verify that the same result (with the same constants) holds for real scalars. The following result is corollary of the previous, now for polynomials, and it will also be important for our purpose.

**Theorem 4.5** (Kahane–Salem–Zygmund’s inequality - Polynomial version, [15]). *Let  $m, n$  be positive integers. Then there exists a  $m$ -homogeneous polynomial  $P : \ell_\infty^n \rightarrow \mathbb{K}$  of the form*

$$P_{m,n}(\mathbf{z}) = \sum_{|\alpha|=d} \pm \binom{m}{\alpha} \mathbf{z}^\alpha$$

such that

$$\|P_{m,n}\| \leq \sqrt{32 \log(6m)} \times n^{\frac{m+1}{2}} \times \sqrt{m!}.$$

### 4.3 A corollary to Minkowski’s inequality

Minkowski’s inequality is a very well-known result that helps to prove that  $L_p$  spaces are Banach spaces: it is the triangle inequality for  $L_p$  spaces. We need a somewhat well known result, which is a corollary of one of the many versions of Minkowski’s inequality, whose proof can be found, for instance, in [40].

**Proposition 4.6** (Corollary to Minkowski’s inequality). *For any  $0 < p \leq q < \infty$  and for any matrix of complex numbers  $(c_{ij})_{i,j=1}^\infty$ ,*

$$\left( \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} |c_{ij}|^p \right)^{q/p} \right)^{1/q} \leq \left( \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |c_{ij}|^q \right)^{p/q} \right)^{1/p}.$$

## 5 Recent “unexpected” applications to classical problems

### 5.1 The Bohnenblust–Hille inequality with subpolynomial constants

The Riemann hypothesis certainly motivated and inspired many prestigious mathematicians from the 20th century to study Dirichlet sums in a more extensive fashion (for instance, Bourgain, Enflo, or Montgomery [20, 37, 49]). In the first decades of the 20th century Harald Bohr was immersed in the study of Dirichlet series (see [17–19]). One of his main interests was to determine the width of the maximal strips on which a Dirichlet series can converge absolutely but non uniformly. More precisely, for a Dirichlet series  $D(s) := \sum_n a_n n^{-s}$ , where  $a_n$  are complex coefficients and  $s$  is a complex variable, Bohr defined

$$\sigma_a(D) := \inf \{r \in \mathbb{R} : D(s) \text{ converges absolutely for } \operatorname{Re}(s) > r\},$$

$$\sigma_u(D) := \inf \{r \in \mathbb{R} : D(s) \text{ converges uniformly in } \operatorname{Re}(s) > r + \varepsilon \text{ for every } \varepsilon > 0\},$$

and

$$T := \sup \{\sigma_a(D) - \sigma_u(D) : D \text{ is a Dirichlet series}\}.$$

Bohr’s question was: What is the value of  $T$ ?

The Bohnenblust–Hille inequality, proved in 1931 by H.F. Bohnenblust and E. Hille, is a crucial tool to answer Bohr’s problem: the precise value of  $T$  is  $1/2$ .

When dealing with the Bohnenblust–Hille inequality it is elucidative to start with proving Littlewood’s  $4/3$  inequality, a predecessor of the Bohnenblust–Hille

inequality. Littlewood's 4/3 inequality was proved in 1930 to solve a problem posed by P.J. Daniell. It is worth noticing how Holder's inequality plays a fundamental role in the argument used in the proof. We include (for the sake of completeness) a proof of the optimality of the power 4/3 using the Kahane–Salem–Zygmund inequality.

**Theorem 5.1** (Littlewood's 4/3 inequality). *There is a constant  $L_{\mathbb{K}} \geq 1$  such that*

$$\left( \sum_{i,j=1}^N |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq L_{\mathbb{K}} \|U\| \tag{5.1}$$

for every bilinear form  $U : \ell_{\infty}^N \times \ell_{\infty}^N \rightarrow \mathbb{K}$  and every positive integer  $N$ . Moreover, the power 4/3 is optimal.

