Non-Archimedean meromorphic solutions of functional equations

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Abstract

We discuss meromorphic solutions of functional equations over non-Archimedean fields, and prove difference analogues of the Clunie lemma, Malmquist-type theorems and the Mokhon'ko theorem.

1 Introduction

Value distribution theory established by R. Nevanlinna, also called Nevanlinna theory, is a very useful tool for studying both the growth of meromorphic functions in the complex plane $\mathbb{C}$ and meromorphic solutions of differential equations, see for instance the Clunie lemma (cf. [2],[11]), Malmquist-type theorems (cf. [13],[17]) and the Mokhon'ko theorem (cf. [15]). These theorems also have analogues for meromorphic functions over non-Archimedean fields (cf. [9]). Detailed information about Nevanlinna theory over non-Archimedean fields can be found in [9].

Recently, some authors started studying meromorphic solutions of difference equations based on Nevanlinna theory over $\mathbb{C}$ (cf. [4], [5], [12]). In this paper, we obtain difference analogues of the theorems stated above by using Nevanlinna theory over non-Archimedean fields.

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2 Main Results

Let κ be an algebraically closed field of characteristic zero, which is complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Let $\mathcal{A}(\kappa)$ (resp. $\mathcal{M}(\kappa)$) denote the set of entire (respectively meromorphic) functions over κ. As usual, if R is a ring, we use $R[X_0, X_1, ..., X_n]$ to denote the ring of polynomials over R, depending on the variables $X_0, X_1, ..., X_n$. We will make the following assumption (fixing at the same time the notations):

(A) Let n be a positive integer, and take a_i, b_i in κ such that $|a_i| = 1$, $i = 0, 1, ..., n$, with $a_0 = 1, b_0 = 0$, and such that

$$L_i(z) = a_i z + b_i \ (i = 0, 1, ..., n)$$

are distinct. Let f be a non-constant meromorphic function over κ and write $f_i = f \circ L_i$, $i = 0, 1, ..., n$, where $f_0 = f$. Moreover, consider non-zero elements

$$B \in \mathcal{M}(\kappa)[X]; \qquad \Omega, \Phi \in \mathcal{M}(\kappa)[X_0, X_1, ..., X_n].$$

Then, there exist $\{b_0, ..., b_q\} \subset \mathcal{M}(\kappa)$ with $b_q \not\equiv 0$ such that

$$B(X) = \sum_{k=0}^{q} b_k X^k. \tag{1}$$

Similarly, we can write

$$\Omega\left(X_0, X_1, ..., X_n\right) = \sum_{i \in I} c_i X_0^{i_0} X_1^{i_1} \cdots X_n^{i_n}, \tag{2}$$

where $i = (i_0, i_1, ..., i_n)$ are non-negative integer indices, I is a finite set, and $c_i \in \mathcal{M}(\kappa)$, and also

$$\Phi\left(X_0, X_1, ..., X_n\right) = \sum_{j \in J} d_j X_0^{j_0} X_1^{j_1} \cdots X_n^{j_n}, \tag{3}$$

where $j = (j_0, j_1, ..., j_n)$ are non-negative integer indices, J is a finite set, and $d_j \in \mathcal{M}(\kappa)$.

In this paper, we will use the usual notations and concepts from Nevanlinna theory, see e.g. [9]. For example, $\mu(r, f)$ denotes the maximum term of the power series for $f \in \mathcal{A}(\kappa)$ and its fractional extension to $\mathcal{M}(\kappa)$, $m(r, f)$ is the compensation (or proximity) function of f, $N(r, f)$ is the valence function of f for poles, and finally,

$$T(r, f) = m(r, f) + N(r, f),$$

is the characteristic function of f. Then we can state our results as follows.

Theorem 2.1. *Under the assumption* (A), *if f is a solution of the functional equation*

$$B(f)\Omega(f, f_1, ..., f_n) = \Phi(f, f_1, ..., f_n) \tag{4}$$

with $\deg B \geq \deg \Phi$*, then*

$$m(r,\Omega) \leq \sum_{i\in I} m(r,c_i) + \sum_{j\in J} m(r,d_j) + l\, m\left(r,\frac{1}{b_q}\right) + l\sum_{j=0}^{q} m(r,b_j), \tag{5}$$

where $l = \max\{1, \deg \Omega\}$*,* $\Omega = \Omega(f, f_1, ..., f_n)$*. Furthermore, if* Φ *is a polynomial of* f*, we also have*

$$N(r,\Omega) \leq \sum_{i\in I} N(r,c_i) + \sum_{j\in J} N(r,d_j) + O\left(\sum_{j=0}^{q} N\left(r,\frac{1}{b_j}\right)\right). \tag{6}$$

