String links with the same closure and group diagrams

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Abstract

We give a direct proof that if two string links have isotopic closures, then there is a braid-special isomorphism between their n-level group diagrams, for every $n \geqslant 2$. In the case of link-homotopy, we give an alternative proof to our previous result that there is a braid-special isomorphism between the group diagrams for the homotopy classes of two string links if and only if they have link-homotopic closures.

1 Introduction

An approach to study links is to consider them as closures of string links (see definition 1). String links are a generalization of pure braids. Although more complicated than pure braids, string links still have a natural multiplication and a kind of Artin representation (see [5]) and, contrary to braids, every link of k components can be obtained as a closure of a string link of k components. That simplifies the question of when two string links have the same closure, as compared with the same question for braids that, despite the Theorem of Markov, is very difficult to answer due to possible changes in the number of strings. Using string links, Habbeger and Lin were able to classify links up to link-homotopy (see [6]).

In our paper [1] we associated to a homotopy class of a string link a certain group diagram and showed that two string links have the same closure, up to

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link-homotopy, if and only if there is a certain type of isomorphism between their group diagrams. We called such type of isomorphism a braid-special isomorphism (there was a small mistake in the diagram which we corrected in [2, section 4]). In [2] we used our techniques to study the case of link concordance. In this case we have n-level group diagrams, $n \geqslant 2$, and, not like for link-homotopy, we get only that if the string links have the same closure, up to concordance, then there is, for every $n \geqslant 2$, a n-level braid-special isomorphism between the string links n-level group diagrams.

In this paper we consider first the case of ambient isotopy (see Theorem 4). Despite the fact that our assumption (ambient isotopy) is stronger than that in our paper [2] (concordance), we provide a different, more direct proof. The proof is based in a result from Habegger-Lin [5] (see Theorem 3 below) different from that used in [2]. We also provide an alternative proof for the main Theorem in [1] (see Theorem 12 below), which allows some clear geometric interpretation of that result.

In the case of string links f and g we show also that if we have a braid-special isomorphism between the n-level group diagrams for f and g, then there is a certain relation between f and g that depends on the kernel of the "Artin representation" (Theorem 6). In particular, if f and g are concordant, we have that relation between f and g (Corollary 7).

2 Ambient Isotopy

We will remember some results and notation from [2]. I is the interval $[0,1]$, D is the unit disk $\{x \in \mathbb{R}^2 \mid \|x\| \leqslant 1\}$, $k \geqslant 1$ is an integer number, $\underline{k}$ is the set $\{1, 2, \ldots, k\}$, $(\forall i \in \underline{k})$ a_i is the point $\left(-1 + \frac{2i}{k+1}, 0\right) \in D$ and $j_0 : \underline{k} \times I \longrightarrow D \times I$ is the map defined by $(i, x)j_0 = (a_i, x)$. Note that, as above, if f is a map, we will usually write $(x)f$ instead of $f(x)$.

Definition 1. A *k-string link* is a (smooth or piecewise linear) proper embedding $f : \underline{k} \times I \longrightarrow D \times I$ such that $f|_{\underline{k}\times\partial I} = j_0|_{\underline{k}\times\partial I}$ (see Fig. 1).

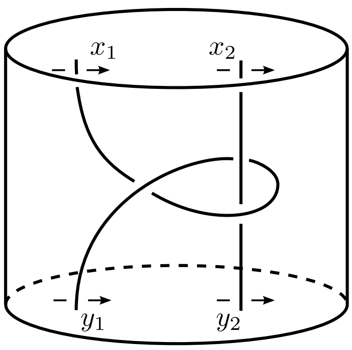


Figure 1: Top and bottom meridians of a 2-string link.

The product of two k-string links f and g, denoted by fg, is given by stacking f on the top of g and reparametrizing (see [1]). This product induces a monoid structure on the set $SL(k)$ of (ambient) isotopy classes of k-string links.

Habegger-Lin (see [5]) introduced a left and a right action of the monoid $SL(2k)$ on the set $SL(k)$ that we will call Habegger-Lin's actions. We use a slightly different notation (see Fig. 2 below).

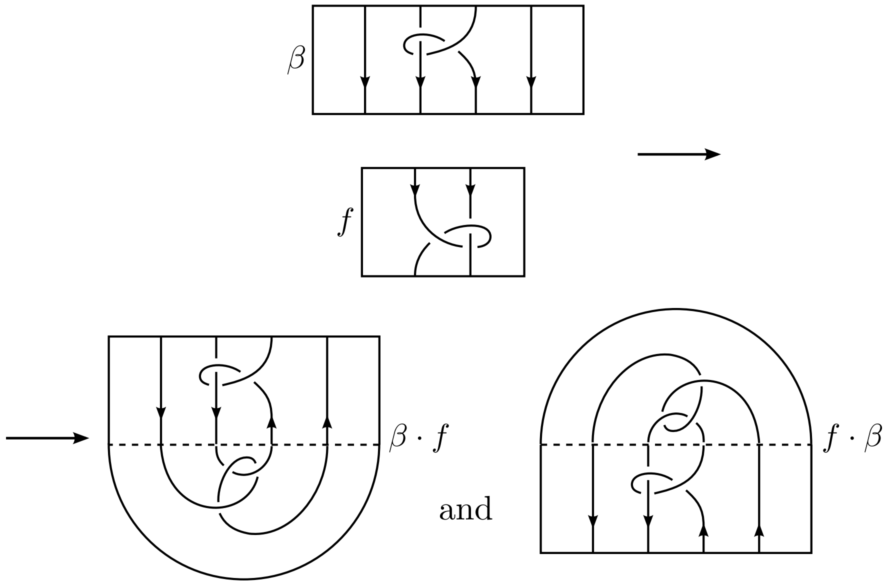


Figure 2: A 4-string link β acts from the left and from the right on a 2-string link f.