*Proof.* Note that

$$\sum_{i,j=1}^N |U(e_i, e_j)|^{\frac{4}{3}} \leq \left[ \sum_{i=1}^N \left( \sum_{j=1}^N |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[ \left( \sum_{i=1}^N \left( \sum_{j=1}^N |U(e_i, e_j)| \right)^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}}$$

is a particular case of the procedure from Section 2. Now we just need to estimate the two factors above. From the Khinchine inequality we have

$$\begin{aligned} \sum_{i=1}^N \left( \sum_{j=1}^N |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} &\leq \sqrt{2} \sum_{i=1}^N \int_0^1 \left| \sum_{j=1}^N r_j(t) U(e_i, e_j) \right| dt \\ &= \sqrt{2} \int_0^1 \sum_{i=1}^N \left| U(e_i, \sum_{j=1}^N r_j(t) e_j) \right| dt \\ &\leq \sqrt{2} \sup_{t \in [0,1]} \sum_{i=1}^N \left| U(e_i, \sum_{j=1}^N r_j(t) e_j) \right| \\ &\leq \sqrt{2} \|U\|. \end{aligned}$$

By symmetry, the same is true if we swap  $i$  and  $j$ . From Minkowski's inequality we have

$$\left( \sum_{i=1}^N \left( \sum_{j=1}^N |U(e_i, e_j)| \right)^2 \right)^{\frac{1}{2}} \leq \sum_{j=1}^N \left( \sum_{i=1}^N |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|U\|$$

and combining all of these inequalities we obtain

$$L_{\mathbb{K}} = \sqrt{2}.$$

In order to prove the optimality of the exponent  $4/3$  we can use the Kahane–Salem–Zygmund inequality. Let  $T_{2,N} : \ell_\infty^N \times \ell_\infty^N \rightarrow \mathbb{C}$  be the bilinear form satisfying the multilinear Kahane–Salem–Zygmund inequality (Theorem 4.4). Then, if (5.1) holds for an exponent  $q > 0$ , we have

$$\left( \sum_{i,j=1}^N |T_{2,N}(e_i, e_j)|^q \right)^{\frac{1}{q}} \leq \sqrt{2}CN^{\frac{3}{2}}$$

and thus

$$N^{\frac{2}{q}} \leq \sqrt{2}CN^{\frac{3}{2}}.$$

Next, letting  $N \rightarrow \infty$  we conclude that  $q \geq \frac{4}{3}$ .  $\blacksquare$

The natural generalization of Littlewood’s  $4/3$  inequality is the Bohnenblust–Hille inequality. This inequality essentially says that for  $m > 2$  the exponent  $\frac{4}{3}$  can be replaced by  $\frac{2m}{m+1}$ , and this exponent is optimal. More precisely, it asserts that, for any  $m \geq 2$ , there exists a constant  $C_{\mathbb{K},m} \geq 1$  such that, for all  $N$  and all  $m$ -linear forms  $T : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{K}$ ,

$$\left( \sum_{i_1, \dots, i_m=1}^N |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \|T\|. \quad (5.2)$$

This result was overlooked and, sometimes, rediscovered during the last 80 years. Different approaches led to different values of the constants  $C_{\mathbb{K},m}$ . Let us denote the optimal constants satisfying equation (5.2) above by  $B_{\mathbb{K},m}^{\text{mult}}$ . As a matter of fact, controlling the growth of the constants  $B_{\mathbb{K},m}^{\text{mult}}$  is crucial for some applications, as it is being left clear along the paper (Sections 5.2 and 5.3 deal with Quantum information theory and the Bohr radius problem, respectively).

Now we show how a suitable use of Hölder’s inequality (Theorem 3.2) provides a very simple proof of the Bohnenblust–Hille inequality, with (so far!) the best known constants.

With the ingredients of Section 4 we can easily obtain an inductive formula for  $B_{\mathbb{K},m}^{\text{mult}}$ . We present a sketch of the proof (more details can be found in [10]; we also refer to the survey [33] which provides a careful and a deep analysis of the Bohnenblust–Hille inequality).

**Theorem 5.2** (Bohnenblust–Hille inequality). *For any positive integer  $m$ , there exists a constant  $B_{\mathbb{K},m}^{\text{mult}} \geq 1$  such that, for all  $m$ -linear forms  $L : \ell_\infty^N \times \cdots \times \ell_\infty^N \rightarrow \mathbb{K}$  and all  $N$ ,*

$$\left( \sum_{i_1, \dots, i_m=1}^N |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \|L\|, \quad (5.3)$$

with  $B_{\mathbb{K},1}^{\text{mult}} = 1$  and  $B_{\mathbb{K},m}^{\text{mult}} \leq A_{\mathbb{K},\frac{2k}{k+1}}^{-1} B_{\mathbb{K},k}^{\text{mult}}$ , for any  $1 \leq k \leq m-1$ .

