Grassmannians of arbitrary rank

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Abstract

We introduce a generalization of Grassmannians of projective spaces that allows us to consider subspaces of any (possibly infinite) rank as points of the Grassmannian. We show that the spaces that we obtain, carry in a natural way the structure of a twin building.

Introduction

Grassmannians are point-line spaces arising from projective spaces by considering the subspaces of a certain *finite* rank. In this article we introduce a generalization of this concept that allows us to consider subspaces of possibly infinite rank. It turns out that the right approach is to take always two such generalizations, and connect them by an opposition relation. Such a structure will be called a *partial twin Grassmannian*.

In section 3 we introduce a method for constructing a pair of connected chamber systems out of any partial twin Grassmannian. In this fashion, the chambers turn out to be chains of subspaces of the underlying projective space. More precisely, these chains are maximal under the condition that every element has a successor (except for those which are hyperplanes) and a predecessor (except for those which are singletons). However, we do not use all such maximal chains since otherwise Grassmannians of infinite rank would in general give rise to a disconnected chamber system, which is of course not desirable. See [KS96] for a deeper insight into this question.

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In section 4 we show that the chamber systems introduced in section 3 are, in fact, buildings. Since the chamber systems always come in pairs, the buildings also come in pairs, and it is natural to ask whether these pairs form a twin building. This indeed is the case as we show in section 5.

1 Preliminaries and notations

A *point-line space* $\mathscr{S} = (\mathcal{P}, \mathcal{L})$ is a pair consisting of a set $\mathcal{P}$, whose elements are called *points* and a set $\mathcal{L} \subset \mathfrak{P}(\mathcal{P})$ of subsets of $\mathcal{P}$ with cardinality at least 2, which are called *lines*. If all points are subsets of a common set V, we sometimes regard a line as the union of its points and hence as a subset of V. Points on a common line are called *collinear*. We write $p_0 \perp p_1$ to denote that p_0 and p_1 are collinear. If $p_0 \perp p_1$, then we call p_0 a *neighbour* of p_1. By $p^\perp$ we denote the set of all neighbours of a point p, called the *perp* of p. For a set of points M we denote by $M^\perp := \bigcap_{p \in M} p^\perp$ the perp of M, i. e. the set of all common neighbours.

A *subspace* of a point-line space $\mathscr{S} = (\mathcal{P}, \mathcal{L})$ is a point-line space $\mathscr{S}' = (\mathcal{P}', \mathcal{L}')$ with $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$ such that every line in $\mathcal{L} \setminus \mathcal{L}'$ has at most one point with $\mathcal{P}'$ in common and every line in $\mathcal{L}'$ is contained in $\mathcal{L}$. We write $\mathscr{S}' \leq \mathscr{S}$, if $\mathscr{S}'$ is a subspace of $\mathscr{S}$ and $\mathscr{S}' < \mathscr{S}$ if $\mathscr{S}'$ is properly contained. Since $\mathscr{S}'$ is determined by its point set, we call $\mathcal{P}'$ itself a subspace. Correspondingly, we treat $\mathscr{S}$ sometimes as its own point set. A proper subspace is called a *hyperplane* if it intersects every line. For a set of points M, we denote by $\langle M \rangle$ the smallest subspace which contains M, called the *span* of M. For a family of points $p_0, \ldots, p_s$ and a family of sets of points $M_0, \ldots, M_r$ we will write $\langle p_0, \ldots, p_s, M_0, \ldots, M_r \rangle$ rather than $\langle \{p_0, \ldots, p_s\} \cup M_0 \cup \cdots \cup M_r \rangle$.

A *partially linear space*[1] is a point-line space such that no two different lines have two different points in common. Clearly, subspaces of partially linear spaces are again partially linear. For two distinct collinear points p and q of a partially linear space, the unique line joining p and q is denoted by pq. A point-line space where every two points are collinear is called *singular*. Singular partially linear spaces are called *linear*. The *rank* of a singular space $\mathscr{S}$ is denoted by $\mathrm{rk}(\mathscr{S})$ and equals $\alpha - 2$, where α is the supremum over the lengths of well-ordered chains of subspaces of $\mathscr{S}$. Hence, the rank of the empty space is -1 and the rank of a singleton is 0. For a point-line space $\mathscr{S}$ let $\mathfrak{S}(\mathscr{S}) := \{X \leq \mathscr{S} \mid X \subseteq X^\perp\}$ denote the set of all singular subspaces of $\mathscr{S}$. The *singular rank* of $\mathscr{S}$ is defined as $\mathrm{srk}(\mathscr{S}) := \sup\{\mathrm{rk}(X) \mid X \in \mathfrak{S}(\mathscr{S})\}$.

A *(possibly degenerate) projective space* is a linear space satisfying the following property:

(VY) For every pair (l, k) of disjoint lines and every point $p \in \mathcal{P} \setminus (l \cup k)$ there is at most one line through p meeting both l and k.

A projective space is called degenerate if it contains at most one line or at least one short line, i. e. a line of cardinality 2. Very often, projective spaces are required to be non-degenerate, but we will *explicitly allow projective spaces to be degenerate*.

[1] The terminology *partial linear space* is more common, but grammatically incorrect.

It is obvious by the definition that every subspace of a projective space is again a projective space.

Let $\mathscr{S}$ be a projective space and let $(\mathscr{S}_i)_{i\in I}$ be the family of all subspaces of $\mathscr{S}$ that are maximal under the condition of not containing any short line, where I is a suitable index set. Then for $i \in I$, the subspace $\mathscr{S}_i$ is either a point, a line with at least three points or a non-degenerate projective space. It is known (see for instance [Seg48]) that $\mathscr{S}$ equals the direct sum $\bigoplus_{i\in I} \mathscr{S}_i$, where the direct sum is defined as follows: the point set of the direct sum is the disjoint union of the point sets of all components; the line set of the direct sum is the union of the line sets of the components plus all short lines with points of distinct components. A direct sum of projective spaces is again a projective space.

Let $\mathscr{S}$ be a projective space and let $X \subseteq \mathscr{S}$ be a set of points. If $p \notin \langle X \setminus \{p\}\rangle$ for every point $p \in X$, we call X *independent*. An independent set of points $B \subseteq \mathscr{S}$ with $\langle B\rangle = \mathscr{S}$ is called a *basis*. In a projective space every independent set of points is a subset of a basis. Furthermore, every basis of $\mathscr{S}$ has cardinality $\mathrm{rk}(\mathscr{S}) + 1$. For a direct sum $\mathscr{S} = \bigoplus_{i\in I} \mathscr{S}_i$ of projective spaces, every basis of $\mathscr{S}$ is a union of bases of all components $\mathscr{S}_i$ and vice versa.

Let V be a subspace of $\mathscr{S}$ and let B be a basis of $\mathscr{S}$ with $\langle B \cap V\rangle = V$. Then we call $\mathrm{crk}_{\mathscr{S}}(V) := |B \setminus V|$ the *corank* of V in $\mathscr{S}$. Let U be another subspace of $\mathscr{S}$. Then $\mathrm{crk}_U(U \cap V) = \mathrm{crk}_{\langle U,V\rangle} V$.

Let U and V be two subspaces of a projective space $\mathscr{S}$ such that $U \cap V$ has finite corank in both U and V. Then we call U and V *commensurate*, denoted by $U \gtrless V$. If U and V are commensurate with $\mathrm{crk}_U(U \cap V) = \mathrm{crk}_V(U \cap V)$, we write $U \overset{>}{\underset{<}{=}} V$.

A *Grassmannian* of a projective space $\mathscr{S}$ is a point-line space whose point set $\mathcal{P}$ consists of all subspaces of $\mathscr{S}$ of rank $k \in \mathbb{N}$ and whose lines are the maximal subsets of $\mathcal{P}$ whose elements intersect in a common subspace of rank $k-1$ and are contained in a common subspace of rank $k+1$. To be more specific, this point-line space is also called a Grassmannian of k-spaces. The Grassmannian of 0-spaces is canonically isomorphic to $\mathscr{S}$.

For a projective space of infinite rank $\mathscr{S}$, there is an analogous way to define a point-line space whose points are the subspaces of corank k. The so-obtained point-line space can be seen as the Grassmannian of corank-k-spaces. Thus, the Grassmannian of corank-1-spaces is just the dual of $\mathscr{S}$.

2 Grassmannians of arbitrary rank

In this section we introduce point-line spaces that are constructed out of projective spaces and are a natural generalization of Grassmannians. Our new construction will allow us to take subspaces of infinite rank and infinite corank as elements of the point set. In order to make this possible, we first introduce the concept of twin spaces.

Definition 2.1. Let $\mathscr{S}^+ = (\mathcal{P}^+, \mathcal{L}^+)$ and $\mathscr{S}^- = (\mathcal{P}^-, \mathcal{L}^-)$ be two disjoint partially linear spaces. Further let $R \subseteq (\mathcal{P}^+ \times \mathcal{P}^-) \cup (\mathcal{P}^- \times \mathcal{P}^+)$ be a symmetric, total relation on $\mathcal{P}^+ \cup \mathcal{P}^-$ (called the *opposition relation*) such that for every pair $(p, l) \in (\mathcal{P}^+ \times \mathcal{L}^-) \cup (\mathcal{P}^- \times \mathcal{L}^+)$, the following holds:

(OP) If p is opposite to some point of l, then there is exactly one point $q \in l$ such that p is not opposite to q.

Then we call the pair $(\mathscr{S}^+, \mathscr{S}^-)$ a *twin space* with opposition relation R.

Let U be a subspace of a projective space $\mathscr{S}$. Then we call a subspace $V \leq \mathscr{S}$ a *complement* of U if and only if U and V are disjoint and $\langle U, V \rangle = \mathscr{S}$. Since conversely, by this definition U is a complement of V, we call U and V *complementary* subspaces.

Definition 2.2. Let $\mathscr{S}$ be a projective space and let U_+ and U_- be non-trivial subspaces of $\mathscr{S}$ that are complementary. For $\sigma \in \{+,-\}$, set $\mathfrak{U}^\sigma := \{V \leq \mathscr{S} \mid V \gtreqless U_\sigma\}$. Further set:

$$\mathcal{L}_{\mathrm{m}}^\sigma := \big\{\{Z \in \mathfrak{U}^\sigma \mid X \cap Y < Z < \langle X, Y \rangle\} \bigm| \{X, Y\} \subseteq \mathfrak{U}^\sigma \ \wedge\ \mathrm{crk}_X(X \cap Y) = 1\big\}$$
$$R := \{(M,N),(N,M) \mid (M,N) \in \mathfrak{U}^+ \times \mathfrak{U}^- \ \wedge\ M \cap N = \varnothing\}$$

Then we call the pair $((\mathfrak{U}^+, \mathcal{L}_{\mathrm{m}}^+), (\mathfrak{U}^-, \mathcal{L}_{\mathrm{m}}^-))$ with the opposition relation R the *twin Grassmannian* of $\mathscr{S}$ with respect to (U_+, U_-).

For $\sigma \in \{+,-\}$, let $\mathcal{P}_{\mathrm{m}}^\sigma \subseteq \mathfrak{U}^\sigma$ be a subset that contains U_σ such that the following conditions are satisfied:

(TG1) For every subspace $V \in \mathcal{P}_{\mathrm{m}}^\sigma$, there is a subspace $W \in \mathcal{P}_{\mathrm{m}}^{-\sigma}$ such that $V \cap W = \varnothing$.

(TG2) $\langle V \mid V \in \mathcal{P}_{\mathrm{m}}^\sigma \rangle = \mathscr{S}$.

(TG3) Let V and W be two elements of $\mathcal{P}_{\mathrm{m}}^\sigma$. Then $\{X \in \mathfrak{U}^\sigma \mid V \cap W \leq X \leq \langle V, W \rangle\} \subseteq \mathcal{P}_{\mathrm{m}}^\sigma$.

For $\sigma \in \{+,-\}$, set $\mathcal{L}_{\mathrm{m}}'^\sigma := \{L \in \mathcal{L}_{\mathrm{m}}^\sigma \mid L \subseteq \mathcal{P}_{\mathrm{m}}^\sigma\}$ and $R' := R \cap ((\mathcal{P}_{\mathrm{m}}^+ \cup \mathcal{P}_{\mathrm{m}}^-) \times (\mathcal{P}_{\mathrm{m}}^- \times \mathcal{P}_{\mathrm{m}}^+))$. Then $((\mathcal{P}_{\mathrm{m}}^+, \mathcal{L}_{\mathrm{m}}'^+), (\mathcal{P}_{\mathrm{m}}^-, \mathcal{L}_{\mathrm{m}}'^-))$ with the opposition relation R' is called a *partial twin Grassmannian* of $\mathscr{S}$ with respect to (U_+, U_-).

We will see later on that every (partial) twin Grassmannian is a twin space. Therefore we call a twin space a (partial) twin Grassmannian if it is isomorphic to a (partial) twin Grassmannian of a projective space. Throughout this section $\mathscr{S}$ is always a projective space and U_+ and U_- are non-trivial subspaces of $\mathscr{S}$ that are complementary. For $\sigma \in \{+,-\}$, we set $\mathfrak{U}^\sigma := \{V \leq \mathscr{S} \mid V \gtreqless U_\sigma\}$.

The following proposition shows that the concept of partial twin Grassmannians is more general than the one of twin Grassmannians. We will discuss later on the advantages of using the more general concept.

