$f-$statistical convergence, completeness and $f-$cluster points

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Abstract

We study $f-$statistical convergence, which is a generalization of the classical statistical convergence. In terms of it, we give a characterization of completeness in a normed space. We also introduce '$f-$statistical cluster points', which is a richer concept than the classic one. Namely, each (usual) limit point of a sequence is an $f-$statistical cluster point for some f.

1 Introduction and background

The concept of statistical convergence was first defined by Steinhaus ([17]) and also independently by Fast ([4]). Maddox ([10]) extended this concept to sequences in any Hausdorff locally convex topological vector spaces. In [8], Kolk begins to study its applications to Banach space theory. In [3] the authors find a remarkable connection of statistical convergence with some classical properties; concretely, Banach spaces with separable duals are characterized, in a way which cannot be reproduced with usual convergence. Other works studying this convergence are [5], [7], [15] and [11].

Let $A \subset \mathbb{N} = \{1, 2, \dots\}$. We denote by $|A|$ the cardinal of A and if $n \in \mathbb{N}$ we denote $A(n) = \{i \in A : i \leq n\}$. The density of A is defined by

$$d(A) = \lim_n \frac{|A(n)|}{n},$$

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in case this limit exists.

A sequence $(x_n)_n$ in a normed space X is said to be statistically convergent to some $x \in X$, and we write $\operatorname{stlim}_n x_n = x$, if for each $\varepsilon > 0$ we have $d(\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}) = 0$. Analogously, $(x_n)_n$ is said to be statistically Cauchy if for each $\varepsilon > 0$ and $i \in \mathbb{N}$ there exists an integer $m \geq i$ such that $d(\{n \in \mathbb{N} : \|x_n - x_m\| > \varepsilon\}) = 0$.

Fast ([4]) proved that $\operatorname{stlim}_n x_n = x$ if and only if there exists $A \subseteq \mathbb{N}$ with $d(A) = 1$ and $\lim_{n\in A} x_n = x$. Fridy ([5]) proved that a scalar sequence is statistically convergent if and only if it is statistically Cauchy (his proof can be easily generalized to Banach spaces, see [8]).

Moricz ([11]), working with double sequences, proved the analogous versions to Fast's and Fridy's results.

We recall that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called modulus function if it satisfies:

1. $f(x) = 0$ if and only if $x = 0$.
2. $f(x + y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$
3. f is increasing.
4. f is continuous from the right at 0.

From these properties it is clear that a modulus function must be continuous on $\mathbb{R}^+$. Examples of moduli are $f(x) = \frac{x}{1+x}$ and $f(x) = x^p$ with $0 < p \leq 1$.

The notion of modulus function was introduced by Nakano ([12]). Ruckle ([16]) and Maddox ([9]) have introduced and discussed some properties of sequence spaces defined by using a modulus function. Pehlivan ([13]) generalized the strong almost convergence with the help of modulus functions.

In [1], a new concept of density of a subset A of $\mathbb{N}$ is defined by means of an unbounded modulus function f, as $d_f(A) = \lim_n \dfrac{f(|A(n)|)}{f(n)}$, if this limit exists, and we will say that a sequence $(x_n)_n$ is $f-$statistically convergent to x if for every $\varepsilon > 0$ we have $d_f(\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}) = 0$. It is defined also the concept of $f-$statistically Cauchy sequence and it is proved that if X is a complete space and $(x_n)_n$ is an $f-$statistically Cauchy sequence, then $(x_n)_n$ is $f-$statistically convergent. Furthermore in [1] it is also proved that:

1. $(x_n)_n$ is $f-$statistically convergent to x if and only if there exists $A \subseteq \mathbb{N}$ with $d_f(A) = 0$ and $\lim\limits_{n\in\mathbb{N}\setminus A} x_n = x$.
2. If $f - \operatorname{stlim}_n x_n = x$ for every unbounded modulus f then $\lim_n x_n = x$.

In [2], it is defined and studied the $f-$statistical convergence of double sequences for unbounded modulus f and the authors obtain results similar to those given by Moricz in [11] for the statistical convergence of double sequences.

In Section 2 we study some properties of $f-$statistical convergent and $f-$statistical Cauchy sequences, giving a characterization of Banach spaces in those terms.

