The Projective Class Rings of a family of pointed Hopf algebras of Rank two

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Abstract

In this paper, we compute the projective class rings of the tensor product $\mathcal{H}_n(q) = A_n(q) \otimes A_n(q^{-1})$ of Taft algebras $A_n(q)$ and $A_n(q^{-1})$, and its cocycle deformations $H_n(0,q)$ and $H_n(1,q)$, where $n > 2$ is a positive integer and q is a primitive n-th root of unity. It is shown that the projective class rings $r_p(\mathcal{H}_n(q))$, $r_p(H_n(0,q))$ and $r_p(H_n(1,q))$ are commutative rings generated by three elements, three elements and two elements subject to some relations, respectively. It turns out that even $\mathcal{H}_n(q)$, $H_n(0,q)$ and $H_n(1,q)$ are cocycle twist-equivalent to each other, they are of different representation types: wild, wild and tame, respectively.

1 Introduction

Let H be a Hopf algebra over a field $\mathbb{K}$. Doi [18] introduced a cocycle twisted Hopf algebra H^σ for a convolution invertible 2-cocycle σ on H. It is shown in [19, 28] that the Drinfeld double $D(H)$ is a cocycle twisting of the tensor product Hopf algebra $H^{*cop} \otimes H$. The 2-cocycle twisting is extensively employed in various researches. For instance, Andruskiewitsch et al. [1] considered the twists of Nichols algebras associated to racks and cocycles. Guillot, Kassel and Masuoka [21] obtained some examples by twisting comodule algebras by 2-cocycles. It is well known that the monoidal category $\mathcal{M}^H$ of right H-comodules is equivalent to the monoidal category $\mathcal{M}^{H^\sigma}$ of right H^σ-comodules. On the other hand, we

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know that the braided monoidal category ${}_H\mathcal{YD}^H$ of Yetter-Drinfeld H-modules is the center of the monoidal category $\mathcal{M}^H$ for any Hopf algebra H (e.g., see [23]). Hence the monoidal equivalence from $\mathcal{M}^H$ to $\mathcal{M}^{H^\sigma}$ gives rise to a braided monoidal equivalence from ${}_H\mathcal{YD}^H$ to ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$. Chen and Zhang [14] described a braided monoidal equivalent functor from ${}_H\mathcal{YD}^H$ to ${}_{H^\sigma}\mathcal{YD}^{H^\sigma}$. Benkart et al. [3] used a result of Majid and Oeckl [30] to give a category equivalence between Yetter-Drinfeld modules for a finite-dimensional pointed Hopf algebra H and those for its cocycle twisting H^σ. However, the Yetter-Drinfeld module category ${}_H\mathcal{YD}^H$ is also the center of the monoidal category ${}_H\mathcal{M}$ of left H-modules. This gives rise to a natural question:

Is there any relations between the two monoidal categories ${}_H\mathcal{M}$ and ${}_{H^\sigma}\mathcal{M}$ of left modules over two cocycle twist-equivalent Hopf algebras H and H^σ? or how to detect the two monoidal categories ${}_H\mathcal{M}$ and ${}_{H^\sigma}\mathcal{M}$?

This article seeks to address this question through investigating the representation types and projective class rings of a family of pointed Hopf algebras of rank 2, the tensor products of two Taft algebras, and their two cocycle deformations.

In the investigation of the monoidal category of modules over a Hopf algebra H, the decomposition problem of tensor products of indecomposables is of most importance and has received enormous attentions. Our approach is to explore the representation type of H and the projective class ring of H, which is a subring of the representation ring (or Green ring) of H. Originally, the concept of the Green ring $r(H)$ stems from the modular representations of finite groups (see [20], etc.) Since then, there have been plenty of works on the Green rings. For finite-dimensional group algebras, one can refer to [2, 4, 5, 6, 22]. For Hopf algebras and quantum groups, one can see [13, 15, 16, 25, 36, 37].

The n^4-dimensional Hopf algebra $H_n(p,q)$ was introduced in [8], where $n \geqslant 2$ is an integer, $q \in \mathbb{K}$ is a primitive n-th root of unity and $p \in \mathbb{K}$. If $p \neq 0$, then $H_n(p,q)$ is isomorphic to the Drinfeld double $D(A_n(q^{-1}))$ of the Taft algebra $A_n(q^{-1})$. In particular, we have $H_n(p,q) \cong H_n(1,q) \cong D(A_n(q^{-1}))$ for any $p \neq 0$. Moreover, $H_n(p,q)$ is a cocycle deformation of $A_n(q) \otimes A_n(q^{-1})$. For the details, the reader is directed to [8, 9]. When $n = 2$ ($q = -1$), $A_2(-1)$ is exactly the Sweedler 4-dimensional Hopf algebra H_4. Chen studied the finite dimensional representations of $H_n(1,q)$ in [9, 10], and the Green ring $r(D(H_4))$ in [11]. Using a different method, Li and Hu [24] also studied the finite dimensional representations of the Drinfeld double $D(H_4)$, the Green ring $r(D(H_4))$ and the projective class ring $p(D(H_4))$. They also studied two Hopf algebras which are cocycle deformations of $D(H_4)$. By [10], one knows that $D(H_4)$ is of tame representation type. By [24], the two cocycle deformations of $D(H_4)$ are also of tame representation type.

In this paper, we study the three cocycle twist-equivalent Hopf algebras $\mathcal{H}_n(q) = A_n(q) \otimes A_n(q^{-1})$, $H_n(0,q)$ and $H_n(1,q)$ by investigating their representation types and projective class rings, where $n \geqslant 3$. In Section 2, we introduce the Taft algebras $A_n(q)$, the tensor product $\mathcal{H}_n(q) = A_n(q) \otimes A_n(q^{-1})$ and the Hopf algebras $H_n(p,q)$. In Section 3, we first show that $\mathcal{H}_n(q)$ is of wild representation type. With a complete set of orthogonal primitive idempotents, we classify the simple modules and indecomposable projective modules over $\mathcal{H}_n(q)$, and decompose the tensor products of these modules. This leads the description of the projective class ring $r_p(\mathcal{H}_n(q))$, the Jacobson radical $J(R_p(\mathcal{H}_n(q)))$ of the projective class algebra $R_p(\mathcal{H}_n(q))$ and the quotient algebra $R_p(\mathcal{H}_n(q))/J(R_p(\mathcal{H}_n(q)))$. In Section 4, we first show that $H_n(0,q)$ is a symmetric algebra of wild representation type. Then we give a complete set of orthogonal primitive idempotents with the Gabriel quiver, and classify the simple modules and indecomposable projective modules over $H_n(0,q)$. We also describe the projective class ring $r_p(H_n(0,q))$, the Jacobson radical $J(R_p(H_n(0,q)))$ of the projective class algebra $R_p(H_n(0,q))$ and the quotient algebra $R_p(H_n(0,q))/J(R_p(H_n(0,q)))$. In Section 5, using the decompositions of tensor products of indecomposables over $H_n(1,q)$ given in [12], we describe the structure of the projective class ring $r_p(H_n(1,q))$. It is interesting to notice that even the Hopf algebras $\mathcal{H}_n(q)$, $H_n(0,q)$ and $H_n(1,q)$ are cocycle twist-equivalent to each other, they own the different number of blocks with 1, n and $\frac{n(n+1)}{2}$, respectively (see [10, Corollary 2.7] for $H_n(1,q)$). $\mathcal{H}_n(q)$ and $H_n(0,q)$ are basic algebras of wild representation type, but $H_n(1,q)$ is not basic and is of tame representation type. $H_n(0,q)$ and $H_n(1,q)$ are symmetric algebras, but $\mathcal{H}_n(q)$ is not.

2 Preliminaries

Throughout, we work over an algebraically closed field $\mathbb{K}$. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over $\mathbb{K}$; all modules are left modules and finite dimensional; all maps are $\mathbb{K}$-linear; dim and $\otimes$ stand for $\dim_{\mathbb{K}}$ and $\otimes_{\mathbb{K}}$, respectively. Given an algebra A, A-mod denotes the category of finite-dimensional A-modules. For any A-module M and nonnegative integer l, let lM denote the direct sum of l copies of M. For the theory of Hopf algebras and quantum groups, we refer to [23, 29, 31, 34]. Let $\mathbb{Z}$ denote all integers, and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Let H be a Hopf algebra. The Green ring $r(H)$ of H can be defined as follows. $r(H)$ is the abelian group generated by the isomorphism classes $[M]$ of M in H-mod modulo the relations $[M \oplus V] = [M] + [V]$. The multiplication of $r(H)$ is given by the tensor product of H-modules, that is, $[M][V] = [M \otimes V]$. Then $r(H)$ is an associative ring. The projective class ring $r_p(H)$ of H is the subring of $r(H)$ generated by projective modules and simple modules (see [17]). Then the Green algebra $R(H)$ and projective algebra $R_p(H)$ are associative $\mathbb{K}$-algebras defined by $R(H) := \mathbb{K} \otimes_{\mathbb{Z}} r(H)$ and $R_p(H) := \mathbb{K} \otimes_{\mathbb{Z}} r_p(H)$, respectively. Note that $r(H)$ is a free abelian group with a $\mathbb{Z}$-basis $\{[V] | V \in \text{ind}(H)\}$, where $\text{ind}(H)$ denotes the category of finite dimensional indecomposable H-modules.