Proposition 2.3. *Every twin Grassmannian of $\mathscr{S}$ is a partial twin Grassmannian of $\mathscr{S}$.*

Proof. (TG2) and (TG3) are clearly fulfilled. Hence, it remains to verify (TG1). Let V be a subspace of $\mathscr{S}$ that is commensurate to U_+. Then $V' := \langle U_+, V \rangle$ is a subspace with $\mathrm{crk}_{V'} V = \mathrm{crk}_{V'} U_+ = \mathrm{crk}_{U_+}(V \cap U_+) = \mathrm{crk}_V(V \cap U_+) =: r$. Since U_+ and U_- are complements, this implies $\mathrm{rk}(V' \cap U_-) = r - 1$. Thus, U_- contains a complement W' of V' with $\mathrm{crk}_{U_-} W' = r$.

Let $W'' \leq U_+$ be a subspace of rank $r - 1$ that is disjoint from $V \cap U_+$ and set $W := \langle W', W'' \rangle$. Then $\mathrm{crk}_V(V \cap U_+) = r$ together with $\langle V, W'' \rangle = \langle V, V \cap$

$U_+, W''\rangle = \langle V, U_+\rangle = V'$ implies that W'' is a complement of V in V'. Since W' and V' are complements in $\mathscr{S}$, we obtain $W \cap V' = W''$ and therefore $W \cap V = \varnothing$. On the other hand $\langle W, V\rangle = \langle W', W'', V\rangle = \langle W', V'\rangle = \mathscr{S}$ and consequently, W and V are complements in $\mathscr{S}$. ∎

The following remark concerns some immediate consequences of the axioms (TG1) and (TG3).

Remark 2.4. Let V be a subspace with $V \gtreqless U_+$ and let W be a subspace with $W \gtreqless U_-$ such that $V \cap W = \varnothing$. Set $V' := V \cap U_+$ and $W' := W \cap U_-$. Then $\operatorname{crk}_V(V') = \operatorname{crk}_{U_+}(V')$ and $\operatorname{crk}_W(W') = \operatorname{crk}_{U_-}(W')$ and hence, $\operatorname{crk}_{\langle V,W\rangle}(\langle V', W'\rangle)$ $= \operatorname{crk}_{\langle U_+,U_-\rangle}(\langle V', W'\rangle)$. With $\langle U_+, U_-\rangle = \mathscr{S}$ this implies $\langle V, W\rangle = \mathscr{S}$. Therefore, the subspaces V and W of (TG1) are always complements.

From (TG3) and the definition of the lines of the twin Grassmannian it follows directly that every partial twin Grassmannian with respect to (U_+, U_-) is a subspace of the twin Grassmannian with respect to (U_+, U_-). A second consequence of (TG3) is that for a partial twin Grassmannian $((\mathcal{P}^+_{\mathrm{m}}, \mathcal{L}^+_{\mathrm{m}}), (\mathcal{P}^-_{\mathrm{m}}, \mathcal{L}^-_{\mathrm{m}}))$, both point-line spaces $(\mathcal{P}^+_{\mathrm{m}}, \mathcal{L}^+_{\mathrm{m}})$ and $(\mathcal{P}^-_{\mathrm{m}}, \mathcal{L}^-_{\mathrm{m}})$ are connected.

Axiom (TG2) plays a special role. As we will see in the following remark, omitting it does not change anything about the definition of partial twin Grassmannians in itself, but it would change the definition of a partial twin Grassmannian of a given projective space. Nevertheless, we cling to this axiom since it turns out to be useful.

Remark 2.5. For $\sigma \in \{+,-\}$, let $\mathcal{P}^\sigma_1 \subseteq \mathfrak{U}^\sigma$ be a subset with $U_\sigma \in \mathcal{P}^\sigma_1$ such that (TG1) and (TG3) are fulfilled, but (TG2) is not. Then $\mathcal{P}^\sigma_1$ is a subspace of $\mathscr{D}^\sigma$, where $(\mathscr{D}^+, \mathscr{D}^-)$ is the twin Grassmannian of $\mathscr{S}$ with respect to (U_+, U_-). Set $\mathscr{S}' := \langle U \mid U \in \mathcal{P}^+_1\rangle$. Further set $\mathcal{P}^-_0 := \{V \cap \mathscr{S}' \mid V \in \mathcal{P}^-_1\}$. Let $U \in \mathcal{P}^+_1$ and $V \in \mathcal{P}^-_1$ such that U and V are complements. Then U and $V \cap \mathscr{S}'$ are complements in $\mathscr{S}'$. Let $(\mathscr{D}^+_0, \mathscr{D}^-_0)$ be the twin Grassmannian of $\mathscr{S}'$ with respect to $(U_+, U_- \cap \mathscr{S}')$.

For two elements V and W of $\mathcal{P}^-_1$ there are complementary subspaces V' and W' in $\mathcal{P}^+_1$. Since V' and W' are complements of $V \cap \mathscr{S}'$ and $W \cap \mathscr{S}'$ in $\mathscr{S}'$ and furthermore $V' \gtreqless W'$, we conclude $V \cap \mathscr{S}' \gtreqless W \cap \mathscr{S}'$. Therefore we obtain $\mathcal{P}^-_0 \subseteq \mathscr{D}^-_0$. Furthermore, $\operatorname{rk}(U \cap \langle V, W\rangle) = \operatorname{crk}_{\langle V,W\rangle}(V) - 1$ since U is complementary to V. This implies $\operatorname{crk}_{\langle V,W\rangle\cap\mathscr{S}'}(V \cap \mathscr{S}') = \operatorname{crk}_{\langle V,W\rangle}(V)$ and consequently, $\operatorname{crk}_{V\cap\mathscr{S}'}(V \cap W \cap \mathscr{S}') = \operatorname{crk}_V(V \cap W)$. Hence, $\mathcal{P}^-_1 \to \mathcal{P}^-_0 : X \mapsto X \cap \mathscr{S}'$ maps the subspace $\mathcal{P}^-_1$ of $\mathscr{D}^-$ isomorphically onto a subspace of $\mathscr{D}^-_0$.

Let $X' \leq \mathscr{S}'$ be a subspace with $X' \gtreqless V \cap \mathscr{S}'$ such that $V \cap W \cap \mathscr{S}' \leq X' \leq \langle V, W\rangle \cap \mathscr{S}'$. Then for $X := \langle X', V \cap W\rangle$, we obtain $X \gtreqless V$ with $V \cap W \leq X \leq \langle V, W\rangle$ and hence, (TG3) implies $X \in \mathcal{P}^-_1$. Since $X' = X \cap \mathscr{S}'$, we conclude $X' \in \mathcal{P}^-_0$ and that (TG3) holds for $\mathcal{P}^-_0$. Therefore, $\mathcal{P}^-_1 \to \mathcal{P}^-_0 : X \mapsto X \cap \mathscr{S}'$ is surjective and consequently, $\mathcal{P}^-_0$ is a subspace of $\mathscr{D}^-_0$ that is isomorphic to $\mathcal{P}^-_1$ and fulfils (TG1) and (TG3).

Suppose $\mathscr{S}'' := \langle U \mid U \in \mathcal{P}^-_0\rangle < \mathscr{S}'$. Then we set $\mathcal{P}^+_0 := \{V \cap \mathscr{S}'' \mid V \in \mathcal{P}^+_1\}$. By repeating the arguments we conclude that restricting the elements of $\mathcal{P}^+_1$ and $\mathcal{P}^-_1$ to the subspace $\mathscr{S}''$ leads to a structure that induces a pair of point-line spaces isomorphic to $(\mathcal{P}_1^+, \mathcal{P}_1^-)$ such that complementary subspaces are mapped onto complementary subspaces. Since now (TG1), (TG2) and (TG3) are all fulfilled, we conclude that the subspaces contained in $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ are the points of a partial twin Grassmannian. Therefore (TG2) can be seen as a condition that makes sure that $\mathscr{S}$ is "entirely used".

In the following $\mathscr{S}_\mathrm{m} := ((\mathcal{P}_\mathrm{m}^+, \mathcal{L}_\mathrm{m}^+), (\mathcal{P}_\mathrm{m}^-, \mathcal{L}_\mathrm{m}^-))$ is a partial twin Grassmannian of $\mathscr{S}$ with respect to (U_+, U_-). For $\sigma \in \{+,-\}$, we set $\mathscr{S}_\mathrm{m}^\sigma := (\mathcal{P}_\mathrm{m}^\sigma, \mathcal{L}_\mathrm{m}^\sigma)$. Moreover, we denote by $\mathscr{D} = (\mathscr{D}^+, \mathscr{D}^-)$ the twin Grassmannian of $\mathscr{S}$ with respect to (U_+, U_-).

Proposition 2.6. *Every partial twin Grassmannian is a twin space.*

Proof. Let R be the opposition relation of the partial twin Grassmannian $\mathscr{S}_\mathrm{m}$. By (TG1) R is a symmetric, total relation on $\mathcal{P}_\mathrm{m}^+ \cup \mathcal{P}_\mathrm{m}^-$. Let $U \in \mathcal{P}_\mathrm{m}^+$ and $V \in \mathcal{P}_\mathrm{m}^-$ such that U and V are complements. Further let $W \in \mathcal{P}_\mathrm{m}^-$ be a subspace such that V and W are distinct collinear points in $\mathscr{S}_\mathrm{m}^-$ and let $L \in \mathcal{L}_\mathrm{m}^-$ with $\{V, W\} \subseteq L$. Then V and W intersect in a common hyperplane. Hence, U and $\langle V, W\rangle$ intersect in a single point p since U is a complement to V. We conclude that $\langle p, V \cap W\rangle$ is the only element of L that is not disjoint from U. Thus, R fulfils the condition of an opposition relation of twin spaces. By the definition of the lines of a twin Grassmannian we conclude that $\mathscr{S}_\mathrm{m}$ is partially linear. ∎

Let $\mathfrak{F}$ be the set of finite subsets of $\mathcal{P}_\mathrm{m}^+$ and set $\mathscr{S}' := \bigcup_{F \in \mathfrak{F}} \langle F\rangle$, where $\langle F\rangle$ is understood as the span in $\mathscr{S}$. Then $\mathscr{S}'$ is a subspace of $\mathscr{S}$ since the union of two finite sets is again finite. This implies $\langle U \mid U \in \mathcal{P}_\mathrm{m}^+\rangle \le \mathscr{S}'$ and hence, $\mathscr{S}' = \mathscr{S}$ by (TG2). We will make use of this fact for proving the following lemma.

Lemma 2.7. *Let p be a point of $\mathscr{S}$. Then there are elements $U \in \mathcal{P}_\mathrm{m}^+$ and $V \in \mathcal{P}_\mathrm{m}^-$ such that $p \in U \cap V$.*

Proof. Since $\langle U \mid U \in \mathcal{P}_\mathrm{m}^+\rangle = \mathscr{S}$, there is a finite set $F := \{U_i \mid 0 \le i < n\} \subseteq \mathcal{P}_\mathrm{m}^+$ where $n \in \mathbb{N}$ such that $p \in \langle F\rangle$. For $0 \le j < n$, set $\mathscr{S}_j := \langle U_i \mid i \le j\rangle$. We prove by induction over j that every point of $\mathscr{S}_j$ is contained in a subspace $U \in \mathcal{P}_\mathrm{m}^+$.

For $j = 0$ there is nothing to prove since $\mathscr{S}_0 = U_0$. Now assume the claim holds for $j < n-1$. We may assume $U_{j+1} \not\le \mathscr{S}_j$ since otherwise $\mathscr{S}_{j+1} = \mathscr{S}_j$. Let $q \in \mathscr{S}_{j+1} \setminus \mathscr{S}_j$. Then there are points $r \in U_{j+1}$ and $s \in \mathscr{S}_j$ such that q is on the line rs. By the induction hypothesis we know that there is a subspace $W \in \mathcal{P}_\mathrm{m}^+$ such that $s \in W$. Since $r \in \langle U_{j+1}, W\rangle$, (TG3) implies that there is a subspace $U \in \mathcal{P}_\mathrm{m}^+$ with $\langle q, U_{j+1} \cap W\rangle \le U$. A similar argument for $\mathcal{P}_\mathrm{m}^-$ proves the claim. ∎

Lemma 2.8. *Let $n \in \mathbb{N}$ and let $(U_i)_{0 \le i \le n}$ be a family of elements of $\mathcal{P}_\mathrm{m}^+$. Then every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0 \le i \le n} U_i \le V \le \langle U_i \mid 0 \le i \le n\rangle$ is an element of $\mathcal{P}_\mathrm{m}^+$.*

Proof. We prove the claim by induction over n. For $n = 0$ there is nothing to prove and for $n = 1$ this is just (TG3). Hence we may assume $n > 1$ and that every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0 \le i < n} U_i \le V \le \langle U_i \mid 0 \le i < n\rangle$ is an element of $\mathcal{P}_\mathrm{m}^+$. Moreover, we may assume that $\bigcap_{0 \le i \le j} U_i \le U_k \le \langle U_i \mid 0 \le i \le j\rangle$ does not hold for $0 \le j < k \le n$. Set $S := \bigcap_{0 \le i < n} U_i$ and $W := \langle U_i \mid 0 \le i < n\rangle$. In the proof we distinguish three situations:

(I) First consider the case $S \leq U_n$ and $U_n \nleq W$. Since $U_0 \gneqq U_n$, there is a natural number $m \in \mathbb{N}$ and a family of points $(p_i)_{0\leq i<m}$ in $U_n \setminus W$ such that $\langle p_i, W \mid 0 \leq i < m\rangle = \langle U_n, W\rangle$. Let H be a subspace with $U_n \cap W \leq H \leq W$ and $\mathrm{crk}_W(H) = \mathrm{crk}_W(U_0) + 1$ and let $V' \leq W$ such that V' contains H as a hyperplane. Then $V' \in \mathfrak{U}^+$ and thus, $V' \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis. Set $V_i := \langle p_i, H\rangle$ for $0 \leq i < m$. Then $V' \cap U_n \leq V_i \leq \langle V', U_n\rangle$ and thus, $V_i \in \mathcal{P}^+_{\mathrm{m}}$ by (TG3). Furthermore, $S \cap U_n = S \cap (\bigcap_{0\leq i<m} V_i) = S$ since $S \leq U_n \cap W$ and $\langle W, U_n\rangle = \langle W, V_i \mid 0 \leq i < m\rangle$. Therefore we may restrain ourselves to the case $m = 1$.