*Proof.* We present a simple proof for the case  $k = m-1$ , which is the most important, since it provides better constants (and the proof for other values of  $k$  is similar). The proof for  $\mathbb{R}$  is essentially the same as the proof for  $\mathbb{C}$ , so we present

only the proof for the complex case. Let  $n \geq 1$  and let  $L = \sum_{\mathbf{i} \in \mathbb{N}^m} a_{\mathbf{i}} z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$  be an  $m$ -linear form on  $\ell_{\infty}^N \times \dots \times \ell_{\infty}^N$ .

From the Khinchine inequality we have

$$\left( \sum_{\mathbf{i}_S} \left( \sum_{\mathbf{i}_S} |a_{\mathbf{i}}|^2 \right)^{\frac{1}{2} \times \frac{2m-2}{m}} \right)^{\frac{m}{2m-2}} \leq A_{\mathbb{C}, \frac{2m-2}{m}}^{-1} B_{\mathbb{C}, m-1}^{\text{mult}} \|L\|$$

for all  $S \subset \{1, \dots, m\}$  with  $\text{card}(S) = m - 1$ .

From the "Minkowski inequality" (Proposition 4.6) we can obtain analogous estimates if we take the 2 in the last position and move it backwards making it take every position from the last to the first; in other words, considering the following exponents:

$$\left( \frac{2m-2}{m}, \dots, 2, \frac{2m-2}{m} \right), \dots, \left( 2, \frac{2m-2}{m}, \dots, \frac{2m-2}{m} \right)$$

and the same constant. Using the Hölder inequality for multiple exponents we reach the result. ■

Using the values of the constants  $A_{\mathbb{K}, p}$  we conclude that

$$B_{\mathbb{C}, m}^{\text{mult}} \leq \prod_{j=2}^m \Gamma \left( 2 - \frac{1}{j} \right)^{\frac{j}{2-2j}}. \tag{5.4}$$

For real scalars and  $m \geq 14$ ,

$$B_{\mathbb{R}, m}^{\text{mult}} \leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left( \frac{\Gamma \left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}} \tag{5.5}$$

and

$$B_{\mathbb{R}, m}^{\text{mult}} \leq \prod_{j=2}^m 2^{\frac{1}{2j-2}} = \left( \sqrt{2} \right)^{\sum_{j=1}^{m-1} 1/j}.$$

for  $2 \leq m \leq 13$ .

However, a first look at (5.4) and (5.5) gives *a priori* no clues on their behavior. The following consequences of Theorem 5.2 taken from [10] are worthed to be emphasized:

- There exists  $\kappa_1 > 0$  such that, for any  $m \geq 1$ ,

$$B_{\mathbb{C}, m}^{\text{mult}} \leq \kappa_1 m^{\frac{1-\gamma}{2}} < \kappa_1 m^{0.211392}.$$

- There exists  $\kappa_2 > 0$  such that, for any  $m \geq 1$ ,

$$B_{\mathbb{R}, m}^{\text{mult}} \leq \kappa_2 m^{\frac{2-\log 2-\gamma}{2}} < \kappa_2 m^{0.36482}.$$

It is interesting to recall that some old estimates  $B_{\mathbb{K},m}^{\text{mult}}$  can be easily recovered just by choosing different  $(q_1, \dots, q_m)$  when using Hölder’s inequality (or using Theorem 5.2 directly). For instance,

- Davie ([26], 1973).

$$B_{\mathbb{K},m}^{\text{mult}} \leq (\sqrt{2})^{m-1}.$$

Using the Khinchine inequality, we have

$$\left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_i|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq (\sqrt{2})^{m-1} \|L\|$$

for

$$(q_1, \dots, q_m) = (1, 2, \dots, 2)$$

Using the “Minkowski inequality” (Proposition 4.6) we obtain the same estimate for

$$(q_1, \dots, q_m) = (2, 1, \dots, 2), \dots, (q_1, \dots, q_m) = (2, \dots, 2, 1)$$

with the same constant. Now, using Theorem 3.2, we conclude the proof with

$$B_{\mathbb{K},m}^{\text{mult}} \leq (\sqrt{2})^{m-1}.$$

- Pellegrino and Seoane-Sepúlveda ([53], 2012).