The Grothendieck ring $G_0(H)$ of H is defined similarly. $G_0(H)$ is the abelian group generated by the isomorphism classes $[M]$ of M in H-mod modulo the relations $[M] = [N] + [V]$ for any short exact sequence $0 \to N \to M \to V \to 0$ in H-mod. The multiplication of $G_0(H)$ is given by the tensor product of H-modules, that is, $[M][V] = [M \otimes V]$. Then $G_0(H)$ is also an associative ring. Moreover, there is a canonical ring epimorphism from $r(H)$ onto $G_0(H)$.

Let $n \geqslant 2$ be an integer and $q \in \mathbb{K}$ a primitive n-th root of unity. Then the n^2-dimensional Taft Hopf algebra $A_n(q)$ is defined as follows (see [35]): as an algebra, $A_n(q)$ is generated by g and x with relations

$$g^n = 1,\ x^n = 0,\ xg = qgx.$$

The coalgebra structure and antipode are given by

$$\triangle(g) = g \otimes g,\ \triangle(x) = x \otimes g + 1 \otimes x,\ \varepsilon(g) = 1,\ \varepsilon(x) = 0,$$
$$S(g) = g^{-1} = g^{n-1},\ S(x) = -xg^{-1} = -q^{-1}g^{n-1}x.$$

Since q^{-1} is also a primitive n-th root of unity, one can define another Taft Hopf algebra $A_n(q^{-1})$, which is generated, as an algebra, by g_1 and x_1 with relations $g_1^n = 1$, $x_1^n = 0$ and $x_1g_1 = q^{-1}g_1x_1$. The coalgebra structure and antipode are given similarly to $A_n(q)$. Then $A_n(q^{-1}) \cong A_n(q)^{\rm op}$ as Hopf algebras.

The first author Chen introduced a Hopf algebra $H_n(p,q)$ in [8], where $p,q \in \mathbb{K}$ and q is a primitive n-th root of unity. It was shown there that $H_n(p,q)$ is isomorphic to a cocycle deformation of the tensor product $A_n(q) \otimes A_n(q^{-1})$.

The tensor product $A_n(q) \otimes A_n(q^{-1})$ can be described as follows. Let $\mathcal{H}_n(q)$ be the algebra generated by a, b, c and d subject to the relations:

$$\begin{array}{lllll} ba = qab, & db = bd, & ca = ac, & dc = qcd, & cb = bc, \\ a^n = 0, & b^n = 1, & c^n = 1, & d^n = 0, & da = ad. \end{array}$$

Then $\mathcal{H}_n(q)$ is a Hopf algebra with the coalgebra structure and antipode given by

$$\begin{array}{lll} \triangle(a) = a \otimes b + 1 \otimes a, & \varepsilon(a) = 0, & S(a) = -ab^{-1} = -ab^{n-1}, \\ \triangle(b) = b \otimes b, & \varepsilon(b) = 1, & S(b) = b^{-1} = b^{n-1}, \\ \triangle(c) = c \otimes c, & \varepsilon(c) = 1, & S(c) = c^{-1} = c^{n-1}, \\ \triangle(d) = d \otimes c + 1 \otimes d, & \varepsilon(d) = 0, & S(d) = -dc^{-1} = -dc^{n-1}. \end{array}$$

It is straightforward to verify that there is a Hopf algebra isomorphism from $\mathcal{H}_n(q)$ to $A_n(q) \otimes A_n(q^{-1})$ via $a \mapsto 1 \otimes x_1$, $b \mapsto 1 \otimes g_1$, $c \mapsto g \otimes 1$ and $d \mapsto x \otimes 1$. Obviously, $\mathcal{H}_n(q)$ is n^4-dimensional with a $\mathbb{K}$-basis $\{a^ib^jc^ld^k | 0 \leqslant i,j,l,k \leqslant n-1\}$.

Let $p \in \mathbb{K}$. Then one can define another n^4-dimensional Hopf algebra $H_n(p,q)$, which is generated as an algebra by a, b, c and d subject to the relations:

$$\begin{array}{lllll} ba = qab, & db = qbd, & ca = qac, & dc = qcd, & bc = cb, \\ a^n = 0, & b^n = 1, & c^n = 1, & d^n = 0, & da - qad = p(1 - bc). \end{array}$$

The coalgebra structure and antipode are defined in the same way as $\mathcal{H}_n(q)$ before. $H_n(p,q)$ has a $\mathbb{K}$-basis $\{a^ib^jc^ld^k | 0 \leqslant i,j,l,k \leqslant n-1\}$. When $p \neq 0$, $H_n(p,q) \cong H_n(1,q) \cong D(A_n(q^{-1}))$ (see [8, 9]). If $n = 2$ $(q = -1)$, then $H_2(1,-1) \cong D(H_4)$, and $H_2(0,-1)$ is exactly the Hopf algebra $\overline{\mathcal{A}}$ in [24].

By [8, Lemma 3.2], there is an invertible skew-pairing $\tau_p : A_n(q) \otimes A_n(q^{-1}) \to \mathbb{K}$ given by $\tau_p(g^ix^j, x_1^kg_1^l) = \delta_{jk}p^jq^{il}(j)!_q$, $0 \leqslant i,j,k,l < n$. Hence one can form a double crossproduct $A_n(q) \bowtie_{\tau_p} A_n(q^{-1})$. Moreover, $A_n(q) \bowtie_{\tau_p} A_n(q^{-1})$ is isomorphic to $H_n(p,q)$ as a Hopf algebra (see [8, Theorem 3.3]). By [19], τ_p induces an invertible 2-cocycle $[\tau_p]$ on $A_n(q) \otimes A_n(q^{-1})$ such that $A_n(q) \bowtie_{\tau_p} A_n(q^{-1}) = (A_n(q) \otimes A_n(q^{-1}))^{[\tau_p]}$. Thus, there is a corresponding invertible 2-cocycle σ_p on

$\mathcal{H}_n(q)$ such that $\mathcal{H}_n(q)^{\sigma_p} \cong H_n(p,q)$ as Hopf algebras. In particular, we have $\mathcal{H}_n(q)^{\sigma_0} \cong H_n(0,q)$ and $\mathcal{H}_n(q)^{\sigma_1} \cong H_n(1,q)$. In general, if σ is a convolution invertible 2-cocycle on a Hopf algebra H, then σ^{-1} is an invertible 2-cocycle on H^σ and $(H^\sigma)^{\sigma^{-1}} = H$ (see [7, Lemma 1.2]). More generally, if σ is an invertible 2-cocycle on H and τ is an invertible 2-cocycle on H^σ, then $\tau * \sigma$ is an invertible 2-cocycle on H and $H^{\tau*\sigma} = (H^\sigma)^\tau$ (see [7, Lemma 1.4]). Thus, the Hopf algebras $\mathcal{H}_n(q)$, $H_n(0,q)$ and $H_n(1,q)$ are cocycle twist-equivalent to each other.

Throughout the following, fix an integer $n > 2$ and let $q \in \mathbb{K}$ be a primitive n-th root of unity. For any $m \in \mathbb{Z}$, denote still by m the image of m under the canonical projection $\mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

3 The Projective Class Ring of $\mathcal{H}_n(q)$

In this section, we investigate the representations and the projective class ring of $\mathcal{H}_n(q)$, or equivalently, of $A_n(q) \otimes A_n(q^{-1})$.

Let A be the subalgebra of $\mathcal{H}_n(q)$ generated by a and d. Then A is isomorphic to the quotient algebra $\mathbb{K}[x,y]/(x^n,y^n)$ of the polynomial algebra $\mathbb{K}[x,y]$ modulo the ideal (x^n,y^n) generated by x^n and y^n. Let $G = G(\mathcal{H}_n(q))$ be the group of group-like elements of $\mathcal{H}_n(q)$. Then $G = \{b^ic^j | i,j \in \mathbb{Z}_n\} \cong \mathbb{Z}_n \times \mathbb{Z}_n$, and $\mathbb{K}G = \mathcal{H}_n(q)_0$, the coradical of $\mathcal{H}_n(q)$. Clearly, A is a left $\mathbb{K}G$-module algebra with the action given by $b \cdot a = qa$, $b \cdot d = d$, $c \cdot a = a$ and $c \cdot d = q^{-1}d$. Hence one can form a smash product algebra $A\#\mathbb{K}G$. It is easy to see that $\mathcal{H}_n(q)$ is isomorphic to $A\#\mathbb{K}G$ as an algebra. Since $n \geqslant 3$, it follows from [33, p.295(3.4)] that A is of wild representation type. Since char$(\mathbb{K}) \nmid |G|$, $\mathbb{K}G$ is a semisimple and cosemisimple Hopf algebra. It follows from [26, Theorem 4.5] that $A\#\mathbb{K}G$ is of wild representation type. As a consequence, we obtain the following result.

Proposition 3.1. *$\mathcal{H}_n(q)$ is of wild representation type.*

$\mathcal{H}_n(q)$ has n^2 orthogonal primitive idempotents

$$e_{i,j} = \tfrac{1}{n^2}\textstyle\sum_{k,l\in\mathbb{Z}_n} q^{-ik-jl}b^kc^l = \tfrac{1}{n^2}\sum_{k,l=0}^{n-1} q^{-ik-jl}b^kc^l, \ \ i,j \in \mathbb{Z}_n.$$

Lemma 3.2. *Let $i,j \in \mathbb{Z}_n$. Then*

$$be_{i,j} = q^ie_{i,j},\ ce_{i,j} = q^je_{i,j},\ ae_{i,j} = e_{i+1,j}a,\ de_{i,j} = e_{i,j-1}d.$$

Proof. It follows from a straightforward verification. ∎

For $i,j \in \mathbb{Z}_n$, let $S_{i,j}$ be the one dimensional $\mathcal{H}_n(q)$-module defined by $bv = q^iv$, $cv = q^jv$ and $av = dv = 0$, $v \in S_{i,j}$. Let $P_{i,j} = P(S_{i,j})$ be the projective cover of $S_{i,j}$. Let $J = \mathrm{rad}(\mathcal{H}_n(q))$ be the Jacobson radical of $\mathcal{H}_n(q)$.