Let $V \in \mathfrak{U}^+$ with $S \leq V \leq \langle p_0, W\rangle$. If $V \leq W$ we obtain $V \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis. Thus, we may assume $V \nleq W$. Then $V \cap W$ is a hyperplane of V since W is a hyperplane of $\langle p_0, W\rangle$. First assume $p_0 \notin V$. Then $\langle p_0, V\rangle$ intersects W in a subspace $V_W \in \mathfrak{U}^+$. Since $S \leq V \cap W$, we obtain $V_W \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis. Furthermore, V and V_W have the hyperplane $V \cap W$ in common. Now (TG3) implies $V \in \mathcal{P}^+_{\mathrm{m}}$ if and only if $\langle p_0, V \cap W\rangle \in \mathcal{P}^+_{\mathrm{m}}$. Thus, we may assume $p_0 \in V$.

For $V = U_n$ there is nothing to prove. Hence, we may assume that there is a point $p \in V \cap W \setminus U_n$. Let H be a hyperplane of $U_n \cap W$ containing $U_n \cap V \cap W$ and let $q \in W \setminus \langle p, U_n\rangle$. The latter is possible since $U_0 \neq U_1$ and hence, $\mathrm{crk}_W(U_n \cap W) \geq 2$. Set $V' := \langle p, q, H\rangle$. Then $V' \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis and consequently $U'_n := \langle p_0, p, H\rangle \in \mathcal{P}^+_{\mathrm{m}}$ by (TG3) since $V' \cap U_n = H \leq U'_n \leq \langle V', U_n\rangle$. With $U_n \cap V \leq \langle p_0, H\rangle \leq U'_n$ and $p \in U'_n$ we obtain $\mathrm{crk}_V(U'_n \cap V) = \mathrm{crk}_V(U_n \cap V) - 1$. Thus, repeating this argument implies $V \in \mathcal{P}^+_{\mathrm{m}}$.

(II) Now consider the case $U_n \leq W$ and $S \nleq U_n$. Since $S \leq U_0$ there is a hyperplane H of U_0 with $U_0 \cap U_n \leq H$ and $S \nleq H$. We obtain $\langle S, H\rangle = U_0$ and $S \cap U_n \leq H$. Hence, for a point $p \in U_n \setminus U_0$, we conclude $S \nleq \langle p, H\rangle =: U'_n$ and therefore $U_0 \cap U'_n = H$. Set $S' := S \cap U'_n$. Since H intersects S in a hyperplane, we obtain $\mathrm{crk}_S(S') = 1$. Furthermore, $S \cap U_n \leq H$ implies $S \cap U_n \leq S'$ and therefore $\mathrm{crk}_{S'}(S \cap U_n) = \mathrm{crk}_S(S \cap U_n) - 1$. Hence, by the finiteness of $\mathrm{crk}_S(S \cap U_n)$ we may constrain ourselves to the case $\mathrm{crk}_S(S \cap U_n) = 1$.

Let $V \in \mathfrak{U}^+$ with $S \cap U_n \leq V \leq W$. We show $V \in \mathcal{P}^+_{\mathrm{m}}$. Therefore we may assume $V \neq U_n$. If $S \leq V$, the claim follows from the induction hypothesis. Hence, we may assume $S \nleq V$ and thus, $S \cap U_n = S \cap V$ since $S \cap U_n$ is a hyperplane of S. If $S \leq \langle V, U_n\rangle$, then for a subspace $T \leq \langle V, U_n\rangle$ that is a complement of $\langle S, U_n\rangle$ in $\langle V, U_n\rangle$, we obtain $V' := \langle T, S\rangle \gneqq U_n$ since U_n is a hyperplane of $\langle S, U_n\rangle$. Now $V' \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis. With $\langle V', U_n\rangle = \langle V, U_n\rangle$ and $V \cap U_n \leq V'$, we conclude $V \cap U_n = V' \cap U_n$. Hence, $V \in \mathcal{P}^+_{\mathrm{m}}$ follows from (TG3).

It remains to consider the case $S \nleq \langle V, U_n\rangle$. Since $S \cap U_n < S < U_0$, there is a non-empty subspace $T \leq V$ with $T \cap S = \varnothing$ such that $H := \langle T, S \cap V\rangle$ is a hyperplane of V with $H \nleq U_n \cap V$. Thus, $V' := \langle T, S\rangle \gneqq V$ and consequently, $V' \in \mathcal{P}^+_{\mathrm{m}}$ by the induction hypothesis. If $V \cap U_n \leq H$, let $p \in U_n \setminus H$. If $V \cap U_n \nleq H$, let $p \in V \cap U_n \setminus H$. Set $U'_n := \langle p, H\rangle$. Then $U_n \cap V \leq U'_n$ and hence $H \nleq U_n \cap V$ implies $U_n \cap V < U'_n \cap V$. Since $V' \cap \langle V, U_n\rangle = H$, we obtain $V' \cap U_n \leq V \cap U_n$ and therefore $U'_n \in \mathcal{P}^+_{\mathrm{m}}$ by (TG3). Thus, $V \in \mathcal{P}^+_{\mathrm{m}}$ by another induction.

(III) Finally, consider $S \nleq U_n \nleq W$. As in (I) there is a natural number $m \in \mathbb{N}$ and a family of points $(p_i)_{0\leq i<m}$ in $U_n \setminus W$ such that $\langle p_i, W \mid 0 \leq i < m\rangle = \langle U_n, W\rangle$. Let H be a hyperplane of U_0 containing S. Set $V_i := U_i$ for $0 \leq i < n$ and $V_{i+n} := \langle p_i, H\rangle$ for $0 \leq i < m$. Then $\langle V_i \mid 0 \leq i < m+n\rangle = \langle W, U_n\rangle$ and $\bigcap_{0\leq i<n+m} V_i$. Hence, (I) implies $V \in \mathcal{P}^+_{\mathrm{m}}$ for every $V \in \mathfrak{U}^+$ with $S \leq V \leq \langle W, U_n\rangle$. Now the claim follows by (II) since $U_n \leq \langle V_i \mid 0 \leq i < m+n\rangle$ and $\bigcap_{0\leq i\leq n+m} V_i = S \cap U_n$. ∎

The following proposition is axiom (TG3) in a much stronger version.

Proposition 2.9. *Let $n \in \mathbb{N}$ and let $(U_i)_{0\leq i\leq n}$ be a family of elements of $\mathcal{P}^+_{\mathrm{m}}$. Then every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0\leq i\leq n} U_i \leq V$ is an element of $\mathcal{P}^+_{\mathrm{m}}$.*

Proof. Set $S := \bigcap_{0\leq i\leq n} U_i$. Since U_0 is commensurate to every element of $\mathcal{P}^+_{\mathrm{m}}$, the intersection $U_0 \cap U_i$ has finite corank in U_0 for every $i \leq n$. Since n is finite, this implies $U_0 \gtrless S$. Now let $V \in \mathfrak{U}^+$ with $S \leq V$. Then $\mathrm{crk}_V(S) = \mathrm{crk}_{U_0}(S)$. Hence by Lemma 2.7, there is a family $(U_i)_{n<i\leq m}$ of elements of $\mathcal{P}^+_{\mathrm{m}}$ such that $V \leq \langle S, U_i \mid n < i \leq m\rangle$, where $m \in \mathbb{N}$ with $n \leq m$. Now $\bigcap_{0\leq i\leq m} U_i \leq V \leq \langle U_i \mid 0 \leq i \leq m\rangle$ and the claim follows from Lemma 2.8. ∎

The following corollary shows that if the rank of U_+ is finite then the point-line space $\mathscr{S}^+_{\mathrm{m}}$ is uniquely determined. More precisely, $\mathscr{S}^+_{\mathrm{m}}$ is the Grassmannian of $\mathrm{rk}(U_+)$-spaces of $\mathscr{S}$ in this case. This justifies to regard partial twin Grassmannians as a generalization of Grassmannians.

Corollary 2.10. *Let* $\mathrm{rk}(U_+) < \infty$. *Then* $\mathscr{S}^+_{\mathrm{m}} = \mathscr{D}^+$.

Proof. Let $V \in \mathfrak{U}^+$ and take an arbitrary element W_0 of $\mathcal{P}^+_{\mathrm{m}}$. Set $S_0 := W_0$. By Lemma 2.7 there is for every point $p \in S_0 \setminus V$ a subspace $W \in \mathcal{P}^-_{\mathrm{m}}$ with $p \in W$. Hence by (TG1), there is a subspace $W_1 \in \mathcal{P}^+_{\mathrm{m}}$ with $p \notin W_1$ and we obtain $S_1 := S_0 \cap W_1 < S_0$. Since $\mathrm{rk}(S_0) < \infty$, we may repeat this argument to obtain a finite family $(W_i)_{0\leq i\leq n}$ of elements of $\mathcal{P}^+_{\mathrm{m}}$ such that $\bigcap_{0\leq i\leq n} W_i \leq V$. Now the claim follows from Proposition 2.9. ∎

The analogous statement for finite corank does not hold as the following example shows:

Example 2.11. Let V be a vector space over the field $\mathbb{Q}$ of dimension $\aleph_0$ and let $V^* := \mathrm{Hom}(V, K)$ be the dual space of V. Let $B := \{b_i \mid i \in \mathbb{N}\}$ be a basis of V and let $B^* := \{f_i \mid i \in \mathbb{N}\}$ be the set of linearly independent vectors of V^* such that $f_i(b_i) = 1$ for every $i \in \mathbb{N}$ and $f_i(b_j) = 0$ for every $j \in \mathbb{N} \setminus \{i\}$.

Let $\mathcal{P}$ be the set of 1-dimensional subspaces of V and let $\mathcal{L}$ be the set of 2-dimensional subspaces of V. Then $(\mathcal{P}, \mathcal{L})$ is a projective space. Further let $\mathfrak{U}$ be the set of all hyperplanes of $(\mathcal{P}, \mathcal{L})$ such that the corresponding hyperplane in V is the kernel of an element of $\langle B^*\rangle$. Note that $\langle B^*\rangle \nleq V^*$ and hence $\mathfrak{U}$ does not contain all hyperplanes of $(\mathcal{P}, \mathcal{L})$. More precisely, $|\mathfrak{U}| = \aleph_0$, whereas $(\mathcal{P}, \mathcal{L})$ possesses $2^{\aleph_0}$ hyperplanes.

Let $k \in \mathbb{N}$. Then $U^+ := \langle\langle b_i\rangle \mid 0 \leq i \leq k\rangle \leq (\mathcal{P}, \mathcal{L})$ is a subspace of rank k. Further $U_- := \langle\langle b_i\rangle \mid i > k\rangle$ is a subspace of corank $k+1$ in $(\mathcal{P}, \mathcal{L})$ and a complement to U_-. Note that U_- can be obtained as the intersection of finitely many elements of $\mathfrak{U}$. Let $(\mathscr{S}^+, \mathscr{S}^-)$ be the twin Grassmannian of $(\mathcal{P}, \mathcal{L})$ with respect to (U_+, U_-). We will show that the points of $\mathscr{S}^-$ that can be obtained as intersection of finitely many elements of $\mathfrak{U}$ form a subspace of $\mathscr{S}^-$ which we denote by $\mathscr{S}_0^-$. Furthermore, we will show that $(\mathscr{S}^+, \mathscr{S}_0^-)$ is a partial twin Grassmannian of $(\mathcal{P}, \mathcal{L})$ with respect to (U_+, U_-) .

Let $n \in \mathbb{N}$ and let $a_i \in \mathbb{Q}$ for $0 \leq i \leq n$ with $a_n \neq 0$. Then the vector $\sum_{i \leq n} a_i b_i$ is not contained in the kernel of $\sum_{i \leq n} a_i f_i$. This implies that every element of $\mathscr{S}^+$ has a complement in $\mathscr{S}_0^-$ and vice versa. Furthermore, for $i \leq n$, let $g_i \in \langle B^* \rangle$. Then $g \mapsto \ker(g)$ maps the elements of $\langle g_i \mid i \leq n \rangle$ bijectively onto the hyperplanes of V that contain $\bigcap_{i \leq n} \ker(g_i)$. From these facts it follows that $\mathscr{S}_0^-$ is a subspace of $\mathscr{S}^-$ and $(\mathscr{S}^+, \mathscr{S}_0^-)$ is a partial twin Grassmannian.

The point-line space $\mathscr{S}_0^-$ contains $\aleph_0$ points, whereas $\mathscr{S}^-$ contains $2^{\aleph_0}$ points. Moreover, $\mathscr{S}^+$ and $\mathscr{S}_0^-$ are isomorphic since V and $\langle B^* \rangle$ are isomorphic. There are no twin Grassmannians such that both parts are isomorphic to $\mathscr{S}^+$. Therefore it is essential to consider the more general concept of partial twin Grassmannians rather than restraining ourselves to twin Grassmannians.

Remark 2.12. Let $(\mathcal{P}, \mathcal{L})$ be as in Example 2.11. Further let U_+ and U_- be complementary subspaces of $(\mathcal{P}, \mathcal{L})$ that are both of infinite rank. Finally let $(\mathscr{S}^+, \mathscr{S}^-)$ be the twin Grassmannian of $(\mathcal{P}, \mathcal{L})$ with respect to (U_+, U_-). Then both $\mathscr{S}^-$ and $\mathscr{S}^+$ are *infinite Grassmannians* in the sense of [Kac90, Exercise 14.32, p. 339].