Theorem 2.1 is a difference analogue of the Clunie lemma over non-Archimedean fields (cf. [9]). Halburd and Korhonen [5] obtained a difference analogue of the Clunie lemma over the complex numbers (cf. [2]). Theorem 2.1 has numerous applications in the study of non-Archimedean difference equations, and beyond. In order to state one of its applications, we need the following definition:

Definition 2.2. *A solution* $f \in \mathcal{M}(\kappa)$ *of* (4) *is said to be admissible if*

$$\sum_{i\in I} T(r,c_i) + \sum_{j\in J} T(r,d_j) + \sum_{k=0}^{q} T(r,b_k) = o(T(r,f)), \tag{7}$$

or equivalently, the coefficients of B, Φ, Ω *are slowly moving targets with respect to* f*.*

If c_i, d_j, b_k all are rational functions, then each transcendental meromorphic function f over κ must satisfy (7), which means that each transcendental meromorphic solution f over κ is admissible.

Theorem 2.3. *If* Φ *is of the form*

$$\Phi(f, f_1, ..., f_n) = \Phi(f) = \sum_{j=0}^{p} d_j f^j,$$

and if (4) *has an admissible non-constant meromorphic solution* f*, then*

$$q = 0, \quad p \leq \deg(\Omega).$$

Theorem 2.3 is a difference analogue of a Malmquist-type theorem over non-Archimedean fields (cf. [9]) . Malmquist-type theorems were obtained by Malmquist [14], Gackstatter-Laine [3], Laine [10], Toda [16], Yosida [18] (or see He-Xiao [6]) for meromorphic functions on $\mathbb{C}$, and Hu-Yang [8] or [7] for several complex variables.

Corollary 2.4. *Assume condition* (A) *to hold such that the coefficients of* B, Ω, Φ *are rational functions over* κ*, and* Φ *has the form as in Theorem 2.3 . Then, if* (4) *has a transcendental meromorphic solution* f *over* κ*, it holds that* Φ/B *is a polynomial in* f *of degree* $\leq \deg(\Omega)$*.*

Corollary 2.4 is a difference analogue of the non-Archimedean Malmquist-type theorem due to Boutabaa [1].

Theorem 2.5. *Let $f \in \mathcal{M}(\kappa)$ be a non-constant admissible solution of*

$$\Omega\left(f, f', ..., f^{(n)}\right) = 0, \tag{8}$$

where now the solution f is admissible if $\sum_{i\in I} T(r, c_i) = o(T(r, f))$. If a slowly moving target $a \in \mathcal{M}(\kappa)$ with respect to f, that is, $T(r, a) = o(T(r, f))$, does not satisfy the equation (8), then

$$m\left(r, \frac{1}{f-a}\right) = o(T(r, f)).$$

Theorem 2.5 is an analogue of a result due to Mokhon'ko and Mokhon'ko [15] over non-Archimedean fields, which also has a difference analogue as follows:

Theorem 2.6. *Assume the condition (A) to hold and let $f \in \mathcal{M}(\kappa)$ be a non-constant admissible solution of*

$$\Omega(f, f_1, ..., f_n) = 0. \tag{9}$$

If a slowly moving target $a \in \mathcal{M}(\kappa)$ with respect to f does not satisfy the equation (9), then

$$m\left(r, \frac{1}{f-a}\right) = o(T(r, f)).$$

A version of Theorem 2.6 over the complex numbers can be found in [5].

3 A difference analogue of a lemma on logarithmic derivation

Take $a(\neq 0), b \in \kappa$ and consider the linear transformation

$$L(z) = az + b$$

over κ. For a positive integer m, put

$$\Delta_L f = f \circ L - f,\ \Delta_L^m f = \Delta_L(\Delta_L^{m-1} f).$$

Lemma 3.1. *Take $f \in \mathcal{A}(\kappa)$ and assume $|a| \leq 1$. When $r > |b|/|a|$, we have*

$$\mu(r, f \circ L) \leq \mu(r, f).$$

Moreover, we obtain

$$\mu\left(r, \frac{f\circ L}{f}\right) \leq 1,\ \mu\left(r, \frac{\Delta_L^m f}{f}\right) \leq 1.$$