In [6] Fridy introduced the concept of statistical limit point and statistical cluster point of real sequences and gave some properties of the corresponding sets. Afterwards, in [14] Pehlivan et alt. studied the set of statistical cluster points of a sequence in finite-dimensional spaces. In Section 3 we define the $f-$statistical version of those sets giving some properties of the set of $f-$statistical cluster points in an arbitrary normed space.

Let us remark that some of the results proved here only have full meaning when dealing with $f-$statistical convergence. In this respect, consider theorem 3.2 combined with proposition 3.4, which does not have a simpler, statistical version.

Statistical convergence is typically studied in the setting of normed spaces and we have proceeded accordingly. However, all results in what follows can be translated to metric spaces, with minor changes at the most.

2 On $f-$statistical convergence and $f-$statistical Cauchy sequences

In this section we try to make clear the role of Cauchy sequences in the setting of $f-$statistical convergence. Also, we will show that the weak version is related to separability of the dual.

Definition 2.1. *Let X be a normed space and f an unbounded modulus. We will say that a sequence $\bar{x} = (x_n)_n \subseteq X$ is $f-$statistically null if for every $\varepsilon > 0$*

$$d_f(\{n \in \mathbb{N} : \|x_n\| \geq \varepsilon\}) = 0.$$

Theorem 2.2. *Let X be a normed space and f an unbounded modulus. If $\bar{x} = (x_n)_n \subseteq X$ is a sequence $f-$statistically convergent to x, then there exist two sequences $\bar{y} = (y_n)_n$ and $\bar{z} = (z_n)_n$ such that $\bar{y}$ converges to x in the usual sense, $\bar{x} = \bar{y} + \bar{z}$ and $\bar{z}$ is $f-$statistically null.*

Proof. Since $\bar{x} = (x_n)_n \subseteq X$ is $f-$statistically convergent to x, according to [1] there exists $A \subseteq \mathbb{N}$ with $d_f(A) = 0$ and such that $\lim\limits_{n \in \mathbb{N} \setminus A} x_n = x$, so we define the sequences $(y_n)_n$ and $(z_n)_n$ as follows:

$$y_n = \begin{cases} x & \text{if } n \in A \\ x_n & \text{if } n \in \mathbb{N} \setminus A \end{cases}$$

$$z_n = \begin{cases} x_n - x & \text{if } n \in A \\ 0 & \text{if } n \in \mathbb{N} \setminus A \end{cases}$$

Therefore $(y_n)_n$ is convergent to x, it is clear that $\bar{x} = (y_n + z_n)_n$ and $(z_n)_n$ is $f-$statistically null. ∎

The following result is a consequence of the above theorem.

Corollary 2.3. *Let X be a normed space and f an unbounded modulus. If $(x_n)_n \subseteq X$ is an $f-$statistically convergent sequence then it has a convergent subsequence.*

We have a characterization of complete spaces in terms of the $f-$statistical convergence.

Theorem 2.4. *Let X be a normed space. The following are equivalent:*

1. *X is complete.*
2. *For every unbounded modulus f and $f-$statistically Cauchy sequence $(x_n)_n$, $(x_n)_n$ is $f-$statistically convergent.*
3. *There exists an unbounded modulus f such that if a sequence $(x_n)_n$ is $f-$statistically Cauchy then it is $f-$statistically convergent.*

Proof. In [1] theorem 3.3 it is proved that *1.* implies *2.* That *2.* implies *3.* is obvious, and now we will see that *3.* implies *1.*

Let $(x_n)_n$ be a Cauchy sequence, then given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|x_n - x_N\| \leq \varepsilon$.

We consider the set

$$B = \{n \in \mathbb{N} : \|x_n - x_N\| > \varepsilon\}.$$

We have that this set is finite, so $d_f(B) = 0$. Therefore $(x_n)_n$ is an $f-$statistically Cauchy sequence and by hypothesis it will be $f-$statistically convergent. Using corollary 2.3 there will exist a convergent subsequence and by the fact that $(x_n)_n$ is a Cauchy sequence, then the sequence is convergent. ■

Theorem 2.5. *Let X be a normed space, f an unbounded modulus, if $(x_n)_n \subseteq X$ is an $f-$statistically Cauchy sequence, then it has a Cauchy subsequence.*

Proof. Let Y be the completion of X. Since $(x_n)_n \subseteq Y$ is $f-$statistically Cauchy, by theorem 2.4 it is $f-$statistically convergent (to some $y_0 \in Y$).