Remark 2.13. Partial twin Grassmannians can be seen as a generalization of Grassmannians. This is because every subspace of a projective space has a complement and hence, every Grassmannian of k-spaces is together with the Grassmannian of corank-$(k+1)$-spaces a twin Grassmannian.

Conversely, Corollary 2.10 implies that whenever U_+ has finite rank then $\mathscr{D}^+$ is a Grassmannian in the usual sense. Hence, the two parts of a partial twin Grassmannian can be seen as Grassmannians of α-spaces, where α is an arbitrary cardinal. However, the reader should keep in mind that if α is infinite and equals the rank of $\mathscr{S}$, then the corank β of the considered subspaces can be of any possible cardinal between 1 and α. Hence, in this case, it does not suffice to mention the rank of the considered subspace. As long as β is smaller than α one can talk about a Grassmannian of corank-β-spaces. If β equals α one should mention both the rank and the corank.

Remark 2.14. The only case where $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are not disjoint is $\mathrm{rk}(\mathscr{S}) < \infty$ and $\mathrm{rk}(U_+) = \mathrm{rk}(U_-)$. Moreover, by Corollary 2.10 this implies that $\mathcal{P}_\mathrm{m}^+$ and $\mathcal{P}_\mathrm{m}^-$ are disjoint or equal.

3 Chamber systems

An abstract *simplicial complex* Δ is a collection of sets such that $B \in \Delta$ for any subset B with $B \subseteq A \in \Delta$. A partially ordered set of sets that is isomorphic to a simplicial complex is also called a simplicial complex. A simplicial complex possesses a smallest element that we denote by 0. An element that only contains 0 properly is called a *vertex*. An arbitrary element A of a simplicial complex Δ is called a *simplex*. A *subcomplex* Δ' of a simplicial complex Δ is a subset of Δ such that Δ' is again a simplicial complex. Let A be a simplex of a simplicial complex Δ. Then the set of all simplices of Δ containing A is again a simplicial complex called the *residue* of A in Δ and denoted by $\mathrm{res}_\Delta(A)$ or simply by $\mathrm{res}(A)$ if there is no confusion about the underlying simplicial complex.

A simplicial complex Δ is called a *chamber complex* if every element of Δ is contained in a maximal element of Δ and if for two maximal elements C and C' of Δ, there exists a finite sequence $(C_i)_{0\leq i\leq m}$ such that $|C_i \setminus C_{i+1}| = |C_{i+1} \setminus C_i| = 1$ for every $i < m$. The maximal elements of a chamber complex are called *chambers*. Two chambers C and C' are called *adjacent* if $|C \setminus C'| = 1$ and a finite sequence $(C_i)_{0\leq i\leq m}$ such that C_i and C_{i+1} are adjacent for every $i < m$ is called a *gallery* from C_0 to C_m. A chamber complex is call *thick* (respectively *thin*) if for any two adjacent chambers C and C' the subset $C \cap C'$ is contained in at least three (respectively exactly two) chambers. It follows immediately that in a chamber complex every two chambers have the same cardinality. Moreover, $|C \setminus 0| = |C' \setminus 0| =: \alpha$ for every two chambers C and C'. We call α the *rank* of Δ.

As in the previous section let $\mathscr{S}_\mathrm{m} := ((\mathcal{P}^+_\mathrm{m}, \mathcal{L}^+_\mathrm{m}), (\mathcal{P}^-_\mathrm{m}, \mathcal{L}^-_\mathrm{m}))$ be a partial twin Grassmannian of a projective space $\mathscr{S} = (\mathcal{P}, \mathcal{L})$ with respect to (U_+, U_-). Let $\mathfrak{F}^\sigma \subseteq \mathfrak{P}(\mathcal{P}^\sigma_\mathrm{m})$ be the set of finite subsets of $\mathcal{P}^\sigma_\mathrm{m}$ and set $\mathfrak{S}^\sigma := \{\bigcap X, \langle\bigcup X\rangle \mid X \in \mathfrak{F}^\sigma\} \subseteq \mathfrak{P}(\mathcal{P})$ for $\sigma \in \{+,-\}$. Note that every two elements of $\mathfrak{S}^\sigma$ are commensurate. Furthermore, if $\mathrm{rk}(U_\sigma) < \infty$, we obtain $\varnothing \in \mathfrak{S}^\sigma$ and $\mathscr{S} \in \mathfrak{S}^{-\sigma}$.

Lemma 3.1. *Let S and V be subspaces of $\mathscr{S}$ such that $S \leq V$ and $\mathrm{crk}_V(S) < \infty$. Then $S \in \mathfrak{S}^+$ implies $V \in \mathfrak{S}^+$.*

Proof. Let B be a set of $\mathrm{crk}_V(S)$ points such that $\langle B, S\rangle = V$.

First assume $S = \langle\bigcup M\rangle$ for a finite subset M of $\mathcal{P}^+_\mathrm{m}$. Let $U \in M$. Let $b \in B$. For every $b \in B$ there is a subspace $U_b \in \mathcal{P}^+_\mathrm{m}$ that contains b by Lemma 2.7. By (TG3) we may assume that U and U_b have a hyperplane in common. We conclude $V = \langle B, S\rangle = \langle\bigcup_{b\in B} U_b \cup \bigcup M\rangle \in \mathfrak{S}^+$.

Now assume $S = \bigcap M$ for a finite subset M of $\mathcal{P}^+_\mathrm{m}$. Let $U \in \mathfrak{U}^+$ be a subspace with $S \leq U$ such that $U \leq V$ or $V \leq U$ holds. By Proposition 2.9 we know $U \in \mathcal{P}^+_\mathrm{m}$. For $U \leq V$, we are in the situation above. Thus, we may assume $V < U$. Then there is a set $N \subseteq \mathfrak{U}^+$ of cardinality $\mathrm{crk}_U(V) + 1$ such that $V = \bigcap N$. Proposition 2.9 implies $N \subseteq \mathcal{P}^+_\mathrm{m}$ since every element of N contains S. This implies $V \in \mathfrak{S}^+$. ∎

Remark 3.2. Let $V \in \mathfrak{S}^+$ and let $S \leq V$ with $\mathrm{crk}_V(S) < \infty$. Then $V \in \mathfrak{S}^+$ does not necessarily imply $S \in \mathfrak{S}^+$. Indeed, suppose it would; since $\mathrm{crk}_\mathscr{S}(U_+) < \infty$ implies $\mathscr{S} \in \mathfrak{S}^+$, we would have $\mathcal{P}^+_\mathrm{m} = \mathfrak{U}^+$ whenever $\mathrm{crk}_\mathscr{S}(U_+) < \infty$. This contradicts what we have seen in Example 2.11.

We now introduce a type function for the elements of $\mathfrak{S}^+$ and $\mathfrak{S}^-$: Let $\sigma \in \{+,-\}$; then we define the *type function* $\mathrm{typ}\colon \mathfrak{S}^\sigma \to \mathbb{Z}$ such that:

$$\begin{aligned}
&\mathrm{typ}(U) := 0 \\
&\mathrm{typ}(V) := \sigma n \qquad \text{if } \mathrm{crk}_V(U) = n \text{ and} \\
&\mathrm{typ}(V) := -\sigma n \quad\ \ \text{if } \mathrm{crk}_U(V) = n
\end{aligned}$$

for subspaces $U \in \mathcal{P}^\sigma_\mathrm{m}$ and $V \in \mathfrak{S}^\sigma$.

Note that by this definition, two subspaces $W \in \mathfrak{S}^+$ and $W' \in \mathfrak{S}^-$ that are complementary in $\mathscr{S}$ are of the same type.

Let I be an index set and let $\{C_i \mid i \in I\}$ be a family of chains of $\mathfrak{S}^+$ such that for two distinct indices i and j of I, either $C_i \subset C_j$ or $C_j \subset C_i$. Then $\bigcup_{i \in I}$ is again a chain of $\mathfrak{S}^+$. Thus, Zorn's Lemma implies that $\mathfrak{S}^+$ possesses maximal chains. For $\sigma \in \{+, -\}$, let $\mathfrak{M}^\sigma$ denote the set of maximal chains of $\mathfrak{S}^\sigma \setminus \{\varnothing, \mathscr{S}\}$. Since any two subspaces that are elements of $\mathfrak{S}^\sigma$ are commensurate, every element of $\mathfrak{M}^\sigma$ is a set M that contains a unique subspace $U \in \mathfrak{S}^\sigma$ with $\mathrm{typ}(U) = n$ for every $n \in \mathrm{typ}(\mathfrak{S}^\sigma \setminus \{\varnothing, \mathscr{S}\})$. For a chain $M \in \mathfrak{M}^\sigma$, we denote by $\overline{M} := M \cup (\mathfrak{S}^\sigma \cap \{\varnothing, \mathscr{S}\})$ the corresponding maximal chain of $\mathfrak{S}^\sigma$.

Two chains $M_+ \in \mathfrak{M}^+$ and $M_- \in \mathfrak{M}^-$ are called *opposite* if and only if for every subspace $V_+ \in M_+$, there is a subspace $V_- \in M_-$ that is a complement to V_+ in $\mathscr{S}$. By the definition of M_+ and M_- we know that this definition is symmetric.

Lemma 3.3. *Let $M_+ \in \mathfrak{M}^+$ and let $N_- \subseteq \mathfrak{S}^-$ be a chain such that for every subspace $V \in N_-$ there is a complementary subspace $U \leq \mathscr{S}$ with $U \in M_+$. Then there is a chain $M_- \in \mathfrak{M}^-$ with $N_- \subseteq M_-$ that is opposite M_+.*

Proof. First assume $N_- = \varnothing$. Let $U_0 \in M_+ \cap \mathcal{P}^+_{\mathrm{m}}$. Then by (TG1) there is a subspace $V_0 \in \mathcal{P}^-_{\mathrm{m}}$ that is complementary to U_0 in $\mathscr{S}$. We show that there is a chain $M_- \in \mathfrak{M}^-$ with $V_0 \in M_-$ that is opposite M_+. Hence, from now on we may assume $N_- \neq \varnothing$.

Now let $V_0 \in N_-$ such that V_0 is not a hyperplane of any element of $N_- \cup \mathscr{S}$. Further let $U_0 \in M_+$ be the subspace that is a complement to V_0 in $\mathscr{S}$. Let $U_1 \in M_+$ be the element that is a hyperplane of U_0. Then $\langle U_1, V_0 \rangle$ is a hyperplane of $\mathscr{S}$. If U_0 is not the maximal element of N_-, let $V_2 \in N_-$ be the smallest subspace with $V_2 > V_0$. Then V_2 is the complement to an element of M_- and hence $\langle U_1, V_2 \rangle = \mathscr{S}$. Thus, there is a point $p \in V_2 \setminus \langle U_1, V_0 \rangle$. If U_0 is the maximal element of N_-, let p be an arbitrary point of $\mathscr{S} \setminus \langle U_1, V_0 \rangle$. Then $V_1 := \langle p, V_0 \rangle$ is a complement to U_1 in $\mathscr{S}$ and $V_1 \in \mathfrak{S}^-$ by Lemma 3.1. Thus, $N_- \cup \{V_1\}$ is a chain of $\mathfrak{S}^-$ such that every element of N_- is the complement in $\mathscr{S}$ to an element in M_+. Hence by induction, it remains to prove the claim for non-empty chains N_- such that every element of N_- is the hyperplane of an element either of N_- or of $\mathscr{S}$.

If N_- has no smallest element, N_- is maximal and we are done. Thus, we may assume that N_- has a smallest element V_0. Further we may assume $\mathrm{rk}(V_0) > 0$ since otherwise we are done. Let U_0 and U_1 be the elements of M_+ such that U_0 is a complement to V_0 in $\mathscr{S}$ and a hyperplane of U_1. Then U_1 and V_0 intersect in a point p. By Lemma 2.7 there is a subspace $W' \in \mathcal{P}^+_{\mathrm{m}}$ that contains p and hence by (TG1) there is a subspace $W \in \mathcal{P}^-_{\mathrm{m}}$ with $p \notin W$. If $V_0 \geq V$ for a subspace $V \in \mathcal{P}^-_{\mathrm{m}}$, we set $S := V \cap W$; otherwise we set $S := V_0 \cap S$. In both cases we obtain $S \in \mathfrak{S}^-$. Since $S \leq V_0$ and $p \notin S$, there is a hyperplane V_1 of V_0 with $p \notin V_0$ and $S \leq V_1$. Now V_1 is a complement of U_1 in $\mathscr{S}$ and we conclude $V_1 \in \mathfrak{S}^-$ by Lemma 3.1. Now the claim follows by induction. ∎

Note that for every chain $M_+ \in \mathfrak{M}^+$ there is an opposite chain $M_- \in \mathfrak{M}^-$ since in the previous lemma N_- might be empty. In the following let $M_+ \in \mathfrak{M}^+$

and $M_- \in \mathfrak{M}^-$ be opposite chains. By the previous lemma we may assume $U_+ \in M_+$ and $U_- \in M_-$. For $\sigma \in \{+,-\}$, let $\mathfrak{C}^\sigma := \{M \in \mathfrak{M}^\sigma \mid |M \smallsetminus M_\sigma| < \infty\}$ be the set of all maximal chains of $\mathfrak{S}^\sigma$ that differ from M_σ in finitely many positions.