Definition 2. The *reflection* of a k-string link f is the k-string link f^R obtained by reflecting f in $D \times \frac{1}{2}$.

The fundamental group of the complement of a string link f is called the *group of* f and is denoted by $\pi(f)$.

For a group G, let $\{G_n\}, n \geqslant 1$, denote the lower central series of G, that is, $G_1 = G$ and inductively $G_{n+1} = [G, G_n]$ (where for sets $A, B \subseteq G$, $[A, B]$ denotes the group generated by all commutators $[a, b] = aba^{-1}b^{-1}$, $a \in A, b \in B$.)

Let $\widetilde{G} = \varprojlim_n \frac{G}{G_n}$ be the nilpotent completion of G.

Let $F(k)$ denote the free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Let f be a k-string link. We will denote by $x_i = x_i(f) \in \pi(f)$, for all $i \in \underline{k}$, the top meridians of f and by $y_i = y_i(f) \in \pi(f)$, for all $i \in \underline{k}$, the bottom meridians of f (see Fig. 1 and [3]).

For $j = 0, 1$, inclusions $i_j : D \times \{j\} \setminus \partial_j f \to D \times I \setminus f$ induce homomorphisms $\mu_0(f) : F(k) = F(\alpha_1, \alpha_2, \ldots, \alpha_k) \to \pi(f), (\alpha_i)\mu_0(f) = x_i(f)$, and $\mu_1(f) : F(k) \to \pi(f), (\alpha_i)\mu_1(f) = y_i(f)$, called respectively, the *top meridian map for* f and the *bottom meridian map for* f. By Stallings' Theorem [11] (see also [5]), they also induce isomorphisms on the lower central series quotients of fundamental groups:

$$\frac{F(k)}{F(k)_n} \underset{\cong}{\xrightarrow{(\mu_0(f))_n}} \frac{\pi(f)}{\pi(f)_n} \underset{\cong}{\xleftarrow{(\mu_1(f))_n}} \frac{F(k)}{F(k)_n}.$$

Therefore $(\mu_0(f))_n\,(\mu_1(f))_n^{-1}$ is an element $\overline{f}_n \in Aut\left(\frac{F(k)}{F(k)_n}\right)$, the group of automorphisms of $\frac{F(k)}{F(k)_n}$.

$\mu_0(f)$ and $\mu_1(f)$ also induce isomorphisms (see [8]):

$$\widetilde{F(k)} \underset{\cong}{\xrightarrow{\widetilde{\mu_0(f)}}} \widetilde{\pi(f)} \underset{\cong}{\xleftarrow{\widetilde{\mu_1(f)}}} \widetilde{F(k)}.$$

Thus we have $\widetilde{f} = \widetilde{\mu_0(f)}\;\widetilde{\mu_1(f)}^{-1} \in Aut(\widetilde{F(k)})$.

The associations $f \mapsto \overline{f_n}$ and $f \mapsto \widetilde{f}$ are monoid homomorphism from $SL(k)$ into $Aut\left(\frac{F(k)}{F(k)_n}\right)$ and $Aut(\widetilde{F(k)})$, respectively.

Note also that, since $\mu_0(f^R) = \mu_1(f)$ and $\mu_1(f^R) = \mu_0(f)$, we have $\overline{(f^R)}_n = \overline{f}_n^{-1}$ and also $\widetilde{f^R} = \widetilde{f}^{-1}$.

$\overline{f}_n$ is a *braid-like automorphism* of $\frac{F(k)}{F(k)_n}$, that is (i) it sends the class of each generator α_i into a conjugate of itself and (ii) it sends the class of the product $\alpha_1\alpha_2\ldots\alpha_n$ into itself. An automorphism that satisfies (i) above is called *special*.

Let $F(k)$ be the free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $F(2k)$ be the free group in $2k$ generators $\alpha_1, \alpha_2, \ldots, \alpha_k, \widetilde{\alpha_k}, \ldots, \widetilde{\alpha_2}, \widetilde{\alpha_1}$. We will denote by ξ the epimorphism $\xi : F(2k) \to F(k)$ given by $(\alpha_i)\xi = \alpha_i$ and $(\widetilde{\alpha_i})\xi = \alpha_i^{-1}$ for any $i \in \underline{k}$. The kernel of ξ is $\langle \alpha_i\widetilde{\alpha_i} \mid i \in \underline{k}\rangle^N$, the normal subgroup of $F(2k)$ generated by $\{\alpha_i\widetilde{\alpha_i} \mid i \in \underline{k}\}$.

Let $\overline{S_k(1)}_n = \{\beta \in SL(2k) \mid \overline{\beta \cdot 1}_n$ is the identity automorphism of $\frac{F(k)}{F(k)_n}\}$. Clearly $\overline{S_k(1)}_n$ contains the stabilizer of 1 for Habegger-Lin's action $S_k(1) =$ $= \{\beta \in SL(2k) \mid \beta \cdot 1 = 1 \in SL(k)\}$.