Then $(x_n)_n$ has a convergent subsequence by corollary 2.3. This subsequence is a Cauchy sequence in X. ■

In the following result we replace the hypothesis of completeness of the space by a weaker one. For this purpose we need the following definitions, the first one was introduced in [6] and the second one is its $f-$statistical version:

Definition 2.6. *Let X be a Banach space, $\bar{x} = (x_n)_n \subseteq X$ a sequence and f an unbounded modulus then, for an infinite $K \subset \mathbb{N}$,*

- *if $d(K) = 0$, then $(x_n)_{n\in K}$ is said to be a thin subsequence of $\bar{x}$; otherwise, it is said to be a non-thin subsequence.*
- *if $d_f(K) = 0$, then $(x_n)_{n\in K}$ is said to be an $f-$thin subsequence of $\bar{x}$; otherwise, it is said to be a non$-f-$thin subsequence.*

Theorem 2.7. *Let X be a normed space, f an unbounded modulus and $(x_n)_n$ an $f-$statistically Cauchy sequence which has a non$-f-$thin subsequence convergent to x. Then $(x_n)_n$ is $f-$statistically convergent.*

Proof. Let K be the set of indices of the non$-f-$thin subsequence of $(x_n)_n$. Since the sequence is $f-$statistically Cauchy, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the set $B = \{n \in \mathbb{N} : \|x_n - x_N\| \geq \frac{\varepsilon}{3}\}$ has $f-$density zero. We have that $K \cap (\mathbb{N} \backslash B)$ is infinite; otherwise, we can write $K = (K \cap B) \cup (K \cap (\mathbb{N} \backslash B))$,

$K \cap B$ has $f-$density zero because is a subset of B and $K \cap (\mathbb{N}\backslash B)$ cannot have $f-$density zero because K would have $f-$density zero, a contradiction.

Then, there exists $m \in K \cap (\mathbb{N}\backslash B)$ sufficiently large such that $\|x_m - x\| \leq \frac{\varepsilon}{3}$ and $\|x_m - x_N\| \leq \frac{\varepsilon}{3}$, so

$$\|x_N - x\| \leq \|x_N - x_m\| + \|x_m - x\| \leq \frac{2\varepsilon}{3}.$$

Therefore,

$$\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\} \subseteq \{n \in \mathbb{N} : \|x_n - x_N\| > \frac{\varepsilon}{3}\},$$

so $d_f(\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}) = 0$ and the sequence $(x_n)_n$ is $f-$statistically convergent to x.

■

The following result is an $f-$statistical version of the well-known property APO ([7]).

Lemma 2.8. *Let f be an unbounded modulus and $(A_i)_i \subseteq \mathcal{P}(\mathbb{N})$ a sequence of sets of $f-$density zero. There exists a sequence of sets $(B_i)_i \subseteq \mathcal{P}(\mathbb{N})$ such that $B_i \subseteq A_i$ and $A_i\backslash B_i$ is finite for every $i \in \mathbb{N}$, moreover $B = \bigcup_{i\in\mathbb{N}} B_i$ has $f-$density zero.*

Proof. We will work with the sets $(A'_i)_i$ given by $A'_1 = A_1$, $A'_2 = A_2\backslash A_1$, $A'_3 = A_3\backslash(A_1 \cup A_2), \ldots$ Observe that these sets are pairwise disjoint.

For every $j \in \mathbb{N}$ we know that $d_f\left(\bigcup_{i=1}^{j} A'_i\right) = 0$, therefore there exists a sequence of natural numbers $(k_j)_j$ strictly increasing such that if $n \geq k_j$ then

$$\frac{1}{f(n)} f\left(\sum_{i=1}^{n}\sum_{m=1}^{j} \chi_{A'_m}(i)\right) \leq \frac{1}{j}.$$

For every $n \geq k_1$ let $p_n \in \mathbb{N}$ be such that $k_{p_n} \leq n < k_{p_n+1}$, it is clear that $p_n \to \infty$.