Proposition 3.4. *For every chain $N_+ \in \mathfrak{C}^+$, there is a chain $N_- \in \mathfrak{C}^-$ that is opposite N_+.*

Proof. We may assume $N_+ \neq M_+$ since otherwise we are done by setting $N_- := M_-$. Since $N_+ \smallsetminus M_+$ is a finite set, there are subspaces U_0 and U_1 in N_+ such that $U \in N_+$ for every $U \in M_+$ with $U < U_0$ or $U > U_1$. Let V_0 and V_1 be the elements of M_- such that (U_0, V_0) and (U_1, V_1) are pairs of complements in $\mathscr{S}$. By Lemma 3.3 there is a chain $N_- \in \mathfrak{M}^-$ with $N_- \supseteq M_- \smallsetminus \{V \in M_- \mid V_1 \leq V \leq V_0\}$ that is opposite N_+. The claim follows with $\operatorname{crk}_{V_0}(V_1) < \infty$. ∎

For $\sigma \in \{+,-\}$, we denote by Δ^σ the set of all chains of $\mathfrak{S}^\sigma$ that are subchains of an element of $\mathfrak{C}^\sigma$. In other words, a chain C of $\mathfrak{S}^\sigma$ is an element of Δ^σ if and only if $|C \smallsetminus M_\sigma| < \infty$ and $C \cap \{\varnothing, \mathscr{S}\} = \varnothing$. Clearly, Δ^σ is a simplicial complex.

Theorem 3.5. *The simplicial complex Δ^+ is a chamber complex, where $\mathfrak{C}^+$ is the set of chambers.*

Proof. By definition every element of Δ^+ is contained in an element of $\mathfrak{C}^+$. Hence, it suffices to check that for every $C \in \mathfrak{C}^+$ there is a gallery $(C_i)_{0 \leq i \leq n}$ from $C_0 = C$ to $C_n = M_+$.

We may assume $C \neq M_+$. Let $V_1 \in C$ such that $V_1 \notin M_+$ and $U \in M_+$ for every $U \in C$ with $U < V_1$. Let V_0 be the hyperplane of V_1 that is an element of both $\overline{C}$ and $\overline{M}_+$. For $i \in \mathbb{N}$ and $i < \operatorname{crk}_{\mathscr{S}}(V_0)$ let $U_i \in \overline{M}_+$ and $V_i \in \overline{C}$ such that $\operatorname{crk}_{U_i}(V_0) = \operatorname{crk}_{V_i}(V_0) = i$. Further let $k \in \mathbb{N}$ be minimal such that $U \in \overline{M}_+$ for every $U \in \overline{C}$ with $U \geq V_k$ and set $C' := C \smallsetminus \{V_i \mid 0 < i < k\} = M_+ \smallsetminus \{U_i \mid 0 < i < k\}$.

For $0 < i < k$, we define recursively subspaces V_i' in the following way: If $V_i \leq U_{k-1}$, set $V_i' := V_i$. Otherwise set $V_i' := V_{i+1} \cap U_{k-1}$. In the second case $V_i \leq V_{i+1}$ implies $V_{i+1} \not\leq U_{k-1}$. Moreover, since $V_{i+1} \leq U_k$ and U_{k-1} is a hyperplane of U_k, we conclude that U_{k-1} intersects V_{i+1} in a hyperplane. Thus, in any case V_i' is a hyperplane of V_{i+1} and a subspace of U_{k-1}. Setting $V_0' := V_0$ we obtain that V_{i-1}' is a subspace of all V_i, V_{i+1} and U_{k-1}. Thus, in any case V_{i-1}' is a subspace of V_i'. This implies that $C_0 := C$ and $C_i := C' \cup \{V_j' \mid 0 < j \leq i\} \cup \{V_j \mid i < j < k\}$ for $0 < i < k$ are all elements of $\mathfrak{C}^+$. Moreover, $|C_i \smallsetminus C_{i+1}| = |C_{i+1} \smallsetminus C_i| \leq 1$ for $i < k-1$.

Note that $V_{k-1}' = U_{k-1}$. Hence, $U \in \overline{M}_+$ for every $U \in \overline{C}_{k-1}$ with $U \geq V_{k-1}'$ and $U \leq V_0$. Thus, repeating the previous construction at most $k-2$ more times finishes the proof. ∎

4 Buildings

Recall that in Definition 2.2 the projective space may have short lines. Considering only non-degenerate projective spaces, some of the following results can be obtained by applying the results of [Müh98] and [DMVM12]. We will show that the restriction to the non-degenerate case is not necessary. It suffices to claim the axiom (VY).

A *morphism of simplicial complexes* is a map $\varphi\colon \Delta \to \Delta'$ from a simplicial complex into another such that for any simplex $A \in \Delta$ the poset $\{X \in \Delta \mid X \leq A\}$ is mapped isomorphically onto $\{X \in \Delta' \mid X \leq A^{\varphi}\}$. Note that φ induces a map from the set of vertices of Δ into the set of vertices of Δ' which determines φ uniquely. A *morphism of chamber complexes* is a morphism of simplicial complexes such that chambers are mapped onto chambers.

Definition 4.1. Let Δ be a simplicial complex and let $\mathfrak{A}$ be a set of subcomplexes of Δ. The pair $(\Delta, \mathfrak{A})$ is called a *building* of which the elements of $\mathfrak{A}$ are called *apartments* if the following conditions hold:

- **(B1)** Δ is thick.
- **(B2)** The elements of $\mathfrak{A}$ are thin chamber complexes.
- **(B3)** Any two elements of Δ belong to an apartment.
- **(B4)** If two apartments Σ and Σ' contain two common simplices A and A', there exists an isomorphism of Σ onto Σ' which leaves A, A' and all simplices contained in one of them invariant.

A pair $(\Delta, \mathfrak{A})$ is called a *weak building* if it satisfies the axioms (B2), (B3) and (B4).

Let $\mathscr{S}$ be a projective space and for $\sigma \in \{+,-\}$, let U_σ, M_σ, $\mathfrak{C}^\sigma$ and Δ^σ be as in the previous section. By $\mathfrak{I}$ we denote the set of all independent sets of points of $\mathscr{S}$. Now let $\mathfrak{T}^\sigma \subseteq \overline{M_\sigma} \times \mathfrak{I}$ be the set of all pairs (U, B) for which there exists a subspace $V \in \overline{M_\sigma}$ such that the following conditions are fulfilled:

- **(T1)** For any two subspaces $\{X, Y\} \subseteq \{W \in \overline{M_\sigma} \mid W \leq U \vee W \geq V\}$ with $X \leq Y$, the identity $\mathrm{crk}_Y(X) = B \cap Y \setminus X$ holds.
- **(T2)** For every point $p \in B$, there are subspaces X and Y in $\overline{M_\sigma}$ with $p \in Y \setminus X$.

Remark 4.2. Let $(U, B) \in \mathfrak{T}^+$ and let $V \in \overline{M_+}$ such that (T1) and (T2) are fulfilled. Then obviously, (T1) and (T2) are fulfilled for (U, B) and every V' with $V' \in \overline{M_+}$ and $V' \geq V$. It follows that there exists a minimal subspace $V \in \overline{M_\sigma}$ under the condition that (T1) and (T2) are fulfilled and $V \geq U$.

Motivated by this remark we denote for a pair $(U, B) \in \mathfrak{T}^+$ the smallest subspace V with $U \leq V \in \overline{M_\sigma}$ for which (T1) and (T2) are fulfilled by $V_\mu(U, B)$.

Remark 4.3. Let U_0 and V_0 be two arbitrary elements of $\overline{M_+}$ with $U_0 \leq V_0$. For $0 \leq i \leq \mathrm{rk}(U_0)$, let $U_{i+1} \in \overline{M_+}$ be the subspace with $\mathrm{crk}_{U_0}(U_{i+1}) = i+1$ and let p_i be any point of $U_i \setminus U_{i+1}$. For $0 \leq i < \mathrm{crk}_{\mathscr{S}}(V_0)$ let $V_{i+1} \in \overline{M_+}$ be the subspace with $\mathrm{crk}_{V_{i+1}}(V_0) = i+1$ and let q_i be any point of $V_{i+1} \setminus V_i$. Further let $B_0 \subseteq V_0 \setminus U_0$ be an independent set of points with $|B_0| = \mathrm{crk}_{V_0}(U_0)$. Set

$B := B_0 \cup \{p_i \mid i \leq \operatorname{rk}(U_0)\} \cup \{q_i \mid i < \operatorname{crk}_{\mathscr{S}}(V_0)\}$. Then $(U_0, B) \in \mathfrak{T}$. Moreover, every element of $\mathfrak{T}$ can be obtained in this way.

For a pair $(U, B) \in \mathfrak{T}^\sigma$, we set

$$\mathfrak{S}(U,B) := \{\langle I, W\rangle \mid I \subseteq B \ \wedge\ |I| < \infty\,,\ W \in \overline{M_\sigma} \ \wedge\ W \leq U\} \quad \text{and}$$
$$\Sigma(U,B) := \{C \in \Delta^\sigma \mid C \subseteq \mathfrak{S}(U,B)\}$$

Lemma 4.4. *Let $(U,B) \in \mathfrak{T}^+$ and set $\mathfrak{F} := \{I \subseteq B \mid |I| < \infty\}$.*

(a) Let $X \in M_+$ with $X \leq U$ or $X \geq V_\mu(U,B)$. Then $X \in \mathfrak{S}(U,B)$.

(b) Let V and W be elements of $\overline{M_+}$ with $W \leq V \leq U$. Then $\{\langle I, W\rangle \mid I \in \mathfrak{F}\} \supseteq \{\langle I, V\rangle \mid I \in \mathfrak{F}\}$.

(c) Let V and W be elements of $\mathfrak{S}(U,B)$ with $W \leq V$. Then $\langle B \cap V \smallsetminus W, W\rangle = V$ and $|B \cap V \smallsetminus W| = \operatorname{crk}_V(W)$.

Proof. Claim (a) is a direct consequence of (T1) and the fact that B is independent. By the same arguments there is a set of points $I \in \mathfrak{F}$ such that $V = \langle I, W\rangle$ for elements V and W of $\overline{M_+}$. This proves claim (b). Now let V and W be arbitrary elements of $\mathfrak{S}(U,B)$ with $W \leq V$. Then (b) implies that there is a subspace $X \in \overline{M_+}$ such that $\{\langle I, W\rangle \mid I \in \mathfrak{F}\}$ contains both V and W. Now (c) follows since B is independent. ∎

We proceed by introducing certain groups that play an import role in the theory of buildings. See also [AS08, section 2.4] for the following definition.

Definition 4.5. Let (W,S) be a pair consisting of a group W and a set S of reflections such that W admits the presentation $\langle S \mid (st)^{m_{s,t}} = 1\rangle$, where $m_{s,t}$ is the order of $st \in W$ and for any pair $(s,t) \in S \times S$, there is a relation. Then we call (W,S) a *Coxeter system* and W a *Coxeter group*.

Remark 4.6. We denote by $\operatorname{Sym}_0(\mathbb{Z})$ the subgroup of the symmetric group $\operatorname{Sym}(\mathbb{Z})$ that contains all permutations $\pi \in \operatorname{Sym}(\mathbb{Z})$ for which $\{z \in \mathbb{Z} \mid z \neq z^\pi\}$ is finite. Let $s_i \in \operatorname{Sym}_0(\mathbb{Z})$ be the transposition that exchanges i and $i+1$ and fixes all other numbers. Then $(\operatorname{Sym}_0(\mathbb{Z}), \{s_i \mid i \in \mathbb{Z}\})$ is a Coxeter system.

In the following we are interested in permutations of the set $\operatorname{typ}(\overline{M_+} \smallsetminus \{\varnothing\})$; note that this set might differ from $\mathbb{Z}$. Therefore we denote by S_Δ the set of all transpositions s_i with $i \in \operatorname{typ}(M_+)$. Then $\langle S_\Delta\rangle$ is the subgroup of $\operatorname{Sym}_0(\mathbb{Z})$ that stabilises $\mathbb{Z} \smallsetminus \operatorname{typ}(\overline{M_+} \smallsetminus \{\varnothing\})$ pointwise. Note that there are the following four cases:

$$\begin{aligned}
\overline{M_+} \smallsetminus \{\varnothing\} &= \{-n, \ldots, m\}\\
\overline{M_+} \smallsetminus \{\varnothing\} &= \{i \in \mathbb{Z} \mid i \geq -n\}\\
\overline{M_+} \smallsetminus \{\varnothing\} &= \{i \in \mathbb{Z} \mid i \leq m\}\\
\overline{M_+} \smallsetminus \{\varnothing\} &= \mathbb{Z}\,,
\end{aligned}$$

where n and m are positive Integers. One of the first two cases occurs when the underlying point-line space is a Grassmannian in the usual sense. The third case occurs for Grassmannian of corank-m-spaces and the last case occurs whenever both $\mathrm{rk}(M_+)$ and $\mathrm{crk}_{\mathscr{S}}(M_+)$ are infinite.

Let $C \in \mathfrak{C}^+$ be a chamber and for $i \in \mathrm{typ}(\overline{C})$, let $U_i \in \overline{C}$ denote the element of type i. Further let $(U, B) \in \mathfrak{T}^+$ such that $C \in \Sigma(U, B)$. Then for every $i \in \mathrm{typ}(\overline{C} \setminus \{\varnothing\})$, there is exactly one point $p_i \in B$ with $p_i \in U_i \setminus U_{i-1}$.

Now let $\pi \in \langle S_\Delta \rangle$ be a permutation. Then there is exactly one chamber D in $\Sigma(U, B)$ such that $p_{i^\pi} \in V_i \setminus V_{i-1}$, where $V_j \in \overline{D}$ denotes the element of type j for every $j \in \mathrm{typ}(\overline{C})$. We define $C^{(\pi,B)} := D$.