Let $\widetilde{S_k(1)} = \{\beta \in SL(2k) \mid \widetilde{\beta \cdot 1}$ is the identity automorphism of $\widetilde{F(k)}\}$. Then $S_k(1) \subseteq \widetilde{S_k(1)} = \bigcap_n \overline{S_k(1)}_n$.

If $\beta \in SL(2k)$ and $f \in SL(k)$ we have also a previously defined Habegger-Lin's action $f \cdot \beta$. Thus we can consider ${}_k\overline{S(1)}_n = \{\beta \in SL(2k) \mid \overline{1 \cdot \beta}_n$ is the identity automorphism of $\frac{F(k)}{F(k)_n}\}$. Then ${}_k\overline{S(1)}_n$ contains the stabilizer of 1 for Habegger-Lin's action ${}_kS(1) = \{\beta \in SL(2k) \mid 1 \cdot \beta = 1 \in SL(k)\}$. Similarly we can define ${}_k\widetilde{S(1)} = \{\beta \in SL(2k) \mid \widetilde{1 \cdot \beta}$ is the identity automorphism of $\widetilde{F(k)}\} = \bigcap_n {}_k\overline{S(1)}_n$.

Theorem 1. *Let $\beta \in SL(2k)$. $\beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ if and only if there exists an automorphism $\overline{\overline{\beta}}_n : \frac{F(k)}{F(k)_n} \to \frac{F(k)}{F(k)_n}$ such that the diagram*

$$\begin{array}{ccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\quad \overline{\beta}_n \quad} & \dfrac{F(2k)}{F(2k)_n} \\ {\scriptstyle \xi_n}\Big\downarrow & & \Big\downarrow{\scriptstyle \xi_n} \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\quad \overline{\overline{\beta}}_n \quad} & \dfrac{F(k)}{F(k)_n} \end{array}$$

is commutative, where ξ_n is induced from ξ.

Proof. see [2]. ∎

Corollary 2. *$\beta \in \overline{S_k(1)} \cap {}_k\overline{S(1)}$ if and only if there exists an automorphism $\tilde{\tilde{\beta}} : \widetilde{F(k)} \longrightarrow \widetilde{F(k)}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{F(2k)} & \xrightarrow{\ \tilde{\beta}\ } & \widetilde{F(2k)} \\ {\scriptstyle \tilde{\xi}}\downarrow & & \downarrow{\scriptstyle \tilde{\xi}} \\ \widetilde{F(k)} & \xrightarrow{\ \tilde{\tilde{\beta}}\ } & \widetilde{F(k)} \end{array}$$

is commutative, where $\tilde{\xi}$ is induced from ξ.

Proof. see [2]. ∎

Definition 3. A *k-link* (or a *link of k components*) is an embedding of a disjoint union of ordered oriented circles $\bigsqcup_{i=1}^{k} S^1$ into S^3.

To a k-string link f it is associated a k-link $\hat{f}$ called its *closure* (see [1]).

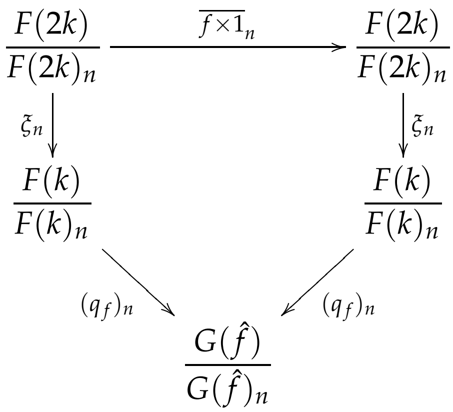
Definition 4. If L is a link, the fundamental group of the complement of L is called the *group of* L and is denoted by $G(L)$.

Let $f \times 1$ denote the 2k-string link obtained from a k-string link f by adding k straight strings at its end (see [2]).

By [8], the natural homomorphism $p_f : \pi(f) \longrightarrow G(\hat{f})$ is onto with kernel normally generated by the commutators $[x_i(f), \lambda_i(f)]$, for any $i \in \underline{k}$, where $x_i(f)$ are the top meridians of f and $\lambda_i(f)$ are the correspondent longitudes of f. Thus we have, for every $n \geqslant 2$, an induced epimorphism $(p_f)_n : \frac{\pi(f)}{\pi(f)_n} \longrightarrow \frac{G(\hat{f})}{G(\hat{f})_n}$. Let $(q_f)_n = (\mu_0(f)_n)(p_f)_n : \frac{F(k)}{F(k)_n} \longrightarrow \frac{G(\hat{f})}{G(\hat{f})_n}$. Then $(q_f)_n$ is an epimorphism that sends $\alpha_i F(k)_n$ to $u_i G(\hat{f})_n$, where $u_i \in G(\hat{f})$ is a meridian for the link $\hat{f}$ arising from $x_i(f)$.