For every $m \in \mathbb{N}$ we define $B'_m = A'_m\backslash\{1, 2, \ldots, k_m\}$, we denote by $B = \bigcup_{i\in\mathbb{N}} B'_i$ and we have that

$$\limsup_n \frac{1}{f(n)} f\left(\sum_{i=1}^{n} \chi_B(i)\right) = \limsup_n \frac{1}{f(n)} f\left(\sum_{i=1}^{n}\sum_{m=1}^{\infty} \chi_{B'_m}(i)\right) =$$

$$= \limsup_n \frac{1}{f(n)} f\left(\sum_{i=1}^{n}\sum_{m=1}^{p_n} \chi_{B'_m}(i)\right) \leq \limsup_n \frac{1}{f(n)} f\left(\sum_{i=1}^{n}\sum_{m=1}^{p_n} \chi_{A'_m}(i)\right) \leq$$

$$\leq \limsup_n \frac{1}{p_n} = 0,$$

where the second equality holds due to the fact that if $m > p_n$ then $\min B'_m > k_m \geq k_{p_n+1} > n$, so $\chi_{B'_m}(i) = 0$ if $i \leq n$ and therefore $\sum\limits_{m=p_n+1}^{+\infty} \chi_{B'_m}(i) = 0$ in such case.

We sum up that $\lim\limits_{n} \frac{1}{f(n)} f\left(\sum\limits_{i=1}^{n} \chi_B(i)\right) = 0$, i.e. $d_f(B) = 0$.

Now we take $B_1 = B'_1$, $B_2 = B'_1 \cup B'_2$, $B_3 = B'_1 \cup B'_2 \cup B'_3, \ldots$ So $B = \bigcup\limits_{i \in \mathbb{N}} B_i$ and it is easy to see that the family $(B_i)_i$ satisfy every requested property. ∎

Definition 2.9. *Let f be an unbounded modulus, the sequence $(x_n)_n \subseteq X$ is said to be weakly $f-$statistically convergent to x if $f - \operatorname{stlim} x^*(x_n) = x^*(x)$ for every $x^* \in X^*$. In this situation we write $w - f - \operatorname{stlim} x_n = x$.*

Now we have this result based on the one given by Connor, Ganichev and Kadets in [3].

Theorem 2.10. *Let X be a Banach space with separable dual X^* and f an unbounded modulus. If $(x_n)_n \subseteq X$ is a bounded, $w - f-$statistically convergent sequence then there exists a $w-$convergent sequence $(y_n)_n$ such that the set*

$$\{n \in \mathbb{N} : x_n \neq y_n\}$$

has $f-$density zero.

Proof. Without loss of generality, we suppose that $w - f - \operatorname{stlim} x_n = 0$. Let $D \subseteq X^*$ be a dense and countable set, then for every $d^* \in D$ it will be $f - \operatorname{stlim} d^*(x_n) = 0$, i.e., for every $d^* \in D$ there exists $A_{d^*} \subseteq \mathbb{N}$ with $d_f(A_{d^*}) = 0$ such that $d^*(x_n)$ tends to 0 as $n \in \mathbb{N} \backslash A_{d^*}$ tends to infinity.

By lemma 2.8 there exists $B \subseteq \mathbb{N}$ with $d_f(B) = 0$ and $A_{d^*} \backslash B$ finite, for every $d^* \in D$.

Let $k \geq 0$ such that $\|x_n\| \leq k$ for every $n \in \mathbb{N}$; given $\varepsilon > 0$ and $x^* \in X^*$, by density we can find $d^* \in D$ such that $\|x^* - d^*\| < \frac{\varepsilon}{2k}$. There exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ and $n \in \mathbb{N} \backslash A_{d^*}$ then $|d^*(x_n)| < \frac{\varepsilon}{2}$, on the other hand there exists $n_2 \in \mathbb{N}$ such that if $n \geq n_2$ and $n \in \mathbb{N} \backslash B$ then $n \in \mathbb{N} \backslash A_{d^*}$.

Let $n_0 = \max\{n_1, n_2\}$, for every $n \geq n_0$ and $n \in \mathbb{N} \backslash B$ and making use of the triangle inequality,

$$|x^*(x_n)| \leq |x^*(x_n) - d^*(x_n)| + |d^*(x_n)| \leq \|x^* - d^*\| \|x_n\| + |d^*(x_n)| < \frac{\varepsilon}{2k} k + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, defining the sequence

$$y_n = \begin{cases} x_n & \text{if} \quad n \in \mathbb{N} \backslash B \\ 0 & \text{if} \quad n \in B \end{cases} ;$$

we have the desired result. ∎

We do not know if the conclusion of the above theorem characterizes the Banach spaces with separable dual. If this were true it would generalize the analogous result for the statistical convergence as stated in [3].