Lemma 4.7. *Let $C \in \mathfrak{C}^+ \cap \Sigma(U, B)$, where $(U, B) \in \mathfrak{T}^+$, and let $\pi \in \langle S_\Delta \rangle$. Then for every $i \in \mathbb{Z}$, the chambers $C^{(\pi,B)}$ and $C^{(s_i\pi,B)}$ are adjacent and differ in the element of type i.*

Proof. For $i \in \mathrm{typ}(\overline{C})$, let $U_i \in \overline{C}$, $V_i \in \overline{C^{(\pi,B)}}$ and $W_i \in \overline{C^{(s_i\pi,B)}}$ be the subspaces of type i. Further let $(U, B) \in \mathfrak{T}^+$ such that $C \in \Sigma(U, B)$ and let $p_i \in B$ denote the point with $p_i \in U_i \setminus U_{i-1}$. Now let $i \in \mathrm{typ}(C)$ and let $j \in \mathrm{typ}(C) \setminus \{i, i+1\}$. Then $j^{s_i} = j$ and hence, $p_{j^{s_i\pi}} = p_{j^\pi}$. Analogously, $p_{i^{s_i\pi}} = p_{(i+1)^\pi}$ and $p_{(i+1)^{s_i\pi}} = p_{i^\pi}$. This implies $V_j = W_j$ for every $j \in \mathbb{Z} \setminus \{i\}$. ∎

Proposition 4.8. *Let $(U, B) \in \mathfrak{T}^+$. Then $\Sigma(U, B)$ is a thin chamber complex whose set of chambers is $\mathfrak{C}^+ \cap \Sigma(U, B)$.*

Proof. Let $A \in \Sigma(U, B)$. Since $A \in \Delta^+$, there are subspaces $U' \leq U$ and $V' \geq V_\mu(U, B)$ in $\overline{M_+}$ such that $U' < X < V'$ for every $X \in A \setminus \overline{M_+}$. By Lemma 4.4(a) we obtain $X \in \mathfrak{S}(U, B)$ for every subspace $X \in M_+$ with $X \leq U'$ or $X \geq V'$. Furthermore, $A' := A \cup \{X \in M_+ \mid X \leq U' \ \vee \ X \geq V'\}$ is a chain and therefore an element of $\Sigma(U, B)$.

Let W' and W be two elements of A' with $U' \leq W' < W \leq V'$ such that there is no subspace $X \in A'$ with $W' < X < W$. By Lemma 4.4(c) we know that W' and W are both contained in $\{\langle I, U' \rangle \mid I \subseteq B \ \wedge \ |I| < \infty\}$ and furthermore, $|B \cap W \setminus W'| = \mathrm{crk}_W(W')$. Thus, there are subspaces $W_i \in \{\langle I, U' \rangle \mid I \subseteq B \ \wedge \ |I| < \infty\}$ for $0 < i < \mathrm{crk}_W(W')$ with $W' \leq W_i < W$ such that $\mathrm{crk}_{W_i}(W') = i$. Adding for each choice of W' and W the subspaces W_i to A' leads to a chain C that is contained in $\mathfrak{C}^+ \cap \Sigma(U, B)$. Since C is maximal in Δ^+ it is also maximal in $\Sigma(U, B)$. Thus, $\mathfrak{C}(U, B) := \mathfrak{C}^+ \cap \Sigma(U, B)$ is the set of maximal elements of $\Sigma(U, B)$.

Let C and C' be two elements of $\mathfrak{C}(U, B)$. Then again there are subspaces $U' < U$ and $V' > V_\mu(U, B)$ in $\overline{M_+}$ such that $M_C := \{X \in M_+ \mid X \leq U' \ \vee \ X \geq V'\} \subseteq C \cap C'$. Thus, there is a permutation $\pi \in \langle S_\Delta \rangle$ such that $C^{(\pi,B)} = C'$. It follows from Remark 4.6 and Lemma 4.7 that there is a sequence of adjacent chambers from C to C'. This proves that $\Sigma(U, B)$ is a chamber complex whose set of chambers is $\mathfrak{C}(U, B)$.

Now assume that C and C' are adjacent. Then there are subspaces $W \in C$ and $W' \in C'$ such that $C \setminus \{W\} = C' \setminus \{W'\}$. Let X and Y be the elements of C with $\mathrm{crk}_X(W) = \mathrm{crk}_W(Y) = 1$. Then $X = \langle W, W' \rangle$ and $Y = W \cap W'$. By Lemma 4.4(c) there are exactly two points p and q in $B \cap X \setminus Y$ and hence, every element of $\mathfrak{C}(U, B)$ containing $C \setminus \{W\}$ contains either $\langle p, Y \rangle$ or $\langle q, Y \rangle$. We conclude that Σ is a thin chamber complex. ∎

Let $(U,B) \in \mathfrak{T}^+$. Then for a subspace $W \in M_+$ with $U < W < V_\mu(U,B)$, the singleton $\{W\}$ is a vertex of $\Sigma(U,B)$ if and only if $\langle W \cap B, U\rangle = W$. This implies that M_+ is not necessarily a chamber of $\Sigma(U,B)$.

Lemma 4.9. *Let $\mathscr{S}$ be a projective space of finite rank n and let $\{C_i \mid 0 \le i \le n\}$ and $\{D_i \mid 0 \le i \le n\}$ be chains of subspaces of $\mathscr{S}$ with $\mathrm{rk}(C_i) = \mathrm{rk}(D_i) = i$ and let $C_{-1} = D_{-1} = \varnothing$. Set $\pi\colon \{0,\dots,n\} \to \{0,\dots,n\}$ such that $i^\pi = \min\{j \le n \mid \langle C_{i-1}, D_j\rangle \ge C_i\}$. Then π is a permutation. Furthermore, $i = \min\{j \le n \mid \langle C_j, D_{i^\pi-1}\rangle \ge D_{i^\pi}\}$.*

Proof. Since $\langle C_{i-1}, D_n\rangle \ge C_i$ for every $i \le n$, we obtain $i^\pi \in \{0,\dots,n\}$. Suppose there are indices $0 \le j < i < n$ with $j^\pi = i^\pi =: k$. Then there are points $p \in C_j \setminus C_{j-1}$ and $q \in C_i \setminus C_{i-1}$ that are both contained in $D_k \setminus D_{k-1}$. This implies $p \ne q$. Thus, the line pq intersects the hyperplane D_{k-1} of D_k in a point $r \ne p$. With $q \in C_i \setminus C_{i-1}$ and $p \in C_{i-1}$, we obtain $r \notin C_{i-1}$ and therefore $r \in (C_i \cap D_{k-1}) \setminus C_{i-1}$, a contradiction to the definition of i^π. Thus, π is a permutation.

Analogously, there is a permutation ψ such that $i^\psi = \min\{j \le n \mid \langle C_j, D_{i-1}\rangle \ge D_i\}$ for every $i \le n$. Since there is a point $p \in C_i \setminus C_{i-1}$ with $p \in D_{i^\pi} \setminus D_{i^\pi-1}$. Hence, $(i^\pi)^\psi \le i$. This implies that the permutation $\psi \circ \pi$ is the identity. The claim follows. ∎

Proposition 4.10. *For any two simplices C and D of Δ^+ there is a pair $(U,B) \in \mathfrak{T}^+$ such that $\{C,D\} \subseteq \Sigma(U,B)$.*

Proof. Since both C and D are contained in a chamber of Δ^+, it suffices to prove the claim for the case that both C and D are chambers. Let U and V be elements of $C \cap D$ such that $\{X \in C \mid X \le U \ \vee\ X \ge V\} \subseteq D \cap M_+$. If $C = D = M_+$, we choose $U = V = U_+$.

Set $k := \mathrm{crk}_V(U)$. For $0 \le i \le k$, let $X_i \in C$ and $Y_i \in D$ be the subspaces with $\mathrm{crk}_{X_i}(U) = \mathrm{crk}_{Y_i}(U) = i$. Let $S \le V$ be a subspace of rank $k-1$ that is complementary to U in V. Then $\mathrm{rk}(S \cap X_i) = \mathrm{rk}(S \cap Y_i) = i-1$. Thus we may apply Lemma 4.9 to conclude that there are points r_i for $0 \le i < k$ and a permutation π of $\{0,\dots,k-1\}$ such that $r_i \in S \cap X_{i+1} \cap Y_{i^\pi+1}$ and $r_i \notin X_i \cup Y_{i^\pi}$. This implies $X_i = \langle r_j, U \mid j < i\rangle$ and $Y_i = \langle r_j, U \mid j^\pi < i\rangle$ for $i \le k$.

Now for $i \in \mathbb{N}$ with $i \le \mathrm{rk}(U)$, let $U_{i+1} \in \overline{M}_+$ be the subspace with $\mathrm{crk}_U(U_{i+1}) = i+1$ and choose a point $p_i \in U_i \setminus U_{i+1}$. Correspondingly, for $i \in \mathbb{N}$ and $i < \mathrm{crk}_{\mathscr{S}}(V)$, let $V_{i+1} \in \overline{M}_+$ be the subspace with $\mathrm{crk}_{V_{i+1}}(V) = i+1$ and let $q_i \in V_{i+1} \setminus V_i$. Set $B := \{r_i \mid i < k\} \cup \{p_i \mid i \le \mathrm{rk}(U)\} \cup \{q_i \mid i < \mathrm{crk}_{\mathscr{S}}(V)\}$. Then $(U,B) \in \mathfrak{T}^+$ and C and D are both contained in $\Sigma(U,B)$. ∎

Lemma 4.11. *Let $(U,B) \in \mathfrak{T}^+$. Further let V and W be elements of $\mathfrak{S}(U,B)$. Then $V \cap W$ and $\langle V, W\rangle$ are again contained in $\mathfrak{S}(U,B)$.*

Proof. By definition of $\mathfrak{S}(U,B)$ there is a maximal subspace $U' \in \overline{M_+}$ with $U' \le V \cap W$. By Lemma 4.4(c) there are finite subsets B_V and B_W of $B \setminus U'$ such that $V = \langle B_V, U'\rangle$ and $W = \langle B_W, U'\rangle$. This implies $\langle V, W\rangle = \langle B_V \cup B_W, U'\rangle \in \mathfrak{S}(U,B)$. Since B is an independent set of points, we conclude $|B_W \setminus B_V| = \mathrm{crk}_{\langle V,W\rangle}(V) = \mathrm{crk}_W(V \cap W)$. Thus, $\mathrm{crk}_{V\cap W}(U') = |B_W| - |B_W \setminus B_V| = |B_W \cap B_V|$ and therefore $V \cap W = \langle B_V \cap B_W, U'\rangle \in \mathfrak{S}(U,B)$. ∎

Lemma 4.12. *Let $(U, B) \in \mathfrak{T}^+$ and let $U' \in \overline{M_+}$ with $U' \leq U$. Then $\Sigma(U', B) = \Sigma(U, B)$.*

Proof. As a direct consequence of Lemma 4.4(b) we conclude $\mathfrak{S}(U', B) = \mathfrak{S}(U, B)$. The claim follows. ∎

Proposition 4.13. *Let C and D be two simplices of Δ^+ and let (U_0, B_0) and (U_1, B_1) be two pairs of $\mathfrak{T}^+$ such that $\{C, D\} \subseteq \Sigma(U_0, B_0) \cap \Sigma(U_1, B_1)$. Then there is an isomorphism $\varphi\colon \Sigma(U_0, B_0) \to \Sigma(U_1, B_1)$ that leaves every simplex that is contained in C or D invariant.*

Proof. First of all, we may assume $U_0 = U_1 =: U$ by Lemma 4.12. Moreover, since both C and D are elements of Δ^+ we may assume $X > U$ for every subspace $X \in (C \cup D) \smallsetminus M_+$. Correspondingly, let $V \in \overline{M_+}$ such that $X < V$ for every subspace $X \in (C \cup D) \smallsetminus M_+$. If $C \cup D \subseteq M_+$, we set $V := U$. We denote by P_C (resp. P_D) the set of all pairs (W, W'), where W and W' are subspaces contained in C (resp. D) such that $U \leq W < W' \leq V$ and $X \geq W'$ for every $X \in C$ (resp. $X \in D$) with $X > W$.

We may prove the claim for simplices $\{C', D'\} \subseteq \Sigma(U, B_0) \cap \Sigma(U, B_1)$ with $C \subseteq C'$ and $D \subseteq D'$. Hence, by Lemma 4.4(a) we may assume $\{X \in M_+ \mid X \leq U \ \vee \ X \geq V\} \subseteq C \cap D$. Let $(X_0, X_1) \in P_C$ and let $(Y_0, Y_1) \in P_D$. By Lemma 4.11 we know that $X' := \langle X_0, X_1 \cap Y_0 \rangle$ is contained in both $\mathfrak{S}(U, B_0)$ and $\mathfrak{S}(U, B_1)$. Furthermore, X' is comparable to every element of C. Thus, we may assume $X' \in C$ or, equivalently, that $X_1 \cap Y_0 \nsubseteq X_0$ implies $\langle X_0, X_1 \cap Y_0 \rangle = X_1$. Analogously, we may assume that $Y_1 \cap X_0 \nsubseteq Y_0$ implies $\langle Y_0, Y_1 \cap X_0 \rangle = Y_1$. We refer to this assumption as $(*)$.