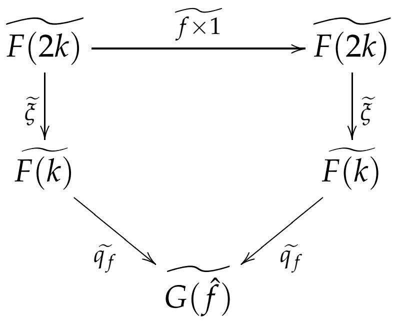
As we saw in [2], since $\overline{f}_n$ conjugates the classes of the meridians $\alpha_i F(k)_n$ by the classes of the correspondent longitudes (see [5]), and remembering that $\mu_0(f)_n$ is an isomorphism, we have that $\ker(q_f)_n = \left\langle (\alpha_i F(k)_n)\overline{f}_n \alpha_i^{-1} F(k)_n \mid i \in \underline{k} \right\rangle^N$ and that $\ker(\xi_n (q_f)_n) = \left\langle (\alpha_i F(2k)_n)\overline{f \times 1}_n \alpha_i^{-1} F(2k)_n,\ \alpha_i \tilde{\alpha}_i F(2k)_n \mid i \in \underline{k} \right\rangle^N$.

Definition 5. Given a k-string link f, the *n-level group diagram* for f, $n \geqslant 2$, is the commutative diagram

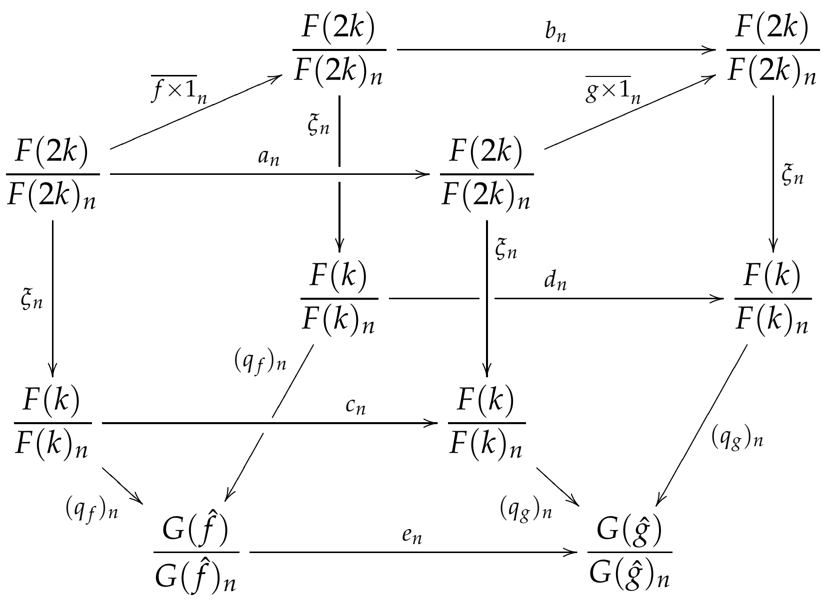


That the previous diagram is commutative follows from [10].

Given n-level group diagrams for all n and taking inverse limits, we have the *group diagram* for f:

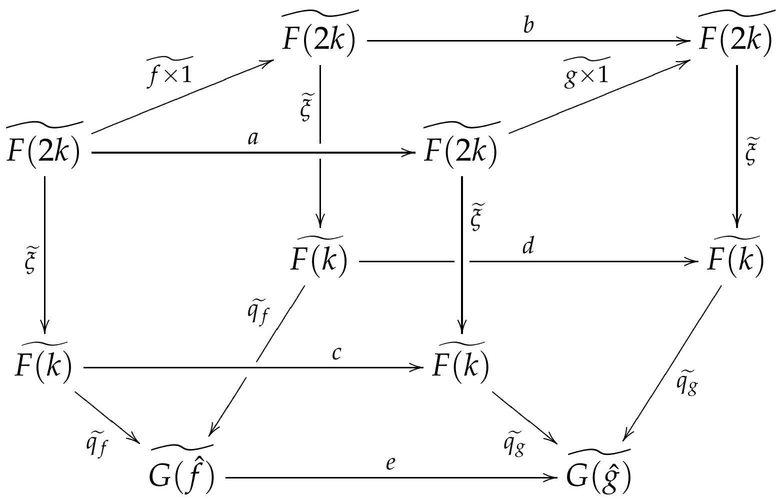


Definition 6. Given n-level group diagrams for k-string links f and g, an n-level *braid-special isomorphism* between them is a commutative diagram



where a_n, b_n are braid-like isomorphisms and c_n, d_n, e_n are special isomorphisms (see [7] for the meaning of e_n being special).

Definition 7. Given group diagrams for k-string links f and g, a *braid-special isomorphism* between them is a commutative diagram



where a, b are braid-like isomorphisms and c, d, e are special isomorphisms.

Given k–string links f and g, it is geometrically clear that if there is a 2k-string link β such that $1 \cdot \beta$ is isotopic to f and $\beta \cdot 1$ is isotopic to g, then $\hat{f}$ and $\hat{g}$ are isotopic. The converse was proved by Habegger-Lin [5, Proposition 2.1]. Thus we have

Theorem 3. *(Habegger-Lin) Let f, g be k-string links. Then $\hat{f}$ is isotopic to $\hat{g}$ if and only if there is a 2k-string link β such that $1 \cdot \beta$ is isotopic to f and $\beta \cdot 1$ is isotopic to g.*

Theorem 4. *If f and g are k-string links with the same closure then, for every $n \geqslant 2$, there exists a n-level braid-special isomorphism between the n-level group diagrams of f and g.*

Proof. If f and g have the same closure then their reflections f^R and g^R also have the same closure. By Theorem 3, there is a 2k-string link β such that $\beta \cdot 1 = f^R$ and $1 \cdot \beta = g^R$.