3 $f-$cluster points

Now we intend to study the $f-$statistical version of cluster points and limit points, and relate them to the classical limit points.

In this section f will be an arbitrary unbounded modulus. Let us define the $f-$cluster and related points for a sequence in a Banach space.

Definitions 3.1. *Let X be a Banach space and $\bar{x} = (x_n)_n \subseteq X$ a sequence in X. It is said that*

- *$x \in X$ is a limit point of $\bar{x}$ if there exists a subsequence of $\bar{x}$ which is convergent to x. The set of all limit points of $\bar{x}$ is denoted by $L_{\bar{x}}$.*
- *$x \in X$ is a statistical limit point of $\bar{x}$ if there exists a non$-$thin subsequence of $\bar{x}$ which converges to x. The set of all statistical limit points of $\bar{x}$ is denoted by $\Lambda_{\bar{x}}$.*
- *$x \in X$ is an $f-$statistical limit point of $\bar{x}$ if there exists a non$-f-$thin subsequence of $\bar{x}$ which converges to x. The set of all $f-$statistical limit points of $\bar{x}$ is denoted by $\Lambda^f_{\bar{x}}$.*
- *$x \in X$ is a statistical cluster point of $\bar{x}$ if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\}$ does not have density zero. The set of all statistical cluster points of $\bar{x}$ is denoted by $\Gamma_{\bar{x}}$.*
- *$x \in X$ is an $f-$statistical cluster point of $\bar{x}$ if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\}$ does not have $f-$density zero. The set of all $f-$statistical cluster points of $\bar{x}$ is denoted by $\Gamma^f_{\bar{x}}$.*

Firstly, let us recall ([1]) that for any unbounded modulus f and any $A \subseteq \mathbb{N}$ we have that $d_f(A) = 0$ implies $d(A) = 0$. Indeed, if $d_f(A) = 0$ then for every $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $f(|A(n)|) \leq \frac{1}{p}f(n) \leq \frac{1}{p}pf\left(\frac{1}{p}n\right) = f\left(\frac{1}{p}n\right)$, which implies $|A(n)| \leq \frac{1}{p}n$ and so $d(A) = 0$.

Now, we will see the relation between those sets.

Theorem 3.2. *Let $\bar{x}$ be a sequence in a Banach space X, then we have*

$$\begin{array}{ccccc} \Lambda_{\bar{x}} & \subseteq & \Lambda^f_{\bar{x}} & & \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\ \Gamma_{\bar{x}} & \subseteq & \Gamma^f_{\bar{x}} & \subseteq & L_{\bar{x}} \end{array}$$

Proof. Let $\bar{x}$ be a sequence in a Banach space X.

- $\Lambda_{\bar{x}} \subseteq \Gamma_{\bar{x}}$ was proved in [6], proposition 1. Alternatively, this is an immediate consequence of the third assertion using as modulus the identity function.
- Having $f-$density zero implies having density zero, therefore $\Lambda_{\bar{x}} \subseteq \Lambda^f_{\bar{x}}$ and $\Gamma_{\bar{x}} \subseteq \Gamma^f_{\bar{x}}$.

- We will see that $\Lambda^f_{\bar{x}} \subseteq \Gamma^f_{\bar{x}}$. Let $x \in \Lambda^f_{\bar{x}}$, then there exists $K \subseteq \mathbb{N}$ infinite with $d_f(K) \neq 0$ such that $\lim_{k \in K} x_k = x$.

  For every $\varepsilon > 0$ we have that the set $A = \{n \in K : \|x_n - x\| \geq \varepsilon\}$ is finite, which implies $d_f(K \backslash A) \geq d_f(K) - d_f(A) = d_f(K) \neq 0$.

  Finally, since f is increasing and $K \backslash A \subseteq \{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\}$,

  $$d_f(\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\}) \geq d_f(K \backslash A) \neq 0$$

  therefore $x \in \Gamma^f_{\bar{x}}$.