Lemma 3.3. *The simple modules $S_{i,j}$, $i, j \in \mathbb{Z}_n$, exhaust all simple modules of $\mathcal{H}_n(q)$, and consequently, the projective modules $P_{i,j}$, $i, j \in \mathbb{Z}_n$, exhaust all indecomposable projective modules of $\mathcal{H}_n(q)$. Moreover, $P_{i,j} \cong \mathcal{H}_n(q)e_{i,j}$ for all $i, j \in \mathbb{Z}_n$.*

Proof. Obviously, $a\mathcal{H}_n(q) = \mathcal{H}_n(q)a$ and $d\mathcal{H}_n(q) = \mathcal{H}_n(q)d$. Since $a^n = 0$ and $d^n = 0$, $\mathcal{H}_n(q)a + \mathcal{H}_n(q)d$ is a nilpotent ideal of $\mathcal{H}_n(q)$. Hence $\mathcal{H}_n(q)a + \mathcal{H}_n(q)d \subseteq J$. On the other hand, it is easy to see that the quotient algebra $\mathcal{H}_n(q)/(\mathcal{H}_n(q)a + \mathcal{H}_n(q)d)$ is isomorphic to the group algebra $\mathbb{K}G$, where $G = G(\mathcal{H}_n(q)) = \{b^i c^j | 0 \leqslant i, j \leqslant n-1\}$, the group of all group-like elements of $\mathcal{H}_n(q)$. Since $\mathbb{K}G$ is semisimple, $J \subseteq \mathcal{H}_n(q)a + \mathcal{H}_n(q)d$. Thus, $J = \mathcal{H}_n(q)a + \mathcal{H}_n(q)d$. Therefore, the simple modules $S_{i,j}$ exhaust all simple modules of $\mathcal{H}_n(q)$, and the projective modules $P_{i,j}$ exhaust all indecomposable projective modules of $\mathcal{H}_n(q)$, $i, j \in \mathbb{Z}_n$. The last statement of the lemma follows from Lemma 3.2. ■

Corollary 3.4. *$\mathcal{H}_n(q)$ is a basic algebra. Moreover, J is a Hopf ideal of $\mathcal{H}_n(q)$, and the Loewy length of $\mathcal{H}_n(q)$ is $2n-1$.*

Proof. It follows from Lemma 3.3 that $\mathcal{H}_n(q)$ is a basic algebra. By $J = \mathcal{H}_n(q)a + \mathcal{H}_n(q)d$, one can easily check that J is a coideal and $S(J) \subseteq J$. Hence J is a Hopf ideal. By $a^{n-1} \neq 0$ and $d^{n-1} \neq 0$, one gets $(\mathcal{H}_n(q)a + \mathcal{H}_n(q)d)^{2n-2} \neq 0$. By $a^n = d^n = 0$, one gets $(\mathcal{H}_n(q)a + \mathcal{H}_n(q)d)^{2n-1} = 0$. It follows that the Loewy length of $\mathcal{H}_n(q)$ is $2n-1$. ■

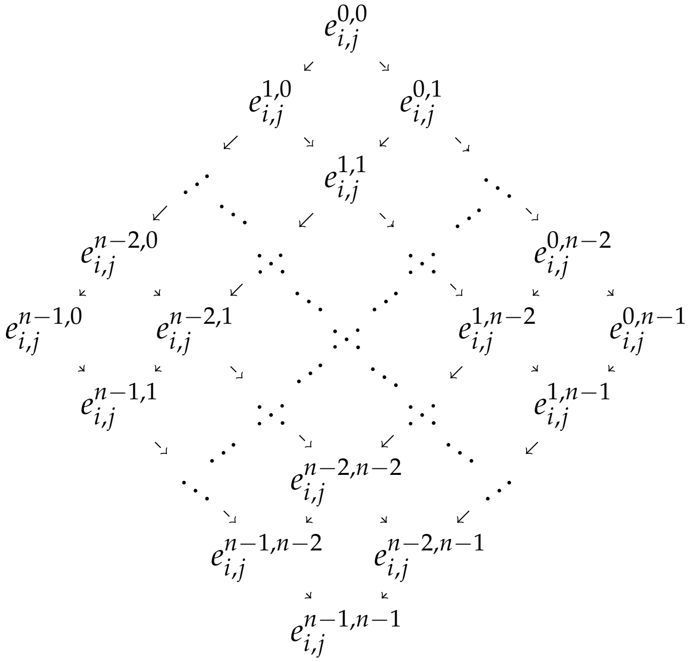
In the rest of this section, we regard that $P_{i,j} = \mathcal{H}_n(q)e_{i,j}$ for all $i, j \in \mathbb{Z}_n$.

Corollary 3.5. *$P_{i,j}$ is n^2-dimensional with a $\mathbb{K}$-basis $\{a^k d^l e_{i,j} | 0 \leqslant k, l \leqslant n-1\}$, $i, j \in \mathbb{Z}_n$. Consequently, $\mathcal{H}_n(q)$ is an indecomposable algebra.*

Proof. By Lemma 3.2, $P_{i,j} = \text{span}\{a^k d^l e_{i,j} | 0 \leqslant k, l \leqslant n-1\}$, and hence $\dim P_{i,j} \leqslant n^2$. Now it follows from $\mathcal{H}_n(q) = \oplus_{i,j\in\mathbb{Z}_n}\mathcal{H}_n(q)e_{i,j}$ and $\dim\mathcal{H}_n(q) = n^4$ that $P_{i,j}$ is n^2-dimensional over $\mathbb{K}$ with a basis $\{a^k d^l e_{i,j} | 0 \leqslant k, l \leqslant n-1\}$. Then by Lemmas 3.2-3.3, one knows that every simple module is a simple factor of $P_{i,j}$ with the multiplicity one. Consequently, $\mathcal{H}_n(q)$ is an indecomposable algebra. ■

Given $M \in \mathcal{H}_n(q)$-mod, for any $\alpha \in \mathbb{K}$ and $u, v \in M$, we use $u \xrightarrow{\alpha} v$ (resp. $u \overset{\alpha}{\dashrightarrow} v$) to represent $a \cdot u = \alpha v$ (resp. $d \cdot u = \alpha v$). Moreover, we omit the decoration of the arrow if $\alpha = 1$.

For $i, j \in \mathbb{Z}_n$, let $e_{i,j}^{k,l} = a^k d^l e_{i,j}$ in $P_{i,j}$, $0 \leqslant k, l \leqslant n-1$. Then the structure of $P_{i,j}$ can be described as follows:



Proposition 3.6. *$S_{i,j} \otimes S_{k,l} \cong S_{i+k,j+l}$ and $S_{i,j} \otimes P_{k,l} \cong P_{k,l} \otimes S_{i,j} \cong P_{i+k,j+l}$ for all $i, j, k, l \in \mathbb{Z}_n$.*

Proof. The first isomorphism is obvious. Note that $S_{0,0}$ is the trivial $\mathcal{H}_n(q)$-module. Since J is a Hopf ideal, it follows from [27, Corollary 3.3] and the first isomorphism that $P_{k,l} \otimes S_{i,j} \cong P_{0,0} \otimes S_{k,l} \otimes S_{i,j} \cong P_{0,0} \otimes S_{i+k,j+l} \cong P_{i+k,j+l}$. Similarly, one can show that $S_{i,j} \otimes P_{k,l} \cong P_{i+k,j+l}$, which also follows from the proof of [17, Lemma 3.3]. ■

Proposition 3.7. *Let $i, j, k, l \in \mathbb{Z}_n$. Then $P_{i,j} \otimes P_{k,l} \cong \oplus_{r,t \in \mathbb{Z}_n} P_{r,t}$.*

Proof. By Proposition 3.6, we only need to consider the case of $i = j = k = l = 0$. For any short exact sequence $0 \to N \to M \to L \to 0$ of modules, the exact sequence $0 \to P_{0,0} \otimes N \to P_{0,0} \otimes M \to P_{0,0} \otimes L \to 0$ is always split since $P_{0,0} \otimes L$ is projective for any module L. By Corollary 3.4 and the proof of Corollary 3.5, $[P_{0,0}] = \sum_{r,t \in \mathbb{Z}_n} [S_{r,t}]$ in $G_0(\mathcal{H}_n(q))$. Then it follows from Proposition 3.6 that $P_{0,0} \otimes P_{0,0} \cong \oplus_{r,t \in \mathbb{Z}_n} P_{0,0} \otimes S_{r,t} \cong \oplus_{r,t \in \mathbb{Z}_n} P_{r,t}$, which is isomorphic to the regular module $\mathcal{H}_n(q)$. ■

By Propositions 3.6 and 3.7, the projective class ring $r_p(\mathcal{H}_n(q))$ is a commutative ring generated by $[S_{1,0}]$, $[S_{0,1}]$ and $[P_{0,0}]$ subject to the relations $[S_{1,0}]^n = 1$, $[S_{0,1}]^n = 1$ and $[P_{0,0}]^2 = \sum_{i,j=0}^{n-1} [S_{1,0}]^i [S_{0,1}]^j [P_{0,0}]$. Hence we have the following proposition.