Note that $1 \cdot \beta^R = (\beta \cdot 1)^R$, therefore $\beta \cdot 1 = f^R$ and $1 \cdot \beta = g^R$ is equivalent to $1 \cdot \beta^R = f$ and $\beta^R \cdot 1 = g$.

We have that, for each $n \geqslant 2$,

(i) $\beta \cdot 1 = f^R \Rightarrow f \times 1\, \beta \cdot 1 = 1 \Rightarrow f \times 1\, \beta \in \overline{S_k(1)}_n$;

(ii) $1 \cdot \beta = g^R \Rightarrow \beta\, g \times 1 \cdot 1 = 1 \Rightarrow \beta\, g \times 1 \in \overline{S_k(1)}_n$;

(iii) $1 \cdot \beta^R = f \Rightarrow \beta^R f^R \times 1 \cdot 1 = 1 \Rightarrow \beta^R f^R \times 1 \in \overline{S_k(1)}_n$;

(iv) $\beta^R \cdot 1 = g \Rightarrow g^R \times 1\, \beta^R \cdot 1 = 1 \Rightarrow g^R \times 1\, \beta^R \in \overline{S_k(1)}_n$.

It follows that $f \times 1\, \beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ and $\beta\, g \times 1 \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$.

By Theorem 1 we have commutative diagrams

$$\begin{array}{ccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{f\times 1}_n\overline{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \xi_n\big\downarrow & & \big\downarrow\xi_n \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\overline{\overline{(f\times 1\ \beta)}}_n} & \dfrac{F(k)}{F(k)_n}\end{array} \qquad \text{and} \qquad \begin{array}{ccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{\beta}_n\overline{g\times 1}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \xi_n\big\downarrow & & \big\downarrow\xi_n \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\overline{\overline{(\beta\ g\times 1)}}_n} & \dfrac{F(k)}{F(k)_n}\end{array}$$

where $\overline{\overline{(f \times 1\ \beta)}}_n$ and $\overline{\overline{(\beta\ g \times 1)}}_n$ are special automorphisms.

Therefore we have a commutative diagram

$$\begin{array}{ccc} \dfrac{G(\hat{f})}{G(\hat{f})_n} & & \dfrac{G(\hat{g})}{G(\hat{g})_n} \\ (q_f)_n\big\uparrow & & \big\uparrow(q_g)_n \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{(\overline{\overline{\beta g\times 1}})_n} & \dfrac{F(k)}{F(k)_n} \\ \xi_n\big\uparrow & & \big\uparrow\xi_n \\ \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{\beta}_n\overline{g\times 1}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \overline{f\times 1}_n\big\uparrow & & \big\uparrow\overline{g\times 1}_n \\ \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{f\times 1}_n\overline{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \xi_n\big\downarrow & & \big\downarrow\xi_n \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{(\overline{\overline{f\times 1\beta}})_n} & \dfrac{F(k)}{F(k)_n} \\ (q_f)_n\big\downarrow & & \big\downarrow(q_g)_n \\ \dfrac{G(\hat{f})}{G(\hat{f})_n} & & \dfrac{G(\hat{g})}{G(\hat{g})_n} \end{array}$$

Since $(\overline{\overline{f \times 1\beta}})_n$ and $(\overline{\overline{\beta g \times 1}})_n$ are induced automorphisms and $\ker \xi_n = \langle \alpha_i\overline{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N$, we know that

1. $\langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N\, \overline{f \times 1}_n\overline{\beta}_n = \langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N$, and

2. $\langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N\, \overline{\beta}_n\overline{g \times 1}_n = \langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N$.

   We know from [2] that $\ker(\xi_n(q_f)_n) = \Big\langle (\alpha_i F(2k)_n)\overline{f \times 1}_n\widetilde{\alpha}_i F(2k)_n,$ $\alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N$ and that $\ker(q_{f^R})_n = \ker(q_f)_n$. From (1) we have

3. $\Big\langle (\alpha_i F(2k)_n)\overline{f \times 1}_n\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\Big\rangle^N \overline{\beta}_n = \langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N$ and from (2) we have

4. $\langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N \overline{\beta}_n = \langle \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k}\rangle^N \overline{g\times 1}_n^{-1}$.

From (3) and (4) we have

$$
\begin{aligned}
&(\ker(\xi_n(q_f)_n))\overline{f\times 1}_n\overline{\beta}_n = \\
&= (\ker(\xi_n(q_{f^R})_n))\overline{f\times 1}_n\overline{\beta}_n = \\
&= \left\langle (\alpha_i F(2k)_n)\overline{f^R\times 1}_n\widetilde{\alpha}_i F(2k)_n, \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i\in\underline{k}\right\rangle^N \overline{f\times 1}_n\overline{\beta}_n = \\
&= \left\langle \alpha_i\widetilde{\alpha}_i F(2k)_n, (\alpha_i F(2k)_n)\overline{f\times 1}_n\widetilde{\alpha}_i F(2k)_n \mid i\in\underline{k}\right\rangle^N \overline{\beta}_n = \\
&= \left\langle (\alpha_i F(2k)_n)\overline{g^R\times 1}_n\widetilde{\alpha}_i F(2k)_n, \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i\in\underline{k}\right\rangle^N = \\
&= \ker(\xi_n(q_{g^R})_n) = \ker(\xi_n(q_g)_n).
\end{aligned}
$$