$$B_{\mathbb{K},m}^{\text{mult}} \leq A_{\mathbb{K},\frac{2m}{m+2}}^{-\frac{m}{2}} B_{\mathbb{K},\frac{m}{2}}^{\text{mult}} \text{ for } m \text{ even, and}$$

$$B_{\mathbb{K},m}^{\text{mult}} \leq \left( A_{\mathbb{K},\frac{2m-2}{m+1}}^{-\frac{m+1}{2}} B_{\mathbb{K},\frac{m-1}{2}}^{\text{mult}} \right)^{\frac{m-1}{2m}} \left( A_{\mathbb{K},\frac{2m+2}{m+3}}^{-\frac{m-1}{2}} B_{\mathbb{K},\frac{m+1}{2}}^{\text{mult}} \right)^{\frac{m+1}{2m}} \text{ for } m \text{ odd.}$$

When  $m$  is even and  $k = m/2$ , we use Khinchine inequality to obtain estimates for the inequalities with exponent

$$(q_1, \dots, q_m) = \left( \frac{2m}{m+2}, \dots, \frac{2m}{m+2}, 2, \dots, 2 \right)$$

and using the Minkowski inequality the same estimate is obtained for

$$(q_1, \dots, q_m) = \left( 2, \dots, 2, \frac{2m}{m+2}, \dots, \frac{2m}{m+2} \right).$$

Using Proposition 5.2 we obtain

$$B_{\mathbb{K},m}^{\text{mult}} \leq A_{\mathbb{K},\frac{2m}{m+2}}^{-m/2} B_{\mathbb{K},m/2}^{\text{mult}}.$$

The case  $m$  odd is somewhat similar, although it needs a little *trick*. It is worth mentioning that these estimates from [53] can be somewhat derived from abstract results appearing in [30].



The Bohnenblust–Hille inequality (multilinear and polynomial) still have interesting versions in the setting of Lorentz spaces. Recall that, given  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$ , the Lorentz space  $\ell_{p,q}(I)$  ( $\ell_{p,q}$  for short) on a nonempty set  $I$  consists of all scalar sequences  $x = (x_i)_{i \in I}$  for which the expression

$$\|x\|_{\ell_{p,q}} = \begin{cases} (\sum_{k \in J} x_k^{*q} (k^{q/p} - (k-1)^{q/p})^q)^{1/q} & \text{if } q < \infty, \\ \sup_{k \in J} k^{1/p} x_k^* & \text{if } q = \infty, \end{cases}$$

is finite. Here, for a given  $x = (x_i)_{i \in I} \in \ell_\infty(I)$ , we denote by  $x^* = (x_j^*)_{j \in J}$  the non-increasing rearrangement of  $x$  defined by

$$x_j^* = \inf\{\lambda > 0; \text{card}(\{i \in I; |x_i| > \lambda\}) \leq j\}, \quad j \in J,$$

where  $J = \{1, \dots, n\}$  whenever  $\text{card}(I) = n$ , and  $J = \mathbb{N}$  whenever  $I$  is infinite.

As mentioned in [39], Littlewood’s 4/3 inequality for Lorentz spaces can be deduced from a unpublished work of G. Pisier. Using Hölder’s inequality for mixed  $\ell_p$  spaces, J. Fournier [39] (see also [13]) was able to provide a more general result: Bohnenblust–Hille’s multilinear inequality for Lorentz spaces. On this environment, the result reads as follows: for every positive integer  $m$ , there is a constant  $C \geq 1$  such that, for every  $n$  and every matrix  $\mathbf{a} = (a_i)_{i \in \mathcal{M}(m,n)}$ , we have

$$\|(a_i)_{i \in \mathcal{M}(m,n)}\|_{\ell_{\frac{2m}{m+1},1}} \leq C \|\mathbf{a}\|.$$

Very interesting multilinear and polynomial Bohnenblust–Hille-type inequalities in Lorentz spaces with subpolynomial and subexponential constants were obtained by A. Defant and M. Mastylo in [29].

## 5.2 Quantum Information Theory

Here we shall briefly describe a result by Montanaro [48, Theorem 5] which provided an application for the optimal Bohnenblust–Hille constants for real scalars within the field of Quantum Physics. This presentation is based on Schwarting’s Ph.D. dissertation [55, Section 2.2.5]. For a more detailed information we refer the interested reader to the Ph.D. dissertation of Briët [21, Chapter 1], which provides a clear introduction to the whole topic of nonlocal games.