- Finally, we will see that $\Gamma^f_{\bar{x}} \subseteq L_{\bar{x}}$. Let $x \in \Gamma^f_{\bar{x}}$, for every $j \in \mathbb{N}$ we have that $d_f(\{n \in \mathbb{N} : \|x_n - x\| < \frac{1}{j}\}) \neq 0$, so let $A_j = \{n \in \mathbb{N} : \|x_n - x\| < \frac{1}{j}\}$ which is an infinite set of natural numbers and $A_{j+1} \subset A_j$ for every $j \in \mathbb{N}$.

  Now, we can take an increasing sequence $n_1 < n_2 < \ldots$ with each $n_i \in A_i$. If $k \geq j$, $j \in \mathbb{N}$, then $\|x_{n_k} - x\| < \frac{1}{k} \leq \frac{1}{j}$. So $(x_{n_k})_k$ is a subsequence of $\bar{x}$ convergent to x, therefore $x \in L_{\bar{x}}$. ■

The previous inclusions are strict in general, as we will see in the following examples (the case of $\Lambda_{\bar{x}} \subsetneq \Gamma_{\bar{x}}$, was proved in the example 3 of [6]):
EXAMPLES 3.3

- If we consider the set $S = \{k^2 : k \in \mathbb{N}\}$ and the unbounded modulus $f(x) = \log(1+x)$ we have that $\Lambda_{\chi_{\mathbb{N}\backslash S(n)}} = \{1\}$ and $\Lambda^f_{\chi_{\mathbb{N}\backslash S(n)}} = \{0,1\}$, because $d_f(\mathbb{N}\backslash S) = 1$, $d_f(S) = \frac{1}{2}$ and $d(S) = 0$.

  On the other hand, we have that $\Gamma_{\chi_{\mathbb{N}\backslash S}(n)} = \{1\}$ and $\Gamma^f_{\chi_{\mathbb{N}\backslash S}(n)} = \{0,1\}$. So $\Lambda_{\bar{x}} \subsetneq \Lambda^f_{\bar{x}}$ and $\Gamma_{\bar{x}} \subsetneq \Gamma^f_{\bar{x}}$.

- We will use again the unbounded modulus $f(x) = \log(1+x)$. Let A be the set of natural numbers such that

  $$|A(n)| = \lfloor n^{\frac{1}{\sqrt{\log(1+n)}}} \rfloor,$$

  and let $(q_n)_n$ be a sequence whose range is the set of rational numbers and define

  $$x_n = \begin{cases} q_n & \text{if } n \in A \\ n & \text{otherwise} \end{cases}$$

  Since the set A has $f-$density zero, then $\Gamma^f_{\bar{x}} = \varnothing$ while the fact that $(q_n)_n$ is dense in $\mathbb{R}$ implies that $L_{\bar{x}} = \mathbb{R}$.

For the next result we will need lemma 3.4 in [1], which asserts that for every infinite set of natural numbers we can find an unbounded modulus g such that the $g-$density of the set is one.

Proposition 3.4. *Let $\bar{x} = (x_n)_n \subseteq X$, then*

$$L_{\bar{x}} = \bigcup \{\Gamma^f_{\bar{x}} : f \text{ is an unbounded modulus}\}.$$

Proof. Since we know that $\Gamma^f_{\bar{x}} \subseteq L_{\bar{x}}$ is true regardless of f, we only have to prove that $L_{\bar{x}} \subseteq \bigcup\{\Gamma^f_{\bar{x}} : f \text{ unbounded modulus}\}$.

Suppose that $x \in L_{\bar{x}}$ and for every unbounded modulus f, $x \notin \Gamma^f_{\bar{x}}$. Then for every f there exists $\varepsilon_f > 0$ such that $d_f(\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon_f\}) = 0$.

On the other hand, since $x \in L_{\bar{x}}$ there exists $A \subseteq \mathbb{N}$ infinite such that $\lim\limits_{n\in A} x_n = x$ and therefore for every $\varepsilon > 0$ the set $C_\varepsilon = \{n \in A : \|x_n - x\| \geq \varepsilon\}$ is finite.