From (2) and (3) we have

$$
\begin{aligned}
&(\ker(\xi_n(q_f)_n))\overline{\beta}_n\overline{g\times 1}_n = \\
&= \left\langle (\alpha_i F(2k)_n)\overline{f\times 1}_n\widetilde{\alpha}_i F(2k)_n, \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i\in\underline{k}\right\rangle^N \overline{\beta}_n\overline{g\times 1}_n = \\
&= \left\langle (\alpha_i F(2k)_n)\overline{g\times 1}_n\widetilde{\alpha}_i F(2k)_n, \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i\in\underline{k}\right\rangle^N = \ker(\xi_n(q_g)_n).
\end{aligned}
$$

Therefore we have a commutative diagram

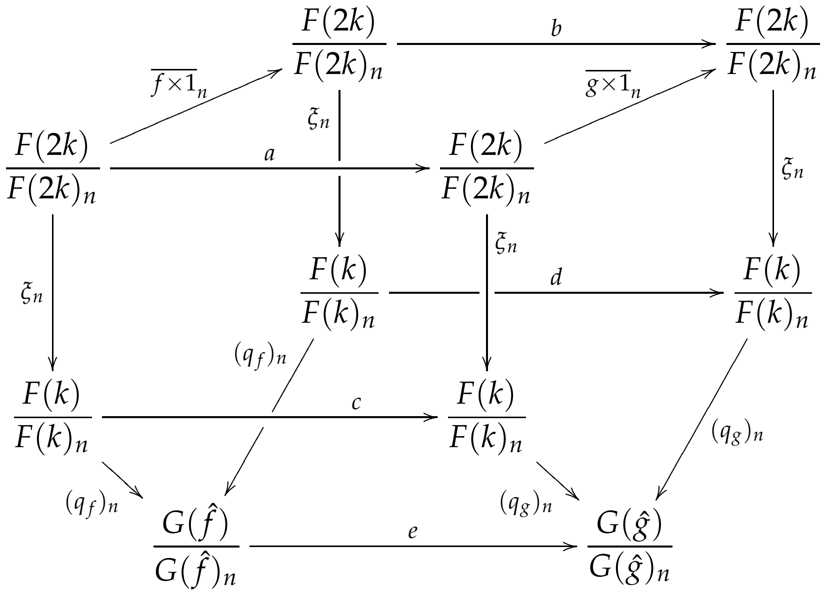
$$
\begin{array}{ccc}
\dfrac{G(\hat{f})}{G(\hat{f})_n} & \xrightarrow{\ a_n\ } & \dfrac{G(\hat{g})}{G(\hat{g})_n} \\
{\scriptstyle (q_f)_n}\uparrow & & \uparrow{\scriptstyle (q_g)_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{(\overline{\overline{\beta g\times 1}})_n} & \dfrac{F(k)}{F(k)_n} \\
{\scriptstyle \xi_n}\uparrow & & \uparrow{\scriptstyle \xi_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{\beta}_n\overline{g\times 1}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \overline{f\times 1}_n}\uparrow & & \uparrow{\scriptstyle \overline{g\times 1}_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{f\times 1}_n\overline{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\downarrow & & \downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{(\overline{\overline{f\times 1\beta}})_n} & \dfrac{F(k)}{F(k)_n} \\
{\scriptstyle (q_f)_n}\downarrow & & \downarrow{\scriptstyle (q_g)_n} \\
\dfrac{G(\hat{f})}{G(\hat{f})_n} & \xrightarrow{\ b_n\ } & \dfrac{G(\hat{g})}{G(\hat{g})_n}
\end{array}
$$

We can show, as in [2, Theorem 21], that $a_n = b_n$ and so we have a n-level braid-special isomorphism as stated. ■

Corollary 5. *If f and g are k-string links with the same closure, then there exists a braid-special isomorphism between the group diagrams of f and g.*

Theorem 6. *Let f and g be k-string links. If there is a braid-special isomorphism between the n-level group diagrams of f and g, then there exists a 2k-string link β such that* $\overline{\beta\cdot g}_n = 1$ *and* $\overline{f\cdot\beta}_n = 1$.

Proof. Suppose there is a braid-special isomorphism



between the n-level group diagrams of f and g.

Let $\gamma_n = b\,\overline{g\times 1}_n^{-1} = \overline{f\times 1}_n^{-1}a$. Since γ_n is braid-like, by [5], there is a 2k-string link β such that $\overline{\beta}_n = \gamma_n$. By Theorem 1, $f\times 1\ \beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ and $\beta\ g\times 1 \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$, since $\overline{f\times 1}_n\overline{\beta}_n = a$ and $\overline{\beta}_n\ \overline{g\times 1}_n = b$. Therefore $\overline{f\cdot\beta}_n = \overline{1\cdot f\times 1\beta}_n = 1$ and $\overline{\beta\cdot g}_n = \overline{\beta\, g\times 1\cdot 1}_n = 1$. ∎

Corollary 7. *If f and g are concordant k-string links, then, for each $n \geqslant 2$, there is a 2k-string link β such that* $\overline{\beta\cdot g}_n = 1$ *and* $\overline{f\cdot\beta}_n = 1$.