A classical nonlocal game is a pair  $\mathcal{G} = (A, \pi)$  consisting on a function (called *predicate*)  $A : \mathcal{A} \times \mathcal{B} \times \mathcal{S} \times \mathcal{T} \rightarrow \{\pm 1\}$ , where  $\mathcal{A}, \mathcal{B}, \mathcal{S}$  and  $\mathcal{T}$  are finite sets, and a probability distribution  $\pi : \mathcal{S} \times \mathcal{T} \rightarrow [0, 1]$ . The game involves three parties: a person called the *referee* and two *players* (usually called Alice and Bob). When the game starts, the referee picks a question  $(s, t) \in \mathcal{S} \times \mathcal{T}$  according to the probability distribution  $\pi$  and, then, sends it to Alice and Bob, who must reply independently (they are not allowed to communicate between each other once the game has begun) by providing an answer  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  each one. The players win the game if  $A(a, b, s, t) = 1$ , and lose otherwise. The players’ goal is to maximize their chance of winning. A XOR game is a nonlocal game in which the answer sets  $\mathcal{A}, \mathcal{B}$  are  $\{\pm 1\}$  and the predicate  $A$  depends only on the exclusive-OR (XOR) of the answers given by the players and the value of a Boolean function

$\mathcal{S} \times \mathcal{T} \rightarrow \{\pm 1\}$ , which from the predicate may be seen as a matrix with entries on  $\{\pm 1\}$ . A game with  $m$ -players is described similarly in the following fashion.

An  $m$ -player XOR (exclusive OR) game is a pair  $\mathcal{G} = (\pi, A)$  consisting of a matrix  $A = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}$ , for which each entry  $a_{\mathbf{i}} \in \{\pm 1\}$ , and a probability distribution  $\pi : \mathcal{M}(m, n) \rightarrow [0, 1]$ . The game consists on *the referee* picking an  $m$ -tuple  $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{M}(m, n)$  according to the probability distribution  $\pi$  and sending each question  $i_k$  to *the player*  $k$ , which, by means of a classical strategy, must reply upon this question with a (deterministic) answer map  $y_k : \{1, \dots, n\} \rightarrow \{\pm 1\}$ . The players win if and only if the product of their answers equals the corresponding entry in the matrix  $A$ , that is if

$$y_1(i_1) \cdots y_m(i_m) = a_{\mathbf{i}}.$$

Concerning the complexity of a XOR game, one defines the bias  $\beta(G)$  to be the greatest difference between the chance of winning and the chance of loosing the game for the optimal classical strategy. Therefore, the classical bias of an  $m$ -player XOR game is given by

$$\beta(G) = \max_{y_1, \dots, y_m \in \{\pm 1\}^n} \left| \sum_{\mathbf{i} \in \mathcal{M}(m, n)} \pi(\mathbf{i}) a_{\mathbf{i}} y_1(i_1) \cdots y_m(i_m) \right|.$$

If we define the  $m$ -linear map  $T : \ell_{\infty}^n \times \cdots \times \ell_{\infty}^n \rightarrow \mathbb{R}$  by  $T(e_{i_1}, \dots, e_{i_m}) := a_{\mathbf{i}} \pi(\mathbf{i})$ , then the bias will be

$$\beta(G) = \|T\|.$$

A natural problem is to find the game for which the classical bias is minimized. It is known that there exists an  $m$ -player XOR game  $\mathcal{G}$  for which

$$\beta(\mathcal{G}) \leq n^{-\frac{m-1}{2}}$$

(see [38]). Using the Bohnenblust–Hille inequality it is straightforward to obtain lower bounds for the classical bias of an  $m$ -player XOR games (see [48, Theorem 5]).

**Theorem 5.3.** [48, Theorem 5] *For every  $m$ -player XOR game  $\mathcal{G} = (\pi, A)$ ,*

$$\beta(\mathcal{G}) \geq \frac{1}{\kappa m^{0.36482}} n^{\frac{1-m}{2}},$$

where  $\kappa > 0$  is an universal constant.