Proof. We can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

since $f \in \mathcal{A}(\kappa)$. Therefore

$$f(L(z)) = \sum_{n=0}^{\infty} a_n (az+b)^n.$$

First of all, we take $r \in |\kappa|$, that is, $r = |z|$ for some $z \in \kappa$. When $r > |b|/|a|$, we find (cf. [9])

$$\mu(r, f \circ L) = |f(L(z))| \leq \max_{n \geq 0} |a_n||az+b|^n = \max_{n \geq 0} |a_n||az|^n \leq \max_{n \geq 0} |a_n||z|^n = \mu(r, f).$$

In particular,

$$\mu\left(r, \frac{f \circ L}{f}\right) = \frac{\mu(r, f \circ L)}{\mu(r, f)} \leq 1,$$

and hence

$$\mu\left(r, \frac{\Delta_L f}{f}\right) = \frac{\mu(r, f \circ L - f)}{\mu(r, f)} \leq \frac{1}{\mu(r, f)} \max\{\mu(r, f \circ L), \mu(r, f)\} \leq 1.$$

By induction, we can prove that

$$\mu\left(r, \frac{\Delta_L^m f}{f}\right) \leq 1.$$

Since $|\kappa|$ is dense in $\mathbb{R}_+ = [0, \infty)$, by using continuity we easily see that these inequalities hold for all $r > |b|/|a|$. ∎

Note that (cf. [9])

$$m(r, f) = \log^+ \mu(r, f) = \max\{0, \log \mu(r, f)\}. \tag{10}$$

Lemma 3.1 immediately implies the following difference analogue of the lemma on the logarithmic derivation:

Corollary 3.2. *Take $f \in \mathcal{A}(\kappa)$ and assume $|a| \leq 1$. When $r > |b|/|a|$, we have*

$$m\left(r, \frac{f \circ L}{f}\right) = 0, \ m\left(r, \frac{\Delta_L^m f}{f}\right) = 0.$$

Lemma 3.3. *Take $f \in \mathcal{M}(\kappa) \setminus \{0\}$ and assume $|a| = 1$. When $r > |b|$, we have that*

$$\mu(r, f \circ L) = \mu(r, f). \tag{11}$$

Moreover, we obtain

$$\mu\left(r, \frac{f \circ L}{f}\right) = 1, \ \mu\left(r, \frac{\Delta_L^m f}{f}\right) \leq 1.$$

Proof. Since $f \in \mathcal{M}(\kappa) \setminus \{0\}$, there exist $g, h (\neq 0) \in \mathcal{A}(\kappa)$ with $f = \frac{g}{h}$. Hence (cf. [9])

$$\mu(r, f \circ L) = \frac{\mu(r, g \circ L)}{\mu(r, h \circ L)}. \tag{12}$$

Take $r \in |\kappa|$. Since $|a| = 1$, we have

$$|L(z)| = |az + b| = |z| = r$$

when $r > |b|$, and so

$$\mu(r, g \circ L) = \mu(r, g).$$

Similarly, we have $\mu(r, h \circ L) = \mu(r, h)$. Thus the formula (11) holds. By using continuity we easily see that the inequality holds for all $r > |b|$. ∎

Corollary 3.4. *Take $f \in \mathcal{M}(\kappa) \setminus \{0\}$ and assume $|a| = 1$. When $r > |b|$, we have*

$$m\left(r, \frac{f \circ L}{f}\right) = 0, \ m\left(r, \frac{\Delta_L^m f}{f}\right) = 0.$$

4 Proof of Theorem 2.1

In order to prove (5), take $z \in \kappa$ with

$$f(z) \neq 0, \infty; \quad b_k(z) \neq 0, \infty \quad (0 \leq k \leq q);$$

$$c_i(z) \neq 0, \infty \quad (i \in I); \quad d_j(z) \neq 0, \infty \quad (j \in J).$$

Write

$$b(z) = \max_{0 \leq k < q} \left\{ 1, \left(\frac{|b_k(z)|}{|b_q(z)|} \right)^{\frac{1}{q-k}} \right\}.$$

If $|f(z)| > b(z)$, we have

$$|b_k(z)||f(z)|^k \leq |b_q(z)|b(z)^{q-k}|f(z)|^k < |b_q(z)||f(z)|^q,$$

and hence

$$|B(f)(z)| = |b_q(z)||f(z)|^q.$$

Then

$$|\Omega\left(f, f_1, ..., f_n\right)(z)| = \frac{|\Phi(f, f_1, ..., f_n)(z)|}{|B(f)(z)|} \leq \frac{1}{|b_q(z)|} \max_{j \in J} |d_j(z)| \left| \frac{f_1(z)}{f(z)} \right|^{j_1} \cdots \left| \frac{f_n(z)}{f(z)} \right|^{j_n}.$$