Proof. If f and g are concordant, it follows from [2] that there is a braid-special isomorphism between the n-level group diagrams of f and g. ∎

3 Link-Homotopy

Definition 8. k-string links f and g are *link-homotopic* if there is a homotopy of the strings in $D\times I$, fixing the endpoints and deforming f to g, such that the images of different strings remain disjoint during the deformation.

We will denote by $H(k)$ the group of link-homotopy classes of k-string links and by 1_k its neutral element.

Habegger-Lin actions induce actions from the group $H(2k)$ on the set $H(k)$.

Definition 9. k-links $L_o, L_1 : \bigsqcup_{i=1}^{k} S_i^1 \longrightarrow S^3$ are *link-homotopic* if there is a homotopy $L_t : \bigsqcup_{i=1}^{k} S_i^1 \longrightarrow S^3$, $0\leqslant t\leqslant 1$, such that $(S_i^1)L_t$ and $(S_j^1)L_t$ are disjoint for any $i\neq j$ and $0\leqslant t\leqslant 1$.

Alternatively, two links are link-homotopic if they are related by a sequence of isotopies and same-component crossing changes (see [6]).

By using Corollary 2.4 and Lemma 2.5 of [6] one can adapt the proof of Theorem 3 above, given in [5], to show that

Theorem 8. *Let f, g be k-string links. Then $\hat{f}$ is link-homotopic to $\hat{g}$ if and only if there is a 2k-string link β such that $1_k \cdot \beta$ is link-homotopic to f and $\beta \cdot 1_k$ is link-homotopic to g.*

Definition 10. Let $F(k)$ be the free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$, and $RF(k)$ the quotient group obtained from $F(k)$ by adding relations which say that each α_i commutes with all of its conjugates. $RF(k)$ is called the *reduced free group* in k generators.

Let f be a k-string link and $x_i = x_i(f)$, $\forall i \in \underline{k}$, its top meridians, then $R\pi(f)$ denotes the quotient group obtained from $\pi(f)$ by adding relations which say that each $x_i(f)$ commutes with all of its conjugates.

Habegger-Lin [6] introduced an Artin-type theorem for $H(k)$; they showed that if f is a k-string link, $\mu_0(f)$ and $\mu_1(f)$ induce isomorphisms

$$RF(k) \underset{\mu_0'(f)}{\overset{\cong}{\longrightarrow}} R\pi(f) \underset{\mu_1'(f)}{\overset{\cong}{\longleftarrow}} RF(k)$$

that provide an automorphism $\overline{f} = \mu_0'(f)\mu_1'(f)^{-1}$. $\overline{f}$ is called the *Artin automorphism* associated to f.

If α_i denotes also the class of α_i in $RF(k)$, we have (1) $(\alpha_i)\overline{f}$ is a conjugate of α_i and (2) $(\alpha_1\alpha_2\ldots\alpha_k)\overline{f} = \alpha_1\alpha_2\ldots\alpha_k$, that is $\overline{f}$ is a *braid-like automorphism* of $RF(k)$. Automorphisms satisfying condition (1) above are called *special*.

The association $f \mapsto \overline{f}$ is an isomorphism between the group $H(k)$ and the subgroup $A_0(RF(k))$ of all braid-like automorphisms of $RF(k)$.

According to Habegger-Lin [5] the stabilizer of 1_k for both actions of $H(2k)$ on $H(k)$ is the same. Let $S(1_k) = \{\beta \in H(2k) \mid \beta \cdot 1_k = 1_k \in H(k)\} = \{\beta \in H(2k) \mid 1_k \cdot \beta = 1_k \in H(k)\}$ be such stabilizer of 1_k.

Let $RF(k)$ be the reduced free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $RF(2k)$ be the reduced free group is $2k$ generators $\alpha_1, \alpha_2, \ldots, \alpha_k, \widetilde{\alpha_k}, \ldots, \widetilde{\alpha_2}, \widetilde{\alpha_1}$. Let $\xi' : RF(2k) \longrightarrow RF(k)$ be the epimorphism given by, $\forall_i \in \underline{k}$, $(\alpha_i)\xi' = \alpha_i$ and $(\widetilde{\alpha_i})\xi = \alpha_i^{-1}$.

We have shown in [1] that

Theorem 9. *Let $\beta \in H(2k)$. $\beta \in S(1_k)$ if and only if there exists a special automorphism $\overline{\overline{\beta}} : RF(k) \longrightarrow RF(k)$ such that the diagram*

$$\begin{array}{ccc} RF(2k) & \xrightarrow{\overline{\beta}} & RF(2k) \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi'} \\ RF(k) & \xrightarrow{\overline{\overline{\beta}}} & RF(k) \end{array}$$

is commutative.

Theorem 10. *The association $\beta \mapsto \overline{\overline{\beta}}$ from $S(1_k)$ into the group $A(RF(k))$, of all special automorphisms of $RF(k)$, is an epimorphism.*

Proof. It is clearly a homomorphism. On the other side, let $\alpha_{ij} \in A(RF(k))$ be defined by $\alpha_i \mapsto \alpha_j\ \alpha_i\ \alpha_j^{-1}$, $\alpha_s \mapsto \alpha_s$ for $s \neq i$. Then $A(RF(k))$ is generated by $\{\alpha_{ij}\}$, for $1 \leqslant i \neq j \leqslant k$ (see [7]). It follows from [3, Proposition 6] that our map is onto. ∎

We have determined its kernel in [4]

Definition 11. A braid-like automorphism ψ of $RF(2k)$ is a *stabilizing automorphism* if $(\ker \xi')\psi \subseteq \ker \xi'$.