Let $(X_0, X_1) \in P_C$ and let $(Y_0, Y_1) \in P_D$ such that Y_1 is minimal under the condition $(X_1 \smallsetminus X_0) \cap Y_1 \neq \varnothing$. By $(*)$ this is equivalent to ($X_1 \cap Y_0 \leq X_0$ and $\langle X_0, X_1 \cap Y_1 \rangle = X_1$). This yields $\mathrm{crk}_{X_1 \cap Y_1}(X_1 \cap Y_0) = \mathrm{crk}_{X_1}(X_0)$. By Lemma 4.11 we know that $X_1 \cap Y_0$ and $X_1 \cap Y_1$ are both contained in $\mathfrak{S}(U, B_0) \cap \mathfrak{S}(U, B_1)$. Thus for $i \in \{0, 1\}$, Lemma 4.4(c) implies $|B_i \cap X_1 \cap Y_1 \smallsetminus Y_0| = |B_i \cap X_1 \smallsetminus X_0|$. Since $\langle X_0, X_1 \cap Y_1 \rangle = X_1$ and $X_1 \cap Y_0 \leq X_0$, we conclude $B_i \cap X_1 \cap Y_1 \smallsetminus Y_0 = B_i \cap X_1 \smallsetminus X_0$.

Particularly, this implies $B_i \cap X_1 \smallsetminus X_0 \subseteq B_i \cap Y_1 \smallsetminus Y_0$ for $i \in \{0, 1\}$. By analogous reasons, there is a pair $(X_0', X_1') \in P_C$ such that $B_i \cap Y_1 \smallsetminus Y_0 \subseteq B_i \cap X_1' \smallsetminus X_0'$. Since $X_1 \smallsetminus X_0$ and $X_1' \smallsetminus X_0'$ are either equal or disjoint, this yields $(X_0', X_1') = (X_0, X_1)$ and thus, $B_i \cap X_1 \smallsetminus X_0 = B_i \cap Y_1 \smallsetminus Y_0$. We conclude that mapping (X_0, X_1) to (Y_0, Y_1) induces a bijection ψ from P_C onto P_D such that $B_i \cap W_1 \smallsetminus W_0 = B_i \cap W_1^{\psi} \smallsetminus W_0^{\psi}$ for every $(W_0, W_1) \in P_C$ and $i \in \{0, 1\}$.

Set $\overline{C} := C \cup (\{\varnothing, \mathscr{S}\} \cap \mathfrak{S}(U, B_0))$. We define a map $\varphi'\colon B_0 \to B_1$ such that for every two successive subspaces $X_0 < X_1$ of $\overline{C}$ the following holds: If $X_1 \leq U$ or $X_0 \geq V$, the unique point of $B_0 \cap X_1 \smallsetminus X_0$ is mapped on the unique point of $B_1 \cap X_1 \smallsetminus X_0$. Otherwise $(X_0, X_1) \in P_C$ and the points of $B_0 \cap X_1 \smallsetminus X_0$ are mapped bijectively onto the points of $B_1 \cap X_1 \smallsetminus X_0$. Finally, we define a map $\varphi\colon \mathfrak{S}(U, B_0) \to \mathfrak{S}(U, B_1)$ such that $\langle I, W \rangle^{\varphi} = \langle I^{\varphi'}, W \rangle$ for every finite subset $I \subseteq B_0$ and every subspace $W \in \overline{M_+}$ with $W \leq U$. It is straightforward to check that this map induces a well-defined isomorphism from $\Sigma(U, B_0)$ onto $\Sigma(U, B_1)$ that leaves every simplex that is contained in C or D invariant. ∎

For $\sigma \in \{+,-\}$, we set $\mathfrak{A}^\sigma := \{\Sigma(U,B) \mid (U,B) \in \mathfrak{T}^\sigma\}$.

Theorem 4.14. *For $\sigma \in \{+,-\}$, the pair $(\Delta^\sigma, \mathfrak{A}^\sigma)$ is a weak building. Moreover, $(\Delta^\sigma, \mathfrak{A}^\sigma)$ is a building if and only if all short lines of $\mathscr{S}$ intersect $\bigcap \overline{M_\sigma}$ or $\mathscr{S} \setminus \bigcup \overline{M_\sigma}$.*

Proof. Axioms (B2), (B3) and (B4) follow from the Propositions 4.8, 4.10 and 4.13. Thus, $(\Delta^\sigma, \mathfrak{A}^\sigma)$ is a weak building.

Set $S := \bigcap M_\sigma$ and $T := \bigcup M_\sigma$. First assume that every line of T that is disjoint to S is a thick line. Let C and C' be two adjacent chambers of Δ^σ and let V and V' be the subspaces of $\mathscr{S}$ with $C \setminus C' = \{V\}$ and $C' \setminus C = \{V'\}$. Set $U := V \cap V'$ and $W := \langle V, V' \rangle$. Then $\mathrm{crk}_W(U) = 2$. Hence, there is a line $l \leq W$ that is disjoint to U. Since $U \geq S$ and $W \leq T$, this implies that l is thick. Now for every point $p \in l$, the subspace $V_p := \langle p, U \rangle$ intersects l in p. Moreover, $(C \setminus \{V\}) \cup \{V_p\} \in \mathfrak{C}^\sigma$. Thus, Δ^σ is thick.

Now assume there is a short line l with $l \subseteq T \setminus S$. Then there are subspaces $U \in \overline{M_\sigma}$ and $W \in \overline{M_\sigma}$ with $l \subseteq W \setminus U$. Thus, there is a chamber $C \in \mathfrak{C}^\sigma$ such that U and $\langle l, U \rangle$ are elements of $\overline{C}$. Let p and q be the two points of l. We may assume $\langle p, U \rangle \in C$. Then there are exactly two chambers in $\mathfrak{C}^\sigma$ that contain $C \setminus \{\langle l, U \rangle\}$. Thus, Δ^σ is not thick. ∎

5 Twin buildings

Throughout this section, let Δ^σ, $\mathfrak{C}^\sigma$, $\mathfrak{A}^\sigma$ and $\mathfrak{T}^\sigma$ be defined as in the previous section, where $\sigma \in \{+,-\}$. Moreover, let $s_i \in S_\Delta$ be the permutation that exchanges i and $i+1$.

In order to introduce twin buildings, we define a certain distance function on $\mathfrak{C}^+$ that is based on the Coxeter system $(\langle S_\Delta \rangle, S_\Delta)$ in the following way: Let C and D be two adjacent chambers of $\mathfrak{C}^+$. Further let $V \in C$ be the subspace that is not contained in D. We say that C and D are s_i-adjacent, where $i = \mathrm{typ}(V)$. Now let $(C_i)_{i \leq n}$ be a gallery, such that C_i and C_{i+1} are t_i-adjacent with $t_i \in S_\Delta$ for $i < n$. Then we call the gallery $(C_i)_{i \leq n}$ of *type* $(t_i)_{i<n}$. A gallery from C to D is called *minimal* if there is no shorter gallery from C to D.

Let (W,S) be a Coxeter system. A word $t_0 t_1 \cdots t_n$, where $n \in \mathbb{N}$ and $t_i \in S$ for $i \leq n$ is called a *reduced word* if for every word $r_0 r_1 \cdots r_m = t_0 t_1 \cdots t_n \in W$ with $r_i \in S$, we have $m \geq n$.

Proposition 5.1. *Let C and D be two chambers of $\mathfrak{C}^+$ and let $(C_i)_{i \leq n}$ and $(D_i)_{i \leq n}$ be two minimal galleries from C to D. Further let $(C_i)_{i \leq n}$ be of type $(t_i)_{i<n}$ and let $(D_i)_{i \leq n}$ be of type $(r_i)_{i<n}$. Then $t_0 t_1 \cdots t_{n-1} = r_0 r_1 \cdots r_{n-1}$ and both are reduced words of (W,S).*

Proof. This is an immediate consequence of [AS08, Lemma 5.16]. ∎

This result motivates the following definition. We define the Weyl distance only for the buildings we are dealing with in this context although this a fundamental concept for buildings in general.

Definition 5.2. Let C and D be two chambers of $\mathfrak{C}^+$ such that there is a minimal gallery from C to D of type $(t_0, \ldots, t_n)$. Then we call $w = t_0 t_1 \cdots t_n \in \langle S_\Delta \rangle$ the *Weyl distance* from C to D, denoted by $\delta_+(C,D)$. By $l(w) := n$ we denote the *length* of w with respect to $(\langle S_\Delta \rangle, S_\Delta)$.

Note that if $\delta(C,D) = w$, then $\delta(D,C) = w^{-1}$. This is an immediate consequence of Proposition 5.1. Let B be a building and let (W,S) be a Coxeter system such that B has a Weyl distance defined by (W,S). Then we call B of type (W,S).

We now introduce twin buildings following [AS08, 5.8].

Definition 5.3. Let (W,S) be a Coxeter system and let $\mathfrak{B}^+$ and $\mathfrak{B}^-$ be the sets of chambers of buildings $\mathfrak{C}^+$ and $\mathfrak{C}^-$ of type (W,S) together with a *codistance* function

$$\delta\colon (B^+ \times B^-) \cup (B^- \times B^+) \to W$$

satisfying the following conditions for each pair of chambers $(C,D) \in B^\sigma \times B^{-\sigma}$, where $\sigma \in \{+,-\}$, $w := \delta(C,D)$ and δ_σ denotes the Weyl distance of B^σ.

- **(TW1)** $\delta(C,D) = \delta(D,C)^-$.
- **(TW2)** Let $C' \in B^\sigma$ with $\delta_\sigma(C',C) = s \in S$ and $l(sw) < l(w)$. Then $\delta(C',D) = sw$.
- **(TW3)** For any $s \in S$, there exists a chamber $C' \in B^\sigma$ with $\delta_\sigma(C',C) = s$ and $\delta(C',D) = sw$.

Then we call $(\mathfrak{C}^+,\mathfrak{C}^-)$ a *twin building* of type (W,S) with *codistance function* δ.

To prove that (Δ^+,Δ^-) is a twin building of type $(\langle S_\Delta\rangle, S_\Delta)$, we define a codistance function for (Δ^+,Δ^-) with values in $\langle S_\Delta\rangle$ and show that it satisfies the axioms (TW1), (TW2) and (TW3).

For $\sigma \in \{+,-\}$, let $C \in \mathfrak{C}^\sigma$ and $D \in \mathfrak{C}^{-\sigma}$. Since M_σ differs in only finitely many elements from C and $M_{-\sigma}$ differs in only finitely many elements from D, there are subspaces $U \in C$ and $V \in C$ of $\mathscr{S}$ such that for every subspace $W \in C$ with $W \geq V$ or $W \leq U$ there is a complementary subspace W' in $\mathscr{S}$ with $W' \in D$. Let $U' \in D$ and $V' \in D$ be the complementary subspaces to U and V, respectively.

Set $S := V \cap U'$. Then S is a complement to U in V and a complement to V' in U'. Now let $D' := \{W \in C \mid W \leq U \ \vee\ W \geq V\} \cup \{\langle S \cap W, U\rangle \mid W \in D\}$ be the chamber that arises by substituting the subspaces of C between U and V by subspaces that are the span of U and $S \cap W$ with $W \in D$ and $V' \leq W \leq U'$. Note that for every subspace $W' \in C$ with $U \leq W' \leq V'$ there is exactly one subspace $W \in D$ and $V' \leq W \leq U'$ such that $W' \gtreqless \langle U, S \cap W\rangle$. We define a partial map $\psi_\sigma\colon \mathfrak{C}^\sigma \times M_\sigma \times M_\sigma \times \mathfrak{C}^{-\sigma} \to \mathfrak{C}^\sigma$ such that $\psi(C,U,V,D) := D'$. In the following, we will write ψ rather than ψ_σ.

In the following lemma for two integers $i < j$, we denote by $w_{i,j} \in \langle S_\Delta\rangle$ the involution that exchanges $i+k$ and $j-k$ for every $k \in \mathbb{N}$ with $2k < i-j$.

Lemma 5.4. *Let $C \in \mathfrak{C}^+$ and $D \in \mathfrak{C}^-$. Further let $U \in C$ and $V \in C$ be subspaces of $\mathscr{S}$ such that for every subspace $W \in C$ with $W \geq V$ or $W \leq U$ there is a complementary subspace W' in $\mathscr{S}$ with $W' \in D$. Set $i :=$ typ(U) and $j :=$ typ(V) and let $U' \in C$ with* typ$(U') = i-1$ *and $V' \in C$ with* typ$(V') = j+1$. *Then*

$$\begin{aligned}\delta_+\left(C, \psi(C,U,V,D)\right)\cdot w_{i+1,j} &= \delta_+\left(C, \psi(C,U',V,D)\right)\cdot w_{i,j}\\ &= \delta_+\left(C, \psi(C,U,V',D)\right)\cdot w_{i+1,j+1}.\end{aligned}$$

Proof. By W_k we denote the element of C of type k. Analogously, we denote the element of D of type k by W'_k. Let $(W, B) \in \mathfrak{T}^+$ such that $\Sigma := \Sigma(W, B)$ is an apartment that contains both C and $\psi(C, U', V, D)$. Since both, C and $\psi(C, U', V, D)$ contain the subspaces V and V', we can choose B such that the point of $B \cap V' \setminus V$ is the unique point of $V' \cap W'_j$, which we denote by p. Set $S := V \cap W'_{i-1}$. Further set $S_k := S \cap W'_{j-k}$ for $0 < k \leq j - i + 1$ and set $S_0 = \varnothing$. Then $\mathrm{rk}(S_k) = k - 1$ for every $k \leq j - i + 1$.