*Proof.* Define the  $m$ -linear form  $T : \ell_{\infty}^n \times \cdots \times \ell_{\infty}^n \rightarrow \mathbb{R}$  by  $T(e_{i_1}, \dots, e_{i_m}) := a_{\mathbf{i}} \pi(\mathbf{i})$ . Then,

$$\sum_{\mathbf{i} \in \mathcal{M}(m, n)} |T(e_{i_1}, \dots, e_{i_m})| = \sum_{\mathbf{i} \in \mathcal{M}(m, n)} \pi(\mathbf{i}) = 1.$$

Applying Hölder's inequality and the Bohnenblust–Hille, we conclude that

$$\begin{aligned} \sum_{\mathbf{i} \in \mathcal{M}(m, n)} |T(e_{i_1}, \dots, e_{i_m})| &\leq \left( \sum_{\mathbf{i} \in \mathcal{M}(m, n)} |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \left( \sum_{\mathbf{i} \in \mathcal{M}(m, n)} 1 \right)^{\frac{m-1}{2m}} \\ &\leq B_{\mathbb{R}, m}^{\text{mult}} n^{\frac{m-1}{2}} \|T\| = B_{\mathbb{R}, m}^{\text{mult}} n^{\frac{m-1}{2}} \beta(\mathcal{G}). \end{aligned}$$

Using the best known estimates for the multilinear Bohnenblust–Hille inequality we conclude that

$$\beta(\mathcal{G}) \geq \frac{1}{\kappa m^{\frac{2-\log 2-\gamma}{2}} n^{\frac{m-1}{2}}} > \frac{1}{\kappa m^{0.36482} n^{\frac{1-m}{2}}}. \quad \blacksquare$$

This result, according to Montanaro (see [48, p.4]), implies a very particular case of a conjecture of Aaronson and Ambainis (see [1]). Also, recent advances on the real polynomial Bohnenblust–Hille inequality (see, e.g., [22,36]), combined with the CHSH inequality (due to Clauser, Horne, Shimony, and Holt in the late 1960's), can be employed in the proof of Bell's theorem, which states that certain consequences of entanglement in quantum mechanics cannot be reproduced by local hidden variable theories. We refer the interested reader to the seminal paper, [25], in which more information regarding this CHSH inequality can be found.

### 5.3 Power series and the Bohr radius problem

The following question was addressed by H. Bohr in 1914:

*How large can the sum of the moduli of the terms of a convergent power series be?*

The answer was given by the following theorem, which was independently obtained by Bohr, Riesz, Schur, and Wiener:

**Theorem 5.4.** *Suppose that a power series  $\sum_{k=0}^{\infty} c_k z^k$  converges for  $z$  in the unit disk, and  $|\sum_{k=0}^{\infty} c_k z^k| < 1$  when  $|z| < 1$ . Then  $\sum_{k=0}^{\infty} |c_k z^k| < 1$  when  $|z| < 1/3$ . Moreover, the radius  $1/3$  is the best possible.*

Following Boas and Khavinson [14], the Bohr radius  $K_n$  of the  $n$ -dimensional polydisk is the largest positive number  $r$  such that all polynomials  $\sum_{\alpha} a_{\alpha} z^{\alpha}$  on  $\mathbb{C}^n$  satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_{\alpha} |a_{\alpha} z^{\alpha}| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_{\alpha} a_{\alpha} z^{\alpha} \right|.$$

The Bohr radius  $K_1$  was estimated by H. Bohr, M. Riesz, I. Schur and F. Wiener, and it was shown that  $K_1 = 1/3$  (Theorem 5.4). For  $n \geq 2$ , exact values of  $K_n$  are unknown. In [14], it was proved that

$$\frac{1}{3} \sqrt{\frac{1}{n}} \leq K_n \leq 2 \sqrt{\frac{\log n}{n}}. \tag{5.6}$$

The paper by Boas and Khavinson, [14], motivated many other works, connecting the asymptotic behavior of  $K_n$  to various problems in Functional Analysis (geometry of Banach spaces, unconditional basis constant of spaces of polynomials, etc.); we refer to [31] for a panorama of the subject. Hence there was a big motivation in recent years in determining the behavior of  $K_n$  for large values of  $n$ .