By the aforementioned lemma, there exists g unbounded modulus such that $d_g(A) = 1$, if we consider its own ε_g and the set C_{ε_g} we have that $d_g(A \backslash C_{\varepsilon_g}) = 1$ and

$$A \backslash C_{\varepsilon_g} = \{n \in A : \|x_n - x\| < \varepsilon_g\} \subseteq \{n \in \mathbb{N} : \|x_n - x\| < \varepsilon_g\}$$

but the second set has $g-$density zero, and this is a contradiction. ∎

Definition 3.5. *A sequence $\bar{x} = (x_n)_n \subseteq X$ is said to be $f-$statistically bounded if there exists a bounded set B such that $d_f(\{n \in \mathbb{N} : x_n \notin B\}) = 0$.*

Theorem 3.6. *Let $\bar{x} = (x_n)_n$ and $\bar{y} = (y_n)_n$ be two sequences in a normed space X such that $d_f(\{n \in \mathbb{N} : x_n \neq y_n\}) = 0$, then $\Lambda^f_{\bar{x}} = \Lambda^f_{\bar{y}}$ and $\Gamma^f_{\bar{x}} = \Gamma^f_{\bar{y}}$.*

Proof. Let $x \in \Lambda^f_{\bar{x}}$, then there exists $B \subseteq \mathbb{N}$ infinite with $d_f(B) \neq 0$ such that $\lim_{n\in B} x_n = x$. We consider the set $A = \{n \in \mathbb{N} : x_n \neq y_n\}$, which has $f-$density zero.

Consider $(y_n)_{n\in B\backslash A}$, it is a subsequence of $\bar{y}$ which converges to x and is non$-f-$thin. Indeed, if $d_f(B\backslash A) = 0$ then

$$d_f(A \cup B) = d_f(A \cup (B\backslash A)) \leq d_f(A) + d_f(B\backslash A) = 0,$$

but $B \subset A \cup B$ and B has non null $f-$density. So $x \in \Lambda^f_{\bar{y}}$ and we can get the other inclusion by symmetry.

Let $x \in \Gamma^f_{\bar{x}}$, then for every $\varepsilon > 0$, $d_f(\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\}) \neq 0$. Given $\varepsilon > 0$, consider

$$B_\varepsilon = \{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\} \text{ and } C_\varepsilon = \{n \in \mathbb{N} : \|y_n - x\| < \varepsilon\}.$$

We have $B_\varepsilon \backslash A \subseteq C_\varepsilon$ and thus $d_f(C_\varepsilon) \geq d_f(B_\varepsilon \backslash A) \geq d_f(B_\varepsilon) - d_f(A) = d_f(B_\varepsilon) \neq 0$. So $x \in \Gamma^f_{\bar{y}}$ and we can get the other inclusion by symmetry. ∎

As a result we have the following corollary:

Corollary 3.7. *Let $\bar{x} = (x_n)_n \subseteq X$ be an $f-$statistically bounded sequence, then the set $\Gamma^f_{\bar{x}}$ is bounded.*

Theorem 3.8. *Let X be a normed space and $\bar{x} = (x_n)_n \subseteq X$, then there exists a sequence $\bar{y} = (y_n)_n$ such that $L_{\bar{y}} = \Gamma^f_{\bar{x}}$ and the terms of $\bar{y}$ are the same as the terms of $\bar{x}$ except on a set of $f-$density zero.*

Proof. We know that $\Gamma^f_{\bar{x}} \subseteq L_{\bar{x}}$. Whenever $u \in L_{\bar{x}} \backslash \Gamma^f_{\bar{x}}$ (in other case it will be trivial), there exists $\varepsilon_u > 0$ such that $d_f(\{n \in \mathbb{N} : \|x_n - u\| < \varepsilon_u\}) = 0$.

We have that $L_{\bar{x}} \backslash \Gamma^f_{\bar{x}}$ is separable and

$$L_{\bar{x}} \backslash \Gamma^f_{\bar{x}} \subseteq \bigcup_{u \in L_{\bar{x}} \backslash \Gamma^f_{\bar{x}}} \{y \in X : \|u - y\| < \varepsilon_u\},$$

by the Lindelöf property there exists $(u_k)_k \subseteq L_{\bar{x}} \backslash \Gamma^f_{\bar{x}}$ such that

$$L_{\bar{x}} \backslash \Gamma^f_{\bar{x}} = \bigcup_{k \in \mathbb{N}} \{y \in X : \|u_k - y\| < \varepsilon_{u_k}\}.$$