Theorem 3.8. *$r_p(\mathcal{H}_n(q)) \cong \mathbb{Z}[x, y, z]/(x^n - 1, y^n - 1, z^2 - \sum_{i,j=0}^{n-1} x^i y^j z)$.*

Proof. By Propositions 3.6 and 3.7, $r_p(\mathcal{H}_n(q))$ is a commutative ring. Moreover, $r_p(\mathcal{H}_n(q))$ is generated, as a $\mathbb{Z}$-algebra, by $[S_{1,0}]$, $[S_{0,1}]$ and $[P_{0,0}]$. Therefore, there exists a ring epimorphism $\phi : \mathbb{Z}[x, y, z] \to r_p(\mathcal{H}_n(q))$ such that $\phi(x) = [S_{1,0}]$, $\phi(y) = [S_{0,1}]$ and $\phi(z) = [P_{0,0}]$. Let $I = (x^n - 1, y^n - 1, z^2 - \sum_{i,j=0}^{n-1} x^i y^j z)$ be the ideal of $\mathbb{Z}[x,y,z]$ generated by x^n-1, y^n-1 and $z^2-\sum_{i,j=0}^{n-1}x^iy^jz$. Then it follows from Propositions 3.6 and 3.7 that $I\subseteq \mathrm{Ker}(\phi)$. Hence ϕ induces a ring epimorphism $\overline{\phi}:\mathbb{Z}[x,y,z]/I\to r_p(\mathcal{H}_n(q))$ such that $\overline{\phi}\circ\pi=\phi$, where $\pi:\mathbb{Z}[x,y,z]\to\mathbb{Z}[x,y,z]/I$ is the canonical projection. Let $\overline{u}=\pi(u)$ for any $u\in\mathbb{Z}[x,y,z]$. Then $\overline{x}^n=1$, $\overline{y}^n=1$ and $\overline{z}^2=\sum_{i,j=0}^{n-1}\overline{x}^i\overline{y}^j\overline{z}$ in $\mathbb{Z}[x,y,z]/I$. Hence $\mathbb{Z}[x,y,z]/I$ is generated, as a $\mathbb{Z}$-module, by $\{\overline{x}^i\overline{y}^j,\overline{x}^i\overline{y}^j\overline{z}|i,j\in\mathbb{Z}_n\}$. Since $r_p(\mathcal{H}_n(q))$ is a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $\{[S_{i,j}],[P_{i,j}]|i,j\in\mathbb{Z}_n\}$, one can define a $\mathbb{Z}$-module map $\psi:r_p(\mathcal{H}_n(q))\to\mathbb{Z}[x,y,z]/I$ by $\psi([S_{i,j}])=\overline{x}^i\overline{y}^j$ and $\psi([P_{i,j}])=\overline{x}^i\overline{y}^j\overline{z}$ for any $i,j\in\mathbb{Z}_n$. Now for any $i,j\in\mathbb{Z}_n$, we have $\psi(\overline{\phi}(\overline{x}^i\overline{y}^j))=\psi(\overline{\phi}(\overline{x})^i\overline{\phi}(\overline{y})^j)=\psi([S_{1,0}]^i[S_{0,1}]^j)=\psi([S_{i,j}])=\overline{x}^i\overline{y}^j$ and $\psi(\overline{\phi}(\overline{x}^i\overline{y}^j\overline{z}))=\psi(\overline{\phi}(\overline{x})^i\overline{\phi}(\overline{y})^j\overline{\phi}(\overline{z}))=\psi([S_{1,0}]^i[S_{0,1}]^j[P_{0,0}])=\psi([P_{i,j}])=\overline{x}^i\overline{y}^j\overline{z}$. This shows that $\overline{\phi}$ is injective, and so $\overline{\phi}$ is a ring isomorphism. ∎

Now we consider the projective class algebra $R_p(\mathcal{H}_n(q))$. By Theorem 3.8, we have

$$R_p(\mathcal{H}_n(q))\cong\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-\textstyle\sum_{i,j=0}^{n-1}x^iy^jz).$$

Put $I=(x^n-1,y^n-1,z^2-\sum_{i,j=0}^{n-1}x^iy^jz)$ and let $J(\mathbb{K}[x,y,z]/I)$ be the Jacobson radical of $\mathbb{K}[x,y,z]/I$. For any $u\in\mathbb{K}[x,y,z]$, let $\overline{u}$ denote the image of u under the canonical projection $\mathbb{K}[x,y,z]\to\mathbb{K}[x,y,z]/I$. Then by the proof of Theorem 3.8, $\mathbb{K}[x,y,z]/I$ is of dimension $2n^2$ with a $\mathbb{K}$-basis $\{\overline{x}^i\overline{y}^j,\overline{x}^i\overline{y}^j\overline{z}|0\leqslant i,j\leqslant n-1\}$. From $\overline{x}^n=1$, $\overline{y}^n=1$ and $\overline{z}^2=\sum_{i,j=0}^{n-1}\overline{x}^i\overline{y}^j\overline{z}$, one gets $(1-\overline{x})\overline{z}^2=(1-\overline{y})\overline{z}^2=0$, and so $((1-\overline{x})\overline{z})^2=((1-\overline{y})\overline{z})^2=0$. Consequently, the ideal $((1-\overline{x})\overline{z},(1-\overline{y})\overline{z})$ of $\mathbb{K}[x,y,z]/I$ generated by $(1-\overline{x})\overline{z}$ and $(1-\overline{y})\overline{z}$ is contained in $J(\mathbb{K}[x,y,z]/I)$. Moreover, $\dim((\mathbb{K}[x,y,z]/I)/((1-\overline{x})\overline{z},(1-\overline{y})\overline{z})=n^2+1$ and

$$\begin{aligned}&(\mathbb{K}[x,y,z]/I)/((1-\overline{x})\overline{z},(1-\overline{y})\overline{z})\\ \cong\ &\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-n^2z,(1-x)z,(1-y)z).\end{aligned}$$

Let $\pi:\mathbb{K}[x,y,z]\to\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-n^2z,(1-x)z,(1-y)z)$ be the canonical projection. For any integers $k,l\geqslant 0$, let $f_{k,l}=\frac{1}{n^2}\sum_{i,j=0}^{n-1}q^{ki+lj}x^iy^j$ in $\mathbb{K}[x,y,z]$. Then a straightforward verification shows that

$$\{\pi(f_{k,l}),\pi(f_{0,k}),\pi(f_{0,0}-\tfrac{1}{n^2}z),\pi(\tfrac{1}{n^2}z)|1\leqslant k\leqslant n-1,0\leqslant l\leqslant n-1\}$$

is a set of orthogonal idempotents, and so it is a full set of orthogonal primitive idempotents in $\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-n^2z,(1-x)z,(1-y)z)$. Therefore,

$$\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-n^2z,(1-x)z,(1-y)z)\cong\mathbb{K}^{n^2+1}.$$

Thus, $J(\mathbb{K}[x,y,z]/I)\subseteq((1-\overline{x})\overline{z},(1-\overline{y})\overline{z})$, and so $J(\mathbb{K}[x,y,z]/I)=((1-\overline{x})\overline{z},(1-\overline{y})\overline{z})$. This shows the following proposition.

Proposition 3.9. *Let $J(R_p(\mathcal{H}_n(q)))$ be the Jacobson radical of $R_p(\mathcal{H}_n(q))$. Then $J(R_p(\mathcal{H}_n(q)))=((1-[S_{1,0}])[P_{0,0}],(1-[S_{0,1}])[P_{0,0}])$ and*

$$\begin{aligned}&R_p(\mathcal{H}_n(q))/J(R_p(\mathcal{H}_n(q)))\\ \cong\ &\mathbb{K}[x,y,z]/(x^n-1,y^n-1,z^2-n^2z,(1-x)z,(1-y)z)\cong\mathbb{K}^{n^2+1}.\end{aligned}$$

4 The Projective Class Ring of $H_n(0,q)$

In this section, we investigate the projective class ring of $H_n(0,q)$.

Proposition 4.1. *$H_n(0,q)$ is a symmetric algebra.*

Proof. By [8, Proposition 3.4] and its proof, $H_n(0,q)$ is unimodular. Moreover, $S^2(a) = qa$, $S^2(b) = b$, $S^2(c) = c$ and $S^2(d) = q^{-1}d$, where S is the antipode of $\mathcal{H}_n(0,q)$. Hence $S^2(x) = bxb^{-1} = cxc^{-1}$ for all $x \in H_n(0,q)$. That is, S^2 is an inner automorphism of $H_n(0,q)$. It follows from [27, 32] that $H_n(0,q)$ is a symmetric algebra. ■

Note that $\mathcal{H}_n(q)$ is not symmetric since it is not unimodular.