If $|f(z)| \leq b(z)$,

$$|\Omega(f, f_1, ..., f_n)(z)| \leq b(z)^{\deg(\Omega)} \max_{i \in I} |c_i(z)| \left| \frac{f_1(z)}{f(z)} \right|^{i_1} \cdots \left| \frac{f_n(z)}{f(z)} \right|^{i_n}.$$

Therefore, in any case, the inequality

$$\mu(r, \Omega) \leq \max_{j \in J, i \in I} \left\{ \frac{\mu(r, d_j)}{\mu(r, b_q)} \prod_{k=1}^{n} \mu\left(r, \frac{f_k}{f}\right)^{j_k}, \mu(r, c_i) \prod_{k=1}^{n} \mu\left(r, \frac{f_k}{f}\right)^{i_k} \max_{0 \leq k < q} \left\{ 1, \mu\left(r, \frac{b_k}{b_q}\right)^{\frac{\deg(\Omega)}{q-k}} \right\} \right\}$$

holds where $r = |z|$, which also holds for all $r > 0$ by continuity of the functions μ. By using Lemma 3.3, we find

$$\mu(r, \Omega) \leq \max_{j \in J, i \in I} \left\{ \frac{\mu(r, d_j)}{\mu(r, b_q)}, \mu(r, c_i) \cdot \max_{0 \leq k < q} \left\{ 1, \mu\left(r, \frac{b_k}{b_q}\right)^{\frac{\deg(\Omega)}{q-k}} \right\} \right\},$$

whence (5) follows from this inequality. Similarly as in the proof of (4.9) in [9], we then easily obtain the inequality (6).

5 Proof of Theorem 2.3

By using the algorithm of division, we have

$$\Phi(f) = \Phi_1(f)B(f) + \Phi_2(f)$$

with $\deg(\Phi_2) < q$. Thus, the equation (4) can be rewritten as follows:

$$\Omega\left(f, f_1, ..., f_n\right) - \Phi_1(f) = \frac{\Phi_2(f)}{B(f)}. \tag{13}$$

Applying Theorem 2.1 to this equation, we obtain

$$m(r, \Omega - \Phi_1) = o(T(r, f)),$$

$$N(r, \Omega - \Phi_1) = o(T(r, f)),$$

and hence

$$T(r, \Omega - \Phi_1) = o(T(r, f)).$$

Then [9, Theorem 2.12] implies

$$T(r, \Omega - \Phi_1) = T\left(r, \frac{\Phi_2}{B}\right) = qT(r, f) + o(T(r, f)),$$

whence it follows that $q = 0$, and (4) takes the form

$$\Omega\left(f, f_1, ..., f_n\right) = \Phi(f).$$

Thus, [9, Theorem 2.12] implies that

$$T(r, \Omega) = T(r, \Phi) = pT(r, f) + o(T(r, f)). \tag{14}$$

On other hand, it is easy to find the estimate

$$N(r, \Omega) \leq \deg(\Omega)N(r, f) + \sum_{i\in I} N(r, c_i). \tag{15}$$

Obviously, we also have

$$m(r, \Omega) \leq \deg(\Omega)m(r, f) + \max_{i\in I}\left\{m(r, c_i) + \sum_{\alpha=1}^{n} i_\alpha m\left(r, \frac{f_\alpha}{f}\right)\right\}. \tag{16}$$

By Lemma 3.3, we then obtain

$$T(r, \Omega) \leq \deg(\Omega)T(r, f) + \sum_{i\in I} T(r, c_i) + O(1), \tag{17}$$

and finally, our result follows from (14) and (17).