In [27], the left hand side inequality of (5.6) was improved to

$$K_n \geq c \sqrt{\log n / (n \log \log n)}.$$

In [28], using the hypercontractivity of the polynomial Bohnenblust–Hille inequality, the authors showed that

$$K_n = b_n \sqrt{\frac{\log n}{n}} \text{ with } \frac{1}{\sqrt{2}} + o(1) \leq b_n \leq 2. \quad (5.7)$$

In this section we sketch how the Hölder inequality for mixed sums played a fundamental role in the final answer to the solution, given in [10], to the Bohr radius problem:

$$\lim_{n \rightarrow \infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1.$$

The solution has several ingredients, including the polynomial Bohnenblust–Hille inequality. Using (4.1), Bohnenblust and Hille were also able to have a polynomial version of this inequality: for any  $m \geq 1$ , there exists a constant  $D_m \geq 1$  such that, for any complex  $m$ -homogeneous polynomial  $P(\mathbf{z}) = \sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$  on  $\mathcal{C}_0$ ,

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_m \|P\|,$$

with

$$D_m = (\sqrt{2})^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

In fact, it is not difficult to use polarization and obtain the polynomial Bohnenblust–Hille inequality by using the multilinear Bohnenblust–Hille inequality, but with *bad* constants (the following approach can be essentially found in [32, Lemma 5]). In fact, if  $L$  is the polar of  $P$ , from (4.2) we have

$$\begin{aligned} \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} &= \sum_{|\alpha|=m} \left( \binom{m}{\alpha} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})| \right)^{\frac{2m}{m+1}} \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2m}{m+1}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2m}{m+1}}. \end{aligned}$$

However, for every choice of  $\alpha$ , the term

$$|L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2m}{m+1}}$$

is repeated  $\binom{m}{\alpha}$  times in the sum

$$\sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}}.$$

Thus

$$\sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2m}{m+1}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2m}{m+1}} = \sum_{i_1, \dots, i_m=1}^n \binom{m}{\alpha}^{\frac{2m}{m+1}} \frac{1}{\binom{m}{\alpha}} |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}}$$

and, since

$$\binom{m}{\alpha} \leq m!$$

we have

$$\sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2m}{m+1}} |L(e_1^{\alpha_1}, \dots, e_n^{\alpha_n})|^{\frac{2m}{m+1}} \leq (m!)^{\frac{m-1}{m+1}} \sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}}.$$

We thus have

$$\begin{aligned} \left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq \left( (m!)^{\frac{m-1}{m+1}} \sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\ &= (m!)^{\frac{m-1}{2m}} \left( \sum_{i_1, \dots, i_m=1}^n |L(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\ &\leq (m!)^{\frac{m-1}{2m}} B_{\mathbb{R},m}^{\text{mult}} \|L\|. \end{aligned}$$

On the other hand, since  $\|L\| \leq \frac{m^m}{m!} \|P\|$  we obtain

$$\begin{aligned} \left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq B_{\mathbb{R},m}^{\text{mult}} (m!)^{\frac{m-1}{2m}} \frac{m^m}{m!} \|P\| \\ &= B_{\mathbb{R},m}^{\text{mult}} \frac{m^m}{(m!)^{\frac{m+1}{2m}}} \|P\|. \end{aligned}$$

Let us denote the best constant  $D_m$  in this inequality by  $B_{\mathbb{C},m}^{\text{pol}}$ . In [28] it was proved that in fact these estimates could be essentially improved to  $(\sqrt{2})^{m-1}$ . However using the variant of Hölder's inequality for mixed  $\ell_p$  spaces, together with some results from Complex Analysis (see [10] for details) and with the sub-polynomial estimates of the multilinear Bohnenblust–Hille inequality (Section 5), one of the main results of [10] shows that we can go much further:

**Theorem 5.5.** *For any  $\varepsilon > 0$ , there exists  $\kappa > 0$  such that, for any  $m \geq 1$ ,*

$$B_{\mathbb{C},m}^{\text{pol}} \leq \kappa(1 + \varepsilon)^m.$$

As we mentioned above, in [28], using the hypercontractivity of the polynomial Bohnenblust–Hille inequality, the authors showed that

$$K_n = b_n \sqrt{\frac{\log n}{n}} \text{ with } \frac{1}{\sqrt{2}} + o(1) \leq b_n \leq 2. \tag{5.8}$$

However, although (5.8) is quite precise, there was still uncertainty in the behavior of the number  $b_n$ . By combining classical tools of Complex Analysis (Harris' inequality [43]), Bayart's inequality [9], Wiener's inequality [10, Lemma 6.1], and the Kahane–Salem–Zygmund inequality (Theorem 4.5) together with Theorem 5.5 the authors, in [10], were finally able to provide the final solution to the Bohr radius problem:

**Theorem 5.6.** *The asymptotic growth of the  $n$ -dimensional Bohr radius is  $\sqrt{\frac{\log n}{n}}$ . In other words,*

$$\lim_{n \rightarrow \infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1.$$

The crucial step to complete the proof was the improvement of the estimates of the polynomial Bohnenblust–Hille inequality that was only achieved by means of the Hölder inequality for mixed sums.