Proposition 4.2. *$H_n(0,q)$ is of wild representation type.*

Proof. It is similar to Proposition 3.1. Let A be the subalgebra of $H_n(0,q)$ generated by a and d. Then A is a $\mathbb{K}G$-module algebra with the action given by $b \cdot a = qa$, $b \cdot d = q^{-1}d$, $c \cdot a = qa$ and $c \cdot d = q^{-1}d$, where $G = G(H_n(0,q)) = \{b^i c^j | i,j \in \mathbb{Z}_n\} \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Moreover, $A \cong \mathbb{K}\langle x,y\rangle/(x^n, y^n, yx - qxy)$ and $H_n(0,q) \cong A\#\mathbb{K}G$, as $\mathbb{K}$-algebras. Since $n \geqslant 3$, it follows from [33, p.295(3.4)] that A is of wild representation type. Since $\mathbb{K}G$ is a semisimple and cosemisimple Hopf algebra by char$(\mathbb{K}) \nmid |G|$, it follows from [26, Theorem 4.5] that $A\#\mathbb{K}G$ is of wild representation type. ■

$H_n(0,q)$ has n^2 orthogonal primitive idempotents

$$e_{i,j} = \tfrac{1}{n^2}\textstyle\sum_{k,l\in\mathbb{Z}_n} q^{-ik-jl}b^k c^l = \tfrac{1}{n^2}\sum_{k,l=0}^{n-1} q^{-ik-jl}b^k c^l, \ \ i,j \in \mathbb{Z}_n.$$

Lemma 4.3. *Let $i,j \in \mathbb{Z}_n$. Then*

$$be_{i,j} = q^i e_{i,j},\ ce_{i,j} = q^j e_{i,j},\ ae_{i,j} = e_{i+1,j+1}a,\ de_{i,j} = e_{i-1,j-1}d.$$

Proof. It follows from a straightforward verification. ■

For $i,j \in \mathbb{Z}_n$, let $S_{i,j}$ be the one dimensional $H_n(0,q)$-module defined by $bv = q^i v$, $cv = q^j v$ and $av = dv = 0$, $v \in S_{i,j}$. Let $P_{i,j} = P(S_{i,j})$ be the projective cover of $S_{i,j}$. Let $J = \mathrm{rad}(H_n(0,q))$ be the Jacobson radical of $H_n(0,q)$.

Lemma 4.4. *The simple modules $S_{i,j}$, $i,j \in \mathbb{Z}_n$, exhaust all simple modules of $H_n(0,q)$, and consequently, the projective modules $P_{i,j}$, $i,j \in \mathbb{Z}_n$, exhaust all indecomposable projective modules of $H_n(0,q)$. Moreover, $P_{i,j} \cong H_n(0,q)e_{i,j}$ for all $i,j \in \mathbb{Z}_n$.*

Proof. It is similar to Lemma 3.3. ■

Corollary 4.5. *$H_n(0,q)$ is a basic algebra. Moreover, J is a Hopf ideal of $H_n(0,q)$, and the Loewy length of $H_n(0,q)$ is $2n-1$.*

Proof. It is similar to Corollary 3.4. ■

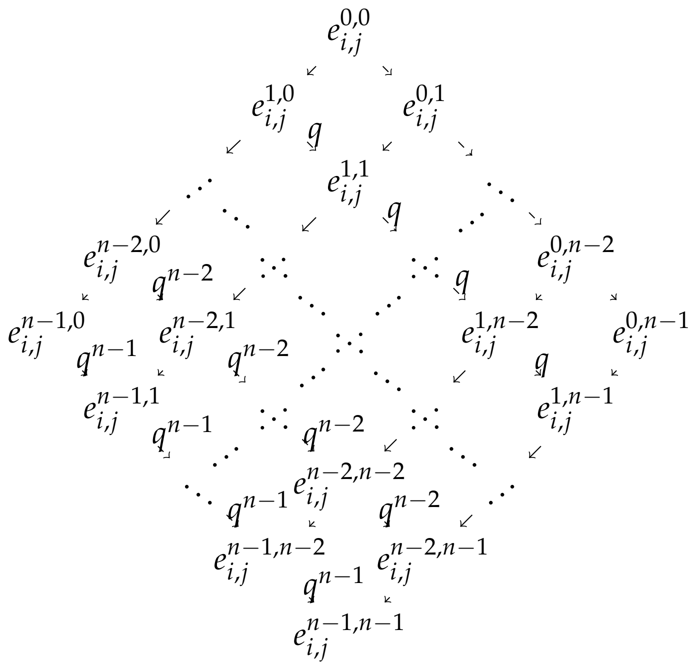
Let $e_i = \sum_{j=0}^{n-1} e_{i+j,j} = \frac{1}{n}\sum_{j=0}^{n-1} q^{-ij}b^j c^{-j}$, $i \in \mathbb{Z}_n$. Then by Lemmas 4.3 and 4.4, $\{e_i | i \in \mathbb{Z}_n\}$ is a full set of central primitive idempotents of $H_n(0,q)$. Hence $H_n(0,q)$ decomposes into n blocks $H_n(0,q)e_i$, $i \in \mathbb{Z}_n$.

In the rest of this section, we regard that $P_{i,j} = H_n(0,q)e_{i,j}$ for all $i, j \in \mathbb{Z}_n$.

Corollary 4.6. *$P_{i,j}$ is n^2-dimensional with a $\mathbb{K}$-basis $\{a^k d^l e_{i,j} | 0 \leqslant k, l \leqslant n-1\}$, $i, j \in \mathbb{Z}_n$.*

Proof. It is similar to Corollary 3.5. ∎

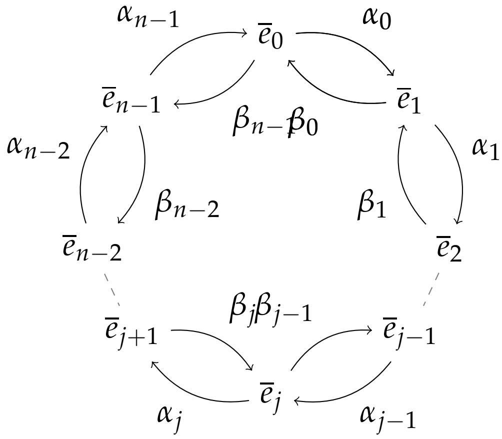
For $i, j \in \mathbb{Z}_n$, let $e_{i,j}^{k,l} = a^k d^l e_{i,j}$ in $P_{i,j}$. Using the same symbols as in the last section, the structure of $P_{i,j}$ can be described as follows:



Proposition 4.7. *The n blocks $H_n(0,q)e_i$, $i \in \mathbb{Z}_n$, are isomorphic to each other.*

Proof. Let $i \in \mathbb{Z}_n$. Since $e_i = \sum_{j=0}^{n-1} e_{i+j,j}$, $H_n(0,q)e_i = \oplus_{j=0}^{n-1} H_n(0,q)e_{i+j,j}$ as $H_n(0,q)$-modules. Then by Corollary 4.6, $\dim(H_n(0,q)e_i) = n^3$. By Lemma 4.3, one gets $be_i = q^i ce_i$. It follows that $H_n(0,q)e_i = \mathrm{span}\{a^j d^k b^l e_i | 0 \leqslant j, k, l\}$, and so $\{a^j d^k b^l e_i | 0 \leqslant j, k, l\}$ is a $\mathbb{K}$-basis of $H_n(0,q)e_i$. Let B be the subalgebra of $H_n(q)$ generated by a, b and d. Then one can easily check that the block $H_n(0,q)e_i$ is isomorphic, as an algebra, to the subalgebra B of $H_n(0,q)$. Thus, the proposition follows. ∎

Let $i \in \mathbb{Z}_n$ be fixed. For any $j \in \mathbb{Z}_n$, let $\overline{e}_j = e_{i+j,j}$. Then the Gabriel quiver $Q = (Q_0, Q_1)$ of the block $H_n(0,q)e_i$ is given by



where for $j \in \mathbb{Z}_n$, the arrows α_j, β_j correspond to $a\overline{e}_j$, $d\overline{e}_{j+1}$, respectively. The admissible ideal I has the following relations:

$$\beta_j\alpha_j - q\alpha_{j-1}\beta_{j-1} = 0,\ \alpha_{j+(n-1)} \cdots \alpha_{j+1}\alpha_j = 0,\ \beta_{j-(n-1)} \cdots \beta_{j-1}\beta_j = 0,\ j \in \mathbb{Z}_n.$$

Proposition 4.8. *$S_{i,j} \otimes S_{k,l} \cong S_{i+k,j+l}$ and $S_{i,j} \otimes P_{k,l} \cong P_{k,l} \otimes S_{i,j} \cong P_{i+k,j+l}$ for all $i,j,k,l \in \mathbb{Z}_n$.*

Proof. It is similar to Proposition 3.6. ∎

Proposition 4.9. *Let $i,j,k,l \in \mathbb{Z}_n$. Then $P_{i,j} \otimes P_{k,l} \cong \oplus_{t\in\mathbb{Z}_n} nP_{i+k+t,j+l+t}$.*

Proof. It is similar to Proposition 3.7. Note that $[P_{0,0}] = \sum_{t=0}^{n-1} n[S_{t,t}]$ in $G_0(H_n(0,q))$ by Corollaries 4.5 and 4.6. ∎

Theorem 4.10. *$r_p(H_n(0,q)) \cong \mathbb{Z}[x,y,z]/(x^n - 1, y^n - 1, z^2 - n\sum_{i=0}^{n-1} x^i z)$.*

Proof. It is similar to Theorem 3.8. Note that $r_p(H_n(0,q))$ is a commutative ring generated by $[S_{1,1}]$, $[S_{0,1}]$ and $[P_{0,0}]$. ∎

Now we consider the projective class algebra $R_p(H_n(0,q))$. By Theorem 4.10, we have

$$R_p(H_n(0,q)) \cong \mathbb{K}[x,y,z]/(x^n - 1, y^n - 1, z^2 - n\textstyle\sum_{i=0}^{n-1} x^i z).$$