As we saw earlier $g \times 1$ represents the 2k-string link obtained from a k-string link g by adding k straight strings at its end. The Artin automorphism $\overline{g \times 1}$ associated to $g \times 1$ will be denoted by $\overline{g} * 1$.

Theorem 11. *k-string links f and g have link-homotopic closures if and only if there is a braid-like automorphism θ of $RF(2k)$ such that we have commutative diagrams*

$$\begin{array}{ccc} RF(2k) & \xrightarrow{\overline{f}*1\ \theta} & RF(2k) \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle \xi'} \\ RF(k) & \longrightarrow & RF(k) \end{array} \quad \textit{and} \quad \begin{array}{ccc} RF(2k) & \xrightarrow{\theta\ \overline{g}*1} & RF(2k) \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle \xi'} \\ RF(k) & \longrightarrow & RF(k) \end{array}$$

where the bottom maps are special automorphisms.

Proof. If $\hat{f}$ and $\hat{g}$ are link-homotopic, then $\widehat{f^{-1}}$ and $\widehat{g^{-1}}$ are link-homotopic. By Theorem 8, there is a 2k-string link β such that $\beta \cdot 1_k$ link-homotopic to f^{-1} and $\beta^{-1} \cdot 1_k = (1_k \cdot \beta)^{-1}$ is link-homotopic to g. Then we have $f \times 1\ \ \beta \cdot 1_k$ link-homotopic to $f \times 1 \cdot f^{-1}$ that is link-homotopic to 1_k and, similarly, $g^{-1} \times 1\ \beta^{-1} \cdot 1_k$ link-homotopic to 1_k. Therefore $f \times 1 \quad \beta \in S(1_k)$ and $\beta\ \ g \times 1 = (g^{-1} \times 1\ \beta^{-1})^{-1} \in S(1_k)$. By Theorem 9 we have commutative diagrams as stated, where $\theta = \overline{\beta}$.

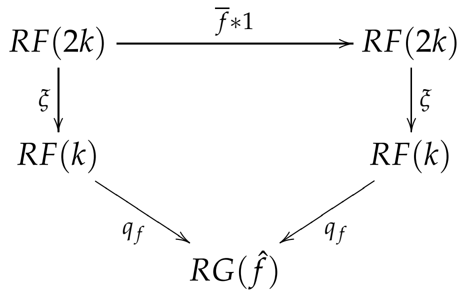
Conversely, suppose we have commutative diagrams

$$\begin{array}{ccc} RF(2k) & \xrightarrow{\overline{f}*1\ \theta} & RF(2k) \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle \xi'} \\ RF(k) & \longrightarrow & RF(k) \end{array} \quad \text{and} \quad \begin{array}{ccc} RF(2k) & \xrightarrow{\theta\ \overline{g}*1} & RF(2k) \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle \xi'} \\ RF(k) & \longrightarrow & RF(k) \end{array}$$

with θ braid-like. Let β be the 2k-string link such that $\overline{\beta} = \theta$. By Theorem 9, $f \times 1\ \ \beta \in S(1_k)$ and $g^{-1} \times 1\ \ \beta^{-1} \in S(1_k)$. Then $f \times 1\ \ \beta \cdot 1_k$ is link-homotopic to 1_k and $g^{-1} \times 1\ \ \beta^{-1} \cdot 1_k$ is link homotopic to 1_k. Then $\beta \cdot 1_k$ is link-homotopic to f^{-1} and $1_k \cdot \beta = (\beta^{-1} \cdot 1_k)^{-1}$ is link-homotopic to g^{-1}. Therefore, by Theorem 8, f^{-1} and g^{-1} have homotopic closures, thus f and g have homotopic closures. ∎

Note that Theorem 11 could be restated as "k-string links f and g have link-homotopic closures if and only if there is a braid-like automorphism θ of $RF(2k)$ such that $\overline{f} * 1\ \theta$ and $\theta\ \overline{g} * 1$ are stabilizing automorphisms of $RF(2k)$."

Given a homotopy class of a k-string link f we have a diagram



That the above diagram is commutative follows from [9]. Such diagram will be called a *group diagram for the homotopy class of* f.

A *braid-special isomorphism* between group diagrams for homotopy classes of k-string links can be defined (see [2, section 4]).

Applying our previous results in the case of link-homotopy we get an alternative proof to the final result of [1] (see also [2, section 4] for a correction), that is

Theorem 12. *Let f and g be k-string links. Then $\hat{f}$ and $\hat{g}$ are link-homotopic if and only if there exists a braid-special isomorphism between the group diagrams for the homotopy classes of f and g.*

Proof. If f and g have link-homotopic closures, from Theorem 8, there is a 2k-string link β such that $\beta \cdot 1_k$ is link-homotopic to f^{-1} and $1_k \cdot \beta$ is link-homotopic to g^{-1}. As in Theorem 4, we obtain a braid-special isomorphism between the group diagrams for the homotopy classes of f and g.

Conversely given a braid-special isomorphism between the group diagrams for the homotopy classes of f and g, we have diagrams as in Theorem 11 for f^{-1} and g^{-1} therefore f and g have link-homotopic closures. ∎

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