#### 5.4 Hardy–Littlewood's inequality constants

Although Hölder's inequality for mixed  $\ell_p$  spaces dates back to the 1960's, its full importance in the subjects mentioned throughout this paper was just very recently realized. New consequences are still appearing (see, for instance [6–8, 23]). The last applications of the Hölder inequality for mixed  $\ell_p$  spaces presented here concern the Hardy–Littlewood inequality and the theory of multiple summing multilinear operators. As in the case of the Bohnenblust–Hille inequality (Section 5) the Hölder inequality for multiple exponents allows a significant improvement in the constants of the Hardy–Littlewood inequality.

Given an integer  $m \geq 2$ , the Hardy–Littlewood inequality (see [4, 42, 54]) asserts that for  $2m \leq p \leq \infty$  there exists a constant  $C_{m,p}^{\mathbb{K}} \geq 1$  such that, for all continuous  $m$ -linear forms  $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$  and all positive integers  $n$ ,

$$\left( \sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|. \quad (5.9)$$

Using the generalized Kahane–Salem–Zygmund inequality (see [4]) one can easily verify that the exponents  $\frac{2mp}{mp+p-2m}$  are optimal. When  $p = \infty$ , using that  $\frac{2mp}{mp+p-2m} = \frac{2m}{m+1}$ , we recover the classical Bohnenblust–Hille inequality (see Theorem 5.2 and [16]).

From [10] we know that  $B_{\mathbb{K},m}^{\text{mult}}$  has a subpolynomial growth. On the other hand, the best known upper bounds for the constants in (5.9) were, until just recently,  $(\sqrt{2})^{m-1}$  (see [4, 5, 34]). However, a suitable use of Theorem 3.2 shows that  $(\sqrt{2})^{m-1}$  can be improved (see [8]) to

$$C_{m,p}^{\mathbb{R}} \leq \left(\sqrt{2}\right)^{\frac{2m(m-1)}{p}} \left(B_{\mathbb{R},m}^{\text{mult}}\right)^{\frac{p-2m}{p}}$$

for real scalars and to

$$C_{m,p}^{\mathbb{C}} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{2m(m-1)}{p}} \left( B_{\mathbb{C},m}^{\text{mult}} \right)^{\frac{p-2m}{p}}$$

for complex scalars. These estimates are substantially better than  $(\sqrt{2})^{m-1}$  because  $B_{\mathbb{K},m}^{\text{mult}}$  has a subpolynomial growth. In particular, if  $p > m^2$  we conclude that  $C_{m,p}^{\mathbb{K}}$  has a subpolynomial growth.

## 5.5 Separately summing operators

Hölder's inequality is also used to generalize recent results on the theory of multiple summing multilinear operators. In [30], and for  $m$ -linear operators on  $q$ -cotype Banach spaces, the authors introduced the notion *separately*  $(r, 1)$ -summing, with  $1 \leq r \leq q < \infty$ , which means that, for any  $(m - 1)$ -coordinates fixed, the resulting linear operator is  $(r, 1)$ -summing. Using separately summing maps, the authors concluded that the initial operator is multiple  $\left( \frac{qrm}{q+(m-1)r}, 1 \right)$ -summing. In [6] it is presented the concept of  $n$ -separability summing, which stands for the  $m$ -linear operators that are multiple summing in  $n$ -coordinates, when there are  $m - n$  other coordinates fixed. Using suitable interpolation, the authors provide  $N$ -separability from  $n$ -separability summing, with  $n < N \leq m$ . This result generalizes the previous one and provides more efficient exponents in some special cases. Moreover, it is also useful to provide estimates for the constants of some variation of Bohnenblust–Hille inequalities introduced in [51, Appendix A] and [52].

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