Put $I = (x^n - 1, y^n - 1, z^2 - n\sum_{i=0}^{n-1} x^i z)$ and let $J(\mathbb{K}[x,y,z]/I)$ be the Jacobson radical of $\mathbb{K}[x,y,z]/I$. For any $u \in \mathbb{K}[x,y,z]$, let $\overline{u}$ denote the image of u under the canonical projection $\mathbb{K}[x,y,z] \to \mathbb{K}[x,y,z]/I$. Then by Theorem 4.10, $\mathbb{K}[x,y,z]/I$ is of dimension $2n^2$ with a $\mathbb{K}$-basis $\{\overline{x}^i\overline{y}^j, \overline{x}^i\overline{y}^j\overline{z} | i,j \in \mathbb{Z}_n\}$. Since $\overline{x}^n = 1$ and $\overline{z}^2 = n\sum_{i=0}^{n-1} \overline{x}^i\overline{z}$, one gets $(1-\overline{x})\overline{z}^2 = 0$, and so $((1-\overline{x})\overline{z})^2 = 0$. Consequently, the ideal $((1-\overline{x})\overline{z})$ of $\mathbb{K}[x,y,z]/I$ generated by $(1-\overline{x})\overline{z}$ is contained in $J(\mathbb{K}[x,y,z]/I)$. Moreover, $\dim((\mathbb{K}[x,y,z]/I)/((1-\overline{x})\overline{z})) = n(n+1)$ and

$$(\mathbb{K}[x,y,z]/I)/((1-\overline{x})\overline{z}) \cong \mathbb{K}[x,y,z]/(x^n - 1, y^n - 1, z^2 - n^2 z, (1-x)z).$$

Let $\pi : \mathbb{K}[x,y,z] \to \mathbb{K}[x,y,z]/(x^n-1, y^n-1, z^2-n^2z, (1-x)z)$ be the canonical projection. For any integer $k \geqslant 0$, let $f_k = \frac{1}{n}\sum_{i=0}^{n-1} q^{ki}x^i$ and $g_k = \frac{1}{n}\sum_{i=0}^{n-1} q^{ki}y^i$ in $\mathbb{K}[x,y,z]$. Then a straightforward verification shows that

$$\{\pi(f_k g_l), \pi((f_0 - \tfrac{1}{n^2}z)g_l), \pi(\tfrac{1}{n^2}zg_l) | 1 \leqslant k \leqslant n-1, 0 \leqslant l \leqslant n-1\}$$

is a set of orthogonal idempotents, and so it is a full set of orthogonal primitive idempotents in $\mathbb{K}[x,y,z]/(x^n-1, y^n-1, z^2-n^2z, (1-x)z)$. Therefore,

$$\mathbb{K}[x,y,z]/(x^n-1, y^n-1, z^2-n^2z, (1-x)z) \cong \mathbb{K}^{n(n+1)}.$$

It follows that $J(\mathbb{K}[x,y,z]/I) \subseteq ((1-\overline{x})\overline{z})$, and so $J(\mathbb{K}[x,y,z]/I) = ((1-\overline{x})\overline{z})$. This shows the following proposition.

Proposition 4.11. *Let $J(R_p(H_n(0,q)))$ be the Jacobson radical of $R_p(H_n(0,q))$. Then $J(R_p(H_n(0,q))) = ((1-[S_{1,1}])[P_{0,0}])$ and*

$$\begin{aligned} & R_p(H_n(0,q))/J(R_p(H_n(0,q))) \\ \cong\ & \mathbb{K}[x,y,z]/(x^n-1, y^n-1, z^2-n^2z, (1-x)z) \cong \mathbb{K}^{n(n+1)}. \end{aligned}$$

5 The Projective Class Ring of $H_n(1,q)$

In this section, we will study the projective class ring of $H_n(1,q)$. The finite dimensional indecomposable $H_n(1,q)$-modules are classified in [9, 10]. There are n^2 simple modules $V(l,r)$ over $H_n(1,q)$, where $1 \leqslant l \leqslant n$ and $r \in \mathbb{Z}_n$. The simple modules $V(n,r)$ are both projective and injective. Let $P(l,r)$ be the projective cover of $V(l,r)$. Then $P(l,r)$ is the injective envelope of $V(l,r)$ as well. Moreover, $P(n,r) \cong V(n,r)$.

Note that $M \otimes N \cong N \otimes M$ for any modules M and N since $H_n(1,q)$ is a quasitriangular Hopf algebra. For any $t \in \mathbb{Z}$, let $c(t) := [\frac{t+1}{2}]$ be the integer part of $\frac{t+1}{2}$. That is, $c(t)$ is the maximal integer with respect to $c(t) \leqslant \frac{t+1}{2}$. Then $c(t) + c(t-1) = t$.

Convention: If $\oplus_{l \leqslant i \leqslant m} M_i$ is a term in a decomposition of a module, then it disappears when $l > m$.

Lemma 5.1. *Let $1 \leqslant l, l' \leqslant n$ and $r, r' \in \mathbb{Z}_n$.*
(1) $V(1,r) \otimes V(l,r') \cong V(l, r+r')$.
(2) $V(1,r) \otimes P(l,r') \cong P(l, r+r')$.
(3) If $l \leqslant l'$ and $l + l' \leqslant n+1$, then $V(l,r) \otimes V(l',r') \cong \oplus_{i=0}^{l-1} V(l+l'-1-2i, r+r'+i)$.
(4) If $l \leqslant l'$ and $t = l + l' - (n+1) > 0$, then

$$\begin{aligned} V(l,r) \otimes V(l',r') \ \cong\ & (\oplus_{i=c(t)}^{t} P(l+l'-1-2i, r+r'+i)) \\ & \oplus(\oplus_{t+1 \leqslant i \leqslant l-1} V(l+l'-1-2i, r+r'+i)). \end{aligned}$$

(5) If $l \leqslant l' < n$ and $l + l' \leqslant n$, then $V(l,r) \otimes P(l',r') \cong \oplus_{i=0}^{l-1} P(l+l'-1-2i, r+r'+i)$.
(6) If $l \leqslant l' < n$ and $t = l + l' - (n+1) \geqslant 0$, then

$$\begin{aligned} V(l,r) \otimes P(l',r') \cong\ & (\oplus_{i=c(t)}^{t} 2P(l+l'-1-2i, r+r'+i)) \\ & \oplus(\oplus_{i=t+1}^{l-1} P(l+l'-1-2i, r+r'+i)). \end{aligned}$$

(7) *If $l' < l < n$ and $l + l' \leqslant n$, then*

$$\begin{aligned} V(l,r) \otimes P(l',r') \;\cong\; & (\oplus_{i=0}^{l'-1} P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=c(l+l'-1)}^{l-1} 2P(n+l+l'-1-2i, r+r'+i)). \end{aligned}$$

(8) *If $l' < l < n$ and $t = l + l' - (n+1) \geqslant 0$, then*

$$\begin{aligned} V(l,r) \otimes P(l',r') \cong \; & (\oplus_{i=c(t)}^{t} 2P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=t+1}^{l'-1} P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=c(l+l'-1)}^{l-1} 2P(n+l+l'-1-2i, r+r'+i)). \end{aligned}$$

(9) *If $l < n$, then*

$$\begin{aligned} V(n,r) \otimes P(l,r') \cong \; & (\oplus_{i=c(l-1)}^{l-1} 2P(n+l-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=1}^{c(n-l)} 2P(l-1+2i, r+r'-i)). \end{aligned}$$

(10) *If $l \leqslant l' < n$ and $l + l' \leqslant n$, then*

$$\begin{aligned} P(l,r) \otimes P(l',r') \cong \; & (\oplus_{i=0}^{l-1} 2P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=l'}^{l'+l-1} 2P(n+l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{c(l'+l-1) \leqslant i \leqslant l'-1} 4P(n+l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{1 \leqslant i \leqslant c(n-l-l')} 4P(l+l'-1+2i, r+r'-i)). \end{aligned}$$

(11) *If $l \leqslant l' < n$ and $t = l + l' - (n+1) \geqslant 0$, then*

$$\begin{aligned} P(l,r) \otimes P(l',r') \cong \; & (\oplus_{i=c(t)}^{t} 4P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=t+1}^{l-1} 2P(l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{i=l'}^{n-1} 2P(n+l+l'-1-2i, r+r'+i)) \\ & \oplus (\oplus_{c(l'+l-1) \leqslant i \leqslant l'-1} 4P(n+l+l'-1-2i, r+r'+i)). \end{aligned}$$

Proof. It follows from [9, 12]. ∎

By Lemma 5.1 or [12, Corollary 3.2], the category consisting of semisimple modules and projective modules in $H_n(1,q)$-mod is a monoidal subcategory of $H_n(1,q)$-mod. Therefore, we have the following corollary.