Note that $S_k = V \cap W'_{j-k}$ and $\langle p, S_k\rangle = V' \cap W'_{j-k}$ for $k \leq j - i + 1$. Thus, the element of $\psi(C, U, V, D)$ of type $i + k$ equals $\langle U, S_k\rangle = \langle U, \langle U', S_k\rangle\rangle$ for $k \leq j - i$. Since both U and $\langle U', S_k\rangle$ are contained in $\mathfrak{S}(U, B)$, Lemma 4.11 implies $\langle U, S_k\rangle \in \mathfrak{S}(U, B)$ and therefore $\psi(C, U, V, D) \in \Sigma$. Finally, for $k \leq j - i$, the element of type $i + k + 1$ of $\psi(C, U, V', D)$ is $\langle U, p, S_k\rangle$. Since $p \in B$ and $\langle U, S_k\rangle \in \mathfrak{S}(U, B)$, we conclude $\langle U, p, S_k\rangle \in \mathfrak{S}(U, B)$. Hence, $\psi(C, U, V', D)$ is an element of Σ.

By Lemma 4.4(c) there is for every $0 \leq k \leq j - i$ exactly one point $p_k \in B$ with $p_k \in \langle U', S_{k+1}\rangle \setminus \langle U', S_k\rangle$. Since $p_k \notin U$ for every $k < j - i$ and $\langle s_k, U' \mid k \leq j - i\rangle$, we obtain $p_{j-i} \in U \setminus U'$ by Lemma 4.4(c). This implies $(\psi(C, U, V, D))^{(\pi_0, B)} = \psi(C, U, V', D)$ for $\pi_0 = s_{i+1}s_{i+2}\cdots s_j$. Let $\pi \in \langle S_\Delta\rangle$ such that $C^{(\pi, B)} = \psi(C, U, V, D)$. By Lemma 4.7, there is a gallery of type $(s_j, s_{j-1}, \ldots, s_{i+1})$ from $\psi(C, U, V, D)$ to $\psi(C, U, V', D)$. Since π^{-1} has $j - i$ inversions, $s_j s_{j-1}\cdots s_{i+1}$ is a reduced word and hence, $\delta_+(C, \psi(C, U, V, D)) = \pi^{-1}$ by Proposition 5.1. By calculation, we obtain $C^{(\pi_0\pi, B)} = \psi(C, U, V', D)$ and hence, $\delta_+(C, \psi(C, U, V, D)) = \pi^{-1}\pi_0{}^{-1}$. With $w_{i+1,j+1} = \pi_0 w_{i+1,j}$ we calculate $\pi^{-1}w_{i+1,j} = \pi^{-1}\pi_0{}^{-1}w_{i+1,j+1}$.

Analogously, $(\psi(C, U, V, D))^{(\pi_1, B)} = \psi(C, U', V, D)$ for $\pi_1 = s_{j-1}s_{j-2}\cdots s_i$ and hence, $\delta_+(C, \psi(C, U', V, D)) = \pi^{-1}\pi_1{}^{-1}$. Finally, $\pi^{-1}w_{i+1,j} = \pi^{-1}\pi_1{}^{-1}w_{i,j}$ completes the proof. ∎

Motivated by the previous lemma, we define a codistance function δ for (Δ^+, Δ^-) in the following way:

Definition 5.5. For $\sigma \in \{+, -\}$, let $C \in \mathfrak{C}^\sigma$ and $D \in \mathfrak{C}^{-\sigma}$. Further let $U \in C$ and $V \in C$ be subspaces of $\mathscr{S}$ such that for every subspace $W \in C$ with $W \geq V$ or $W \leq U$ there is a complementary subspace W' in $\mathscr{S}$ with $W' \in D$. Set $i := \mathrm{typ}(U)$ and $j := \mathrm{typ}(V)$. Then

$$\delta(C, D) := \delta_\sigma\,(C, \psi(C, U, V, D)) \cdot w_{i+1,j}.$$

Correspondingly to what we defined for $\mathfrak{C}^+$, let $C \in \mathfrak{C}^-$ and let $(U, B) \in \mathfrak{T}^-$ such that $C \in \Sigma(U, B)$. For $i \in \mathrm{typ}(M_-)$ let $U_i \in C$ be the the subspace of type i. Further, for $i \in \mathrm{typ}(M_- \setminus \{\mathscr{S}\})$, let $p_i \in B$ be the unique point that is contained in $U_{i-1} \setminus U_i$. Then for a permutation $\pi \in \langle S_\Delta\rangle$, we define $C^{(\pi, B)} := \{\langle p_{i^\pi}, U_{-\infty} \mid i \leq j\rangle \mid j \in \mathrm{typ}(C)\}$, where $U_{-\infty} := \bigcap M_-$.

Proposition 5.6. *Let $C \in \mathfrak{C}^+$ and $D \in \mathfrak{C}^-$. Then $\delta(C, D) = \delta(D, C)^{-1}$.*

Proof. Let $U_i \in C$ and $V_i \in D$ denote the subspaces of type i. Let l and k be integers such that for every i with $i \leq l$ or $i \geq k$, the subspaces U_i and V_i are complementary in $\mathscr{S}$. We may assume $l \leq k$. Set $S := U_k \cap V_l$. By Lemma

4.9 we conclude that there is a basis B' of S such that for every $i \leq k-l$, the set of points $B' \cap U_{l+i}$ is a basis of $S \cap U_{l+i}$ and $B' \cap V_{k-i}$ is a basis of $S \cap V_{k-i}$. Set $D' := \psi(C, U_l, U_k, D)$ and $C' := \psi(D, V_k, V_l, D)$. Finally, let B_+ and B_- be independent sets of points of $\mathscr{S}$ such that $(U_l, B_+) \in \mathfrak{T}^+$, $(V_k, B_-) \in \mathfrak{T}^-$ and furthermore, $\Sigma(U_l, B^+)$ contains both C, and D' and $\Sigma(V_k, B_-)$ contains both D and C'. We may assume $B' \subseteq B_+ \cap B_-$.

For $l < i \leq k$, we denote by p_i the unique point of $B' \cap U_i \setminus U_{i-1}$ and by q_i we denote the unique point of $B' \cap V_{i-1} \setminus V_i$. Let $\pi \in \langle S_\Delta \rangle$ such that $C^{(\pi, B_+)} = D'$. This implies that p_{i^π} is the unique point of $B' \cap V_{j-1} \setminus V_j$ where $j = k-(i-l)+1$. Thus, $p_{i^\pi} = q_j$. Now let $\rho \in \langle S_\Delta \rangle$ such that $D^{(\rho, B_-)} = C'$. Then $q_{j^\rho} = p_i$. This yields $j^\rho = k - (i^{\pi^{-1}} - l) + 1$. Since $j = k-(i-l)+1$, we conclude $\rho = w_{l+1,k}\pi^{-1}w_{l+1,k}$.

By Lemma 5.4 we calculate $\delta(C, D) = \delta_+(C, D') \cdot w_{l+1,k} = \pi^{-1} \cdot w_{l+1,k} = w_{l+1,k} \cdot \rho = (\rho^{-1} \cdot w_{l+1,k})^{-1} = (\delta_-(D, C') \cdot w_{l+1,k})^{-1} = \delta(D, C)^{-1}$. ■

Proposition 5.7. *Let $\sigma \in \{+, -\}$ and let $C \in \mathfrak{C}^\sigma$ and $D \in \mathfrak{C}^{-\sigma}$ with $\delta(C, D) = w$. Then for every $i \in$ typ(C) there exists a chamber $C' \in \mathfrak{C}^\sigma$ with $\delta_\sigma(C', C) = s_i$ and $\delta(C', D) = s_i \cdot \delta(C, D)$.*

Moreover, let $C' \in \mathfrak{C}^\sigma$ with $\delta_\sigma(C', C) = s_i$ and $l(s_i w) < l(w)$. Then $\delta(C', D) = s_i w$.

Proof. By analogous reasons, we restrain ourselves to the case $\sigma = +$. As usual, let $U_j \in \overline{C}$ and $V_j \in \overline{D}$ denote the subspaces of type j and let l and k be integers such that for every j with $j \leq l$ or $j \geq k$, the subspaces U_j and V_j are complementary in $\mathscr{S}$. We may assume $l < i < k$.

Let $B \subseteq \mathscr{S}$ such that $(U_l, B) \in \mathfrak{T}^+$ and both C and $D' := \psi(C, U_l, U_k, D)$ are elements of $\Sigma(U_l, B)$. Then $\delta_+(C, D') = w \cdot w_{l+1,k}$ by Lemma 5.4. Hence, as a consequence of Lemma 4.7 and the fact that $\Sigma(U_l, B)$ is thin, $C^{(w_{l+1,k}w^{-1}, B)} = D'$ and therefore, $(D')^{(ww_{l+1,k}, B)} = C$. Set $C'' := (D')^{(s_i\pi, B)}$. Then $\delta_+(C'', D') = s_i \cdot w \cdot w_{l+1,k}$ and thus, $\delta(C'', D) = s_i \cdot w$ by Lemma 5.4. This proves the first part of the claim.

It is straightforward to show that $l(w)$ equals the number of inversions of w. Furthermore, $w_{l+1,k}$ has $(k-l)(k-l-1)/2$ inversions and the number of inversions of $\pi := ww_{l+1,k}$ is $(k-l)(k-l-1)/2 - l(w)$. Since $l(s_i w) < l(w)$, we obtain by analogous reasons that the number of inversions of $s_i\pi$ is $(k-l)(k-l-1)/2 - l(w) + 1$. By Lemma 4.7 we know C and C'' are s_i-adjacent.

Let p_i and p_{i+1} denote the points of B with $p_i \in U_i \setminus U_{i-1}$ and $p_{i+1} \in U_{i+1} \setminus U_i$. Further let n and m be the integers such that $p_i \in U_n \setminus U_{n-1}$ and $p_{i+1} \in U_m \setminus U_{m-1}$. Then $(n)^{\pi^{-1}} = i$ and $(m)^{\pi^{-1}} = i+1$, since $C^{(\pi^{-1}, B)} = D'$ and consequently, $i^\pi = n$ and $(i+1)^\pi = m$. Since $s_i\pi$ has one inversion more than π, this yields $m > n$.

Let V'_j denote the element of D' of type j. Then

$$\begin{aligned}
\langle U_{i-1}, V'_j \cap U_{i+1} \rangle &= U_{i-1} && \text{if } j < n \\
\langle U_{i-1}, V'_j \cap U_{i+1} \rangle &= U_i && \text{if } n \leq j < m \\
\langle U_{i-1}, V'_j \cap U_{i+1} \rangle &= U_{i+1} && \text{if } j \geq m\,.
\end{aligned}$$

Let $U'_i \in C$ denote the element of type i. Since $\langle U_{i-1}, V'_m \cap U_{i+1}\rangle = U_{i+1}$, we know that $V'_m \cap U_{i-1}$ has corank 2 in $V'_m \cap U_{i+1}$. Hence, there is a point p'_i in $V'_j \cap U'_i \setminus U_{i-1}$. Since $C' \neq C$, we know $\langle p'_i, U_{i-1}\rangle \neq U_i$ and therefore $p'_i \notin U_i$. Thus, $B' := (B \setminus \{p_{i+1}\}) \cup \{p'_i\}$ is again independent and hence, $p'_i \in U_m \setminus U_{m-1}$. This implies that both C' and D' are contained in $\Sigma(U_l, B')$. We calculate $(D')^{(s_i\pi, B')} = C'$ and as above, $\delta_+(C', D') = s_i \cdot w \cdot w_{l+1,k}$ and consequently, $\delta(C', D) = s_i w$. ■

Theorem 5.8. *The pair (Δ^+, Δ^-) is a twin building of type $(\langle S_\Delta\rangle, S_\Delta)$ with codistance function δ.*

Proof. By Theorem 4.14 we know that both Δ^+ and Δ^- are buildings. By Proposition 5.1 both buildings are of type $(\langle S_\Delta\rangle, S_\Delta)$. Thus, the claim follows by the Propositions 5.6 and 5.7. ■

6 Final remarks

In section 3 we introduced the sets of chambers $\mathfrak{C}^+$ and $\mathfrak{C}^-$. It is straightforward to show that $\bigcap C = \bigcap C'$ holds for every two chambers C and C' of $\mathfrak{C}^+$. Furthermore, for a chamber $D \in \mathfrak{C}^-$ the subspaces $\bigcap C$ and $\bigcup D$ are complementary subspaces in $\mathscr{S}$ and so are $\bigcup C$ and $\bigcap D$. Set $\mathscr{S}' := (\bigcup C) \cap (\bigcup D)$.

Intersecting each element of every chamber of $\mathfrak{C}^+ \cup \mathfrak{C}^-$ with $\mathscr{S}'$ leads to a pair of chamber systems that is isomorphic to $(\mathfrak{C}^+ \cup \mathfrak{C}^-)$. Since $\mathscr{S}'$ has countable rank, we could have restrained ourselves to the case where $\mathscr{S}$ has countable rank. Furthermore, we could restrain ourselves to the case $\bigcap M_+ = \varnothing$ and $\bigcup M_+ = \mathscr{S}$. This restriction is only for the part where we dealing with chamber system, hence, starting with section 3. For a partial twin Grassmannians of a projective space $\mathscr{S}_0$ with respect to (U_0, V_0) the rank of U_0 and the corank of U_0 in $\mathscr{S}_0$ matters even in the infinite case.

In section 4 we introduced set of pairs $\mathfrak{T}^+$ each consisting of a subspace U and a countable set of points B. It is possible to be more restrictive at this point. Choose a pair $(U, B_0) \in \mathfrak{T}^+)$. Now define $\mathfrak{T}^+_0 := \{(V, B) \in \mathfrak{T}^+ \mid |B \setminus B_0| < \infty\}$. Let $\mathfrak{T}^-_0$ be defined analogously.

It is easy to check that all proofs remain valid if we would have used $\mathfrak{T}^+_0$ and $\mathfrak{T}^-_0$ in place of $\mathfrak{T}^+$ and $\mathfrak{T}^-$. The only difference is that the set of apartments would be much smaller.

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