For every $k \in \mathbb{N}$, let $A_k = \{n \in \mathbb{N} : \|u_k - x_n\| < \varepsilon_{u_k}\}$ with $d_f(A_k) = 0$. By lemma 2.8 there exists $(B_k)_k \subseteq \mathcal{P}(\mathbb{N})$ such that $A_k \backslash B_k$ is finite for every $k \in \mathbb{N}$ and $B = \cup_{k \in \mathbb{N}} B_k$ has $f-$density zero. We write $\mathbb{N} \backslash B = \{j_1, j_2, \dots\}$ with $j_1 < j_2 < \dots$ and define $\bar{y} = (y_n)_n$ by

$$y_n = \begin{cases} x_{j_n} & \text{if} \quad n \in B \\ x_n & \text{if} \quad n \in \mathbb{N} \backslash B \end{cases}$$

Let $v \in L_{\bar{y}} \subseteq L_{\bar{x}}$, if $v \notin \Gamma^f_{\bar{x}}$ then there exists $m \in \mathbb{N}$ such that $v \in \{y \in X : \|u_m - y\| < \varepsilon_{u_m}\}$, then there exists $M \subseteq \mathbb{N} \backslash B$ infinite such that $(x_n)_{n \in M} \subseteq \{y \in X : \|u_m - y\| < \varepsilon_{u_m}\}$. We have that $M \subseteq A_m$ and

$$M = A_m \cap M \subseteq A_m \backslash B$$

is finite, which is a contradiction. So $v \in \Gamma^f_{\bar{x}}$, and $L_{\bar{y}} \subseteq \Gamma^f_{\bar{x}}$. The reverse inclusion is a consequence of theorem 3.6. ∎

As a result we have the following corollary:

Corollary 3.9. *Let $\bar{x} = (x_n)_n \subseteq X$ be a sequence, then the set $\Gamma^f_{\bar{x}}$ is closed.*

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References

[1] A. Aizpuru, M. C. Listán-García and F. Rambla-Barreno, Density by moduli and statistical convergence, Quaest. Math., **37** (2014), no. 4, 525–530.

[2] A. AIZPURU, M. C. LISTÁN-GARCÍA AND F. RAMBLA-BARRENO, Double density by moduli and statistical convergence, Bull. Belg. Math. Soc. Simon Stevin. **19** (2013), no. 4, 663–673.

[3] J. CONNOR, M. GANICHEV AND V. KADETS, A characterization of Banach spaces with separable duals via weak statistical convergence, J. Math. Anal. Appl. **244** (2000), 251–261.

[4] H. FAST, Sur la convergence statistique, Colloq. Math. **2** (1951), 241–244.

[5] J. FRIDY, On statistical convergence, Analysis **5** (1985), 301–313.

[6] J. FRIDY, Statistical limit points, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.

[7] A.R. FRIEDMAN AND J.J. SEMBER, Densities and Summability, Pacific J. Math. **95** (1981), 293–305.

[8] E. KOLK, The statistical convergence in Banach spaces, Acta Et Commentationes Univ. Tartuensis **928** (1991), 41–52.

[9] I.J. MADDOX, Sequence spaces defined by a modulus, Math. Proc. Camb. Philos. Soc. **100** (1986), no. 1, 161–166

[10] I. J. MADDOX, Statistical convergence in a locally convex space. Math. Proc. Cambridge Philos. Soc. **104** (1988), no. 1, 141–145.

[11] F. MORICZ, Statistical convergence of multiple sequences, Arch. Math. **81** (2003), 82–84.

[12] H. NAKANO, Concave modulars, J. Math. Soc. Japan **5** (1953), 29–49.

[13] S. PEHLIVAN, Strongly almost convergent sequences defined by a modulus and uniformly statistical convergence, Soochow J. Math. **20** (1994), no.2, 205–211.

[14] S. PEHLIVAN, A. GÜNCAN AND M.A MAMEDOV, Statistical cluster points of sequences in finite dimensional spaces, Czechoslovak Math. J. **54(129)** (2004), no. 1, 95–102.

[15] D. RATH AND B.C. TRIPATHY, On statistically convergent and statistically Cauchy sequences, Indian J. pure appl. Math **25(4)** (1994), 381–386.

[16] W.H. RUCKLE, FK spaces in which the sequence of coordinate vectors is bounded, Can. J. Math. **25** (1973), 973–978.

[17] H. STEINHAUS, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. **2** (1951), 73–74.

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