Corollary 5.2. *$r_p(H_n(1,q))$ is a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $\{[V(k,r)], [P(l,r)] | 1 \leqslant k \leqslant n, 1 \leqslant l \leqslant n-1, r \in \mathbb{Z}_n\}$.*

Lemma 5.3. *Let $2 \leqslant m \leqslant n-1$. Then*

$$V(2,0)^{\otimes m} \cong \oplus_{i=0}^{[\frac{m}{2}]} \tfrac{m-2i+1}{m-i+1} \tbinom{m}{i} V(m+1-2i, i).$$

Proof. By Lemma 5.1(3), one can easily check that the isomorphism in the lemma holds for $m = 2$ and $m = 3$. Now let $3 < m \leqslant n-1$ and assume

$$V(2,0)^{\otimes (m-1)} \cong \oplus_{i=0}^{[\frac{m-1}{2}]} \tfrac{m-2i}{m-i} \tbinom{m-1}{i} V(m-2i, i).$$

If $m = 2l$ is even, then by the induction hypothesis and Lemma 5.1(3), we have

$$\begin{aligned} V(2,0)^{\otimes m} &= V(2,0) \otimes V(2,0)^{\otimes(m-1)} \\ &\cong \oplus_{i=0}^{l-1} \tfrac{2l-2i}{2l-i}\tbinom{2l-1}{i} V(2,0) \otimes V(2l-2i,i) \\ &\cong \oplus_{i=0}^{l-1} \tfrac{2l-2i}{2l-i}\tbinom{2l-1}{i} (V(2l+1-2i,i) \oplus V(2l-1-2i,i+1)) \\ &\cong V(2l+1,0) \oplus \tfrac{2}{l+1}\tbinom{2l-1}{l-1} V(1,l) \\ &\quad \oplus (\oplus_{i=1}^{l-1} (\tfrac{2l-2i}{2l-i}\tbinom{2l-1}{i} + \tfrac{2l-2i+2}{2l-i+1}\tbinom{2l-1}{i-1})) V(2l+1-2i,i)) \\ &\cong V(2l+1,0) \oplus \tfrac{2}{l+1}\tbinom{2l-1}{l-1} V(1,l) \\ &\quad \oplus (\oplus_{i=1}^{l-1} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l+1-2i,i)) \\ &\cong \oplus_{i=0}^{l} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l+1-2i,i)) \\ &\cong \oplus_{i=0}^{[\frac{m}{2}]} \tfrac{m+1-2i}{m+1-i}\tbinom{m}{i} V(m+1-2i,i). \end{aligned}$$

If $m = 2l + 1$ is odd, then by the same reason as above, we have

$$\begin{aligned} & V(2,0)^{\otimes m} \\ = \ & V(2,0) \otimes V(2,0)^{\otimes(m-1)} \\ \cong \ & \oplus_{i=0}^{l} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2,0) \otimes V(2l+1-2i,i) \\ \cong \ & (\oplus_{i=0}^{l-1} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l+2-2i,i) \oplus V(2l-2i,i+1)) \oplus \tfrac{1}{l+1}\tbinom{2l}{l} V(2,l) \\ \cong \ & (\oplus_{i=0}^{l} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l+2-2i,i)) \oplus (\oplus_{i=0}^{l-1} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l-2i,i+1)) \\ \cong \ & (\oplus_{i=0}^{l} \tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} V(2l+2-2i,i)) \oplus (\oplus_{i=1}^{l} \tfrac{2l+3-2i}{2l+2-i}\tbinom{2l}{i-1} V(2l+2-2i,i)) \\ \cong \ & V(2l+2,0) \oplus (\oplus_{i=1}^{l} (\tfrac{2l+1-2i}{2l+1-i}\tbinom{2l}{i} + \tfrac{2l+3-2i}{2l+2-i}\tbinom{2l}{i-1})) V(2l+2-2i,i)) \\ \cong \ & V(2l+2,0) \oplus (\oplus_{i=1}^{l} \tfrac{2l+2-2i}{2l+2-i}\tbinom{2l+1}{i} V(2l+2-2i,i)) \\ \cong \ & \oplus_{i=0}^{l} \tfrac{2l+2-2i}{2l+2-i}\tbinom{2l+1}{i} V(2l+2-2i,i) \\ \cong \ & \oplus_{i=0}^{[\frac{m}{2}]} \tfrac{m+1-2i}{m+1-i}\tbinom{m}{i} V(m+1-2i,i). \end{aligned}$$

■

Throughout the following, let $x = [V(1,1)]$ and $y = [V(2,0)]$ in $r_p(H_n(1,q))$.

Corollary 5.4. *The following equations hold in $r_p(H_n(1,q))$ (or $r(H_n(1,q))$):*
(1) $x^n = 1$ *and* $[V(m,i)] = x^i[V(m,0)]$ *for all* $1 \leqslant m \leqslant n$ *and* $i \in \mathbb{Z}$*;*
(2) $[P(m,i)] = x^i[P(m,0)]$ *for all* $1 \leqslant m < n$ *and* $i \in \mathbb{Z}$*;*
(3) $y[V(n,0)] = x[P(n-1,0)]$*;*
(4) $y[P(1,0)] = [P(2,0)] + 2x[V(n,0)]$*;*
(5) $y[P(n-1,0)] = 2[V(n,0)] + x[P(n-2,0)]$*;*
(6) $y[P(m,0)] = [P(m+1,0)] + x[P(m-1,0)]$ *for all* $2 \leqslant m \leqslant n-2$*;*
(7) $[V(m+1,0)] = y^m - \sum_{i=1}^{[\frac{m}{2}]} \frac{m+1-2i}{m+1-i}\binom{m}{i} x^i[V(m+1-2i,0)]$ *for all* $2 \leqslant m < n$*.*

Proof. It follows from Lemmas 5.1 and 5.3. ■

Proposition 5.5. *The commutative ring $r_p(H_n(1,q))$ is generated by x and y.*

Proof. Let R be the subring of $r(H_n(1,q))$ generated by x and y. Then $R \subseteq r_p(H_n(1,q))$. By Corollary 5.4(1), one gets that $[V(1,i)] = x^i \in R$ and $[V(2,i)] = x^i y \in R$ for all $i \in \mathbb{Z}_n$. Now let $2 \leqslant m < n$ and assume $[V(l,i)] \in R$ for all $1 \leqslant l \leqslant m$ and $i \in \mathbb{Z}_n$. Then by Corollary 5.4(1) and (7), one gets that $[V(m+1,i)] = x^i[V(m+1,0)] = x^i y^m - \sum_{j=1}^{[\frac{m}{2}]} \frac{m+1-2j}{m+1-j}\binom{m}{j} x^{i+j}[V(m+1-2j,0)] \in$

R for all $i \in \mathbb{Z}_n$. Thus, we have proven that $[V(m,i)] \in R$ for all $1 \leqslant m \leqslant n$ and $i \in \mathbb{Z}_n$. In particular, $[V(n,i)] \in R$ for all $i \in \mathbb{Z}_n$.

By Corollary 5.4(2) and (3), $[P(n-1,i)] = x^i[P(n-1,0)] = x^{i-1}y[V(n,0)] \in R$ for all $i \in \mathbb{Z}_n$. Then by Corollary 5.4(2) and (5), $[P(n-2,i)] = x^i[P(n-2,0)] = x^{i-1}(y[P(n-1,0)] - 2[V(n,0)]) \in R$ for any $i \in \mathbb{Z}_n$. Now let $1 < m \leqslant n-2$ and assume that $[P(l,i)] \in R$ for all $m \leqslant l < n$ and $i \in \mathbb{Z}_n$. Then by Corollary 5.4(2) and (6), we have $[P(m-1,i)] = x^i[P(m-1,0)] = x^{i-1}(y[P(m,0)] - [P(m+1,0)]) \in R$. Thus, we have shown that $[P(m,i)] \in R$ for all $1 \leqslant m < n$ and $i \in \mathbb{Z}_n$. Then it follows from Corollary 5.2 that $R = r_p(H_n(1,q))$. This completes the proof. ■

Lemma 5.6. (1) $[V(m,0)] = \sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i\binom{m-1-i}{i}x^iy^{m-1-2i}$ *for all* $1 \leqslant m \leqslant n$.
(2) *Let* $1 \leqslant m \leqslant n-1$. *Then*

$$[P(m,0)] = (\sum_{i=0}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m}{n-m-i}\binom{n-m-i}{i}x^{m+i}y^{n-m-2i})[V(n,0)].$$

Proof. (1) It is similar to [38, Lemma 3.2].
(2) Note that $\frac{n-m}{n-m-i}\binom{n-m-i}{i}$ is a positive integer for any $1 \leqslant m \leqslant n-1$ and $0 \leqslant i \leqslant [\frac{n-m}{2}]$. We prove the equality by induction on $n-m$. If $m = n-1$, then by Corollary 5.4(1) and (3), $[P(n-1,0)] = x^{-1}y[V(n,0)] = x^{n-1}y[V(n,0)]$, as desired. If $m = n-2$, then by Corollary 5.4(1) and (5), we have $[P(n-2,0)] = x^{-1}y[P(n-1,0)] - 2x^{-1}[V(n,0)] = (x^{n-2}y^2 - 2x^{n-1})[V(n,0)]$, as desired. Now let $1 \leqslant m < n-2$. Then by Corollary 5.4(1) and (6), and the induction hypotheses, we have

$$\begin{aligned}
[P(m,0)] = \; & x^{-1}y[P(m+1,0)] - x^{-1}[P(m+2,0)] \\
= \; & x^{-1}y(\textstyle\sum_{i=0}^{[\frac{n-m-1}{2}]}(-1)^i\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i}x^{m+1+i}y^{n-m-1-2i})[V(n,0)] \\
& -x^{-1}(\textstyle\sum_{i=0}^{[\frac{n-m-2}{2}]}(-1)^i\frac{n-m-2}{n-m-2-i}\binom{n-m-2-i}{i}x^{m+2+i}y^{n-m-2-2i})[V(n,0)] \\
= \; & (\textstyle\sum_{i=0}^{[\frac{n-m-1}{2}]}(-1)^i\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i}x^{m+i}y^{n-m-2i})[V(n,0)] \\
& +(\textstyle\sum_{i=1}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m-2}{n-m-1-i}\binom{n-m-1-i}{i-1}x^{m+i}y^{n-m-2i})[V(n,0)].
\end{aligned}$$