6 Proof of Theorems 2.5 and 2.6

Substitution of $f = g + a$ into (8) yields $\Psi + P = 0$, where

$$\Psi\left(g, g', ..., g^{(n)}\right) = \sum_i C_i g^{i_0}(g')^{i_1} \cdots (g^{(n)})^{i_n}$$

is a differential polynomial of g such that all of its terms are at least of degree one, and

$$T(r, P) = o(T(r, f)).$$

Also $P \not\equiv 0$, since a does not satisfy (8). Now, take $z \in \kappa$ with

$$g(z) \neq 0, \infty;\ C_i(z) \neq \infty;\ P(z) \neq 0, \infty,$$

and put $r = |z|$. If $|g(z)| \geq 1$, then

$$m\left(r, \frac{1}{g}\right) = \max\left\{0, \log\frac{1}{|g(z)|}\right\} = 0.$$

It is therefore sufficient to consider only the case $|g(z)| < 1$. But then,

$$\begin{aligned} \left|\frac{\Psi\left(g(z), g'(z), ..., g^{(n)}(z)\right)}{g(z)}\right| &= \frac{1}{|g(z)|}\left|\sum_i C_i(z) g(z)^{i_0} g'(z)^{i_1} \cdots g^{(n)}(z)^{i_n}\right| \\ &\leq \max_i |C_i(z)| \left|\frac{g'(z)}{g(z)}\right|^{i_1} \cdots \left|\frac{g^{(n)}(z)}{g(z)}\right|^{i_n} \end{aligned}$$

since $i_0 + \cdots i_n \geq 1$ for all i. Therefore,

$$\begin{aligned} m\left(r, \frac{1}{g}\right) &= \log\frac{1}{|g(z)|} = \log\frac{|P(z)|}{|g(z)|} + \log\frac{1}{|P(z)|} \\ &= \log\frac{|\Psi\left(g(z), g'(z), ..., g^{(n)}(z)\right)|}{|g(z)|} + \log\frac{1}{|P(z)|} \\ &\leq \sum_i \left\{m(r, C_i) + i_1 m\left(r, \frac{g'}{g}\right) + \cdots + i_n m\left(r, \frac{g^{(n)}}{g}\right)\right\} + m\left(r, \frac{1}{P}\right) \\ &= o(T(r, f)). \end{aligned}$$

Since $g = f - a$, the assertion follows.

Obviously, following the method above, we can also prove Theorem 2.6 in a similar way.

7 Final notes

Let us now adopt the following assumption:

(B) Let n be a positive integer, and take a_i, b_i in κ such that $|a_i| = 1$ for each $i = 1, ..., n$, and such that

$$L_i(z) = a_i z + b_i \ (i = 1, ..., n)$$

satisfies $L_i(z) \neq z$ for each $i = 1, ..., n$. Let f be a non-constant meromorphic function over κ and let $\{f_1, ..., f_m\}$ be a finite set consisting of the forms $\Delta^j_{L_i} f$. Take

$$B \in \mathcal{M}(\kappa)[f]; \ \Omega, \Phi \in \mathcal{M}(\kappa)[f, f_1, ..., f_m].$$

According to the methods described in this paper, we can easily prove the following results.

Theorem 7.1. *Under the condition* (B), *if f is a solution of the equation*

$$B(f)\Omega(f, f_1, ..., f_m) = \Phi(f, f_1, ..., f_m) \tag{18}$$

with $\deg B \geq \deg \Phi$, *then*

$$m(r, \Omega) \leq \sum_{i \in I} m(r, c_i) + \sum_{j \in J} m(r, d_j) + l\, m\left(r, \frac{1}{b_q}\right) + l \sum_{j=0}^{q} m(r, b_j), \tag{19}$$

where $l = \max\{1, \deg \Omega\}$, $\Omega = \Omega(f, f_1, ..., f_m)$. Further, if Φ is a polynomial of f, we also have that

$$N(r, \Omega) \leq \sum_{i \in I} N(r, c_i) + \sum_{j \in J} N(r, d_j) + O\left(\sum_{j=0}^{q} N\left(r, \frac{1}{b_j}\right)\right). \tag{20}$$

Theorem 7.2. *If Φ is of the form*

$$\Phi(f, f_1, ..., f_m) = \Phi(f) = \sum_{j=0}^{p} d_j f^j,$$

and if (18) *has an admissible non-constant meromorphic solution f, then*

$$q = 0, \quad p \leq \deg(\Omega).$$

Theorem 7.3. *Assume the condition* (B) *to hold, and let $f \in \mathcal{M}(\kappa)$ be a non-constant admissible solution of*

$$\Omega\,(f, f_1, ..., f_m) = 0, \tag{21}$$

where the solution f is called admissible if

$$\sum_{i \in I} T(r, c_i) = o(T(r, f)).$$

If a slowly moving target $a \in \mathcal{M}(\kappa)$ with respect to f does not satisfy the equation (21), *then*

$$m\left(r, \frac{1}{f - a}\right) = o(T(r, f)).$$

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