If $n-m$ is odd, then $[\frac{n-m-1}{2}] = \frac{n-m-1}{2} = [\frac{n-m}{2}]$, and hence

$$\begin{aligned}
& \textstyle\sum_{i=0}^{[\frac{n-m-1}{2}]}(-1)^i\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i}x^{m+i}y^{n-m-2i} \\
& +\textstyle\sum_{i=1}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m-2}{n-m-1-i}\binom{n-m-1-i}{i-1}x^{m+i}y^{n-m-2i} \\
= \; & x^my^{n-m} + \textstyle\sum_{i=1}^{[\frac{n-m}{2}]}(-1)^i(\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i} \\
& +\frac{n-m-2}{n-m-1-i}\binom{n-m-1-i}{i-1})x^{m+i}y^{n-m-2i} \\
= \; & \textstyle\sum_{i=0}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m}{n-m-i}\binom{n-m-i}{i}x^{m+i}y^{n-m-2i}.
\end{aligned}$$

If $n-m$ is even, then $[\frac{n-m-1}{2}] = \frac{n-m-2}{2} = [\frac{n-m}{2}] - 1$, and hence

$$\begin{aligned}
& \textstyle\sum_{i=0}^{[\frac{n-m-1}{2}]}(-1)^i\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i}x^{m+i}y^{n-m-2i} \\
& +\textstyle\sum_{i=1}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m-2}{n-m-1-i}\binom{n-m-1-i}{i-1}x^{m+i}y^{n-m-2i} \\
= \; & x^my^{n-m} + \textstyle\sum_{i=1}^{[\frac{n-m}{2}]-1}(-1)^i(\frac{n-m-1}{n-m-1-i}\binom{n-m-1-i}{i} \\
& +\frac{n-m-2}{n-m-1-i}\binom{n-m-1-i}{i-1})x^{m+i}y^{n-m-2i} + (-1)^{\frac{n-m}{2}}2x^{\frac{n+m}{2}} \\
= \; & \textstyle\sum_{i=0}^{[\frac{n-m}{2}]}(-1)^i\frac{n-m}{n-m-i}\binom{n-m-i}{i}x^{m+i}y^{n-m-2i}.
\end{aligned}$$

Therefore, $[P(m,0)] = (\sum_{i=0}^{[\frac{n-m}{2}]}(-1)^i \frac{n-m}{n-m-i}\binom{n-m-i}{i}x^{m+i}y^{n-m-2i})[V(n,0)]$. ■

Proposition 5.7. *In $r_p(H_n(1,q))$ (or $r(H_n(1,q))$), we have*

$$(\sum_{i=0}^{[\frac{n}{2}]}(-1)^i \frac{n}{n-i}\binom{n-i}{i}x^iy^{n-2i} - 2)(\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i\binom{n-1-i}{i}x^iy^{n-1-2i}) = 0.$$

Proof. By Lemma 5.6(2), we have

$$x^{-1}y[P(1,0)] = (\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i \frac{n-1}{n-1-i}\binom{n-1-i}{i}x^iy^{n-2i})[V(n,0)].$$

On the other hand, by Corollary 5.4(4) and Lemma 5.6(2), we have

$$\begin{aligned} x^{-1}y[P(1,0)] &= x^{-1}[P(2,0)] + 2[V(n,0)] \\ &= (\sum_{i=0}^{[\frac{n-2}{2}]}(-1)^i \frac{n-2}{n-2-i}\binom{n-2-i}{i}x^{i+1}y^{n-2-2i} + 2)[V(n,0)] \\ &= (\sum_{i=1}^{[\frac{n}{2}]}(-1)^{i-1} \frac{n-2}{n-1-i}\binom{n-1-i}{i-1}x^iy^{n-2i} + 2)[V(n,0)]. \end{aligned}$$

Therefore, one gets

$$\begin{aligned} & (\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i \frac{n-1}{n-1-i}\binom{n-1-i}{i}x^iy^{n-2i})[V(n,0)] \\ = \; & (\sum_{i=1}^{[\frac{n}{2}]}(-1)^{i-1} \frac{n-2}{n-1-i}\binom{n-1-i}{i-1}x^iy^{n-2i} + 2)[V(n,0)], \end{aligned}$$

which is equivalent to

$$\begin{aligned} & (\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i \frac{n-1}{n-1-i}\binom{n-1-i}{i}x^iy^{n-2i} \\ & - \sum_{i=1}^{[\frac{n}{2}]}(-1)^{i-1} \frac{n-2}{n-1-i}\binom{n-1-i}{i-1}x^iy^{n-2i} - 2)[V(n,0)] = 0. \end{aligned}$$

Then a computation similar to the proof of Lemma 5.6 shows that

$$\begin{aligned} & \sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i \frac{n-1}{n-1-i}\binom{n-1-i}{i}x^iy^{n-2i} - \sum_{i=1}^{[\frac{n}{2}]}(-1)^{i-1} \frac{n-2}{n-1-i}\binom{n-1-i}{i-1}x^iy^{n-2i} - 2 \\ = \; & \sum_{i=0}^{[\frac{n}{2}]}(-1)^i \frac{n}{n-i}\binom{n-i}{i}x^iy^{n-2i} - 2. \end{aligned}$$

Thus, the proposition follows from Lemma 5.6(1). ■

Corollary 5.8. *$\{x^ly^m | 0 \leqslant l \leqslant n-1, 0 \leqslant m \leqslant 2n-2\}$ is a $\mathbb{Z}$-basis of $r_p(H_n(1,q))$.*

Proof. By Corollary 5.4(1), $x^n = 1$. By Proposition 5.7, we have

$$\begin{aligned} y^{2n-1} = & - \sum_{i=1}^{[\frac{n-1}{2}]}(-1)^i\binom{n-1-i}{i}x^iy^{2n-1-2i} \\ & - \sum_{i=1}^{[\frac{n}{2}]}(-1)^i \frac{n}{n-i}\binom{n-i}{i}x^iy^{2n-1-2i} + 2y^{n-1} \\ & -(\sum_{i=1}^{[\frac{n}{2}]}(-1)^i \frac{n}{n-i}\binom{n-i}{i}x^iy^{n-2i} - 2)(\sum_{i=1}^{[\frac{n-1}{2}]}(-1)^i\binom{n-1-i}{i}x^iy^{n-1-2i}). \end{aligned}$$

Then it follows from Proposition 5.5 that $r_p(H_n(1,q))$ is generated, as a $\mathbb{Z}$-module, by $\{x^ly^m | 0 \leqslant l \leqslant n-1, 0 \leqslant m \leqslant 2n-2\}$. By Corollary 5.2, $r_p(H_n(1,q))$ is a free $\mathbb{Z}$-module of rank $n(2n-1)$, and hence $\{x^ly^m | 0 \leqslant l \leqslant n-1, 0 \leqslant m \leqslant 2n-2\}$ is a $\mathbb{Z}$-basis of $r_p(H_n(1,q))$. ■

Theorem 5.9. *Let $\mathbb{Z}[x,y]$ be the polynomial ring in two variables x and y, and I the ideal of $\mathbb{Z}[x,y]$ generated by x^n-1 and*

$$(\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^iy^{n-2i}-2)(\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i\binom{n-1-i}{i}x^iy^{n-1-2i}).$$

Then $r_p(H_n(1,q))$ is isomorphic to the quotient ring $\mathbb{Z}[x,y]/I$.

Proof. By Proposition 5.5, there is a ring epimorphism $\phi:\mathbb{Z}[x,y]\to r_p(H_n(1,q))$ given by $\phi(x)=[V(1,1)]$ and $\phi(y)=[V(2,0)]$. By Corollary 5.4(1) and Proposition 5.7, $\phi(I)=0$. Hence ϕ induces a ring epimorphism $\overline{\phi}:\mathbb{Z}[x,y]/I\to r_p(H_n(1,q))$ such that $\phi=\overline{\phi}\circ\pi$, where $\pi:\mathbb{Z}[x,y]\to\mathbb{Z}[x,y]/I$ is the canonical projection. Let $\overline{u}=\pi(u)$ for any $u\in\mathbb{Z}[x,y]$. Then by the definition of I and the proof of Corollary 5.8, one knows that $\mathbb{Z}[x,y]/I$ is generated, as a $\mathbb{Z}$-module, by $\{\overline{x}^l\overline{y}^m|0\leqslant l\leqslant n-1,0\leqslant m\leqslant 2n-2\}$. For any $0\leqslant l\leqslant n-1$ and $0\leqslant m\leqslant 2n-2$, we have $\overline{\phi}(\overline{x}^l\overline{y}^m)=\overline{\phi}(\overline{x})^l\overline{\phi}(\overline{y})^m=\phi(x)^l\phi(y)^m=[V(1,1)]^l[V(2,0)]^m$. By Corollary 5.8, $\{[V(1,1)]^l[V(2,0)]^m|0\leqslant l\leqslant n-1,0\leqslant m\leqslant 2n-2\}$ is a linearly independent set over $\mathbb{Z}$, which implies that $\{\overline{x}^l\overline{y}^m|0\leqslant l\leqslant n-1,0\leqslant m\leqslant 2n-2\}$ is also a linearly independent set over $\mathbb{Z}$. It follows that $\{\overline{x}^l\overline{y}^m|0\leqslant l\leqslant n-1, 0\leqslant m\leqslant 2n-2\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[x,y]/I$. Consequently, $\overline{\phi}$ is a $\mathbb{Z}$-module isomorphism, and so it is a ring isomorphism. ∎

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