Regular 3-dimensional parallelisms of $\mathrm{PG}(3, \mathbb{R})$

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Abstract

In [8] the collineation groups of some known 5-, 4- and 3-dimensional topological regular parallelisms of $PG(3, \mathbb{R})$ were determined. In the present article we concentrate on 3-dimensional regular parallelisms and prove: the 3-dimensional regular parallelisms are exactly those which can be constructed from generalized line stars, see [3]. We determine the collineation groups of 3-dimensional regular parallelisms and show that only group dimension 1 or 2 is possible. If the collineation group is 2-dimensional, then the parallelism is rotational which means that there is a rotation group $SO_2(\mathbb{R})$ about some axis leaving the parallelism invariant. We give a construction method for the generalized line stars which induce these parallelisms.

1 Introduction

This article may be seen as a continuation of [8] and we refer to this paper for the notions of spread, regulus, Plücker map and coordinates, Klein quadric, automorphisms of a topological parallelism. We only repeat the main notion of parallelism:
A *parallelism* is a family $\mathbf{P}$ of spreads such that each line of $\mathrm{PG}(3, \mathbb{R})$ is contained in exactly one spread of $\mathbf{P}$. Spreads which are equivalent to the complex spread are called *regular*. A parallelism of $\mathrm{PG}(3, \mathbb{R})$ all whose members are regular spreads is called *(totally) regular*.

Two lines L_1, L_2 of $\mathrm{PG}(3, \mathbb{R})$ are *parallel with respect to a parallelism* $\mathbf{P}$, abbreviated $\mathbf{P}$-*parallel*, iff L_1 and L_2 are members of the same spread of $\mathbf{P}$. Clearly, the

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parallel axiom holds: For each line L and each point a of the point–line geometry $\mathrm{PG}(3,\mathbb{R}) = (\mathcal{P}_3, \mathcal{L}_3)$ there exists a unique line, say $L^{\|}$, which passes through a and is **P**-parallel to L. Thus we have the mapping

$$p_{\mathbf{P}} : \mathcal{L}_3 \times \mathcal{P}_3 \to \mathcal{L}_3;\ (L,a) \mapsto L^{\|}. \tag{1}$$

A parallelism **P** of $\mathrm{PG}(3,\mathbb{R})$ is called *topological*, if the mapping $p_{\mathbf{P}}$ from (1) is continuous.

In section 2 we introduce modified Plücker coordinates such that the related quadratic form is $(q_0,q_1,q_2,q_3,q_4,q_5) \in \mathbb{R}^6 \mapsto q_0^2+q_1^2+q_2^2-q_3^2-q_4^2-q_5^2 \in \mathbb{R}$. Then the automorphism group of the Klein quadric H_5 is described by the group $PGO_6(\mathbb{R},3)$ with respect to this quadratic form. We give an explicit representation for the isomorphism $\varphi : PSL(4,\mathbb{R}) \to PSO_6(\mathbb{R},3)$, see Proposition 3.

We apply two instruments for the classification of parallelisms: Firstly, for each regular parallelism **P**, a dimension can be defined in a natural way, such that $2 \le \dim \mathbf{P} \le 5$ where dimension 2 characterizes the Clifford parallelism, see [4, Definition 2.4 and Lemma 2.7]. Secondly the collineation group of a topological parallelism is a Lie group G and this group has a dimension. By [7] $\dim G \le 6$ and dim 6 is valid exactly for the Clifford parallelism.

The first step in the classification is to study the known classes and examples under these aspects. In [8] for some of the known parallelisms of dimension 5, 4 and 3 the automorphism groups and their dimensions were determined. In the present article we study 3-dimensional (regular) parallelisms. We prove in section 4 that the 3-dimensional parallelisms are exactly those which can be constructed from a generalized line star, see Theorem 23.

Let $\mathcal{C}$ be the regular spread, then we prove in Theorem 17: The full group $Aut_e(\mathcal{C})$ of automorphic transformations (collineations and dualities) of $\mathcal{C}$ is $PO_4(\mathbb{R},1) \times O_2(\mathbb{R})$. We need this result in section 5 to determine the group $Aut_e\mathbf{P}$ of collineations (and dualities) of a 3-dimensional parallelism **P**. If **P** is induced by the generalized line star $\mathcal{A}$ and $\Lambda(\mathcal{A})$ is the collineation group of the line star, then $Aut_e\mathbf{P} \cong \Lambda(\mathcal{A}) \times O_2(\mathbb{R})$ (Theorem 31). Therefore it remains to determine $\Lambda(\mathcal{A})$. In Theorem 36 we prove: the connected component $\Lambda(\mathcal{A})^1$ is either the identity or the rotation group about some axis. It follows that dim $= 0$ or 1, and from this we get the final Corollary 37, which says that $\dim Aut_e\mathbf{P} \in \{1,2\}$.

In section 6 we study the parallelisms with 2-dimensional group (induced by generalized line stars with 1-dimensional group). We call them rotational since there is a rotation group about some axis (Definition 39). In Theorem 42 we give a construction method for these parallelisms and then we determine the full groups of automorphisms. A special subclass consists of the parallelisms already found in [2].

2 Modified Plücker map and the group $PO_6(\mathbb{R},3)$

We recall the notions of Plücker map, Plücker coordinates and Klein quadric, see [8, 2.2] and [13, p.363-367]. Let $PG(3,\mathbb{R}) = (P_3(\mathbb{R}), \mathcal{L}_3)$ be the 3-dimensional projective point-line geometry, then we define for each line $L \in \mathcal{L}_3$ a point of the projective space $P_5(\mathbb{R})$ in the following way: Choose two different points $(s_0, s_1, s_2, s_3)\mathbb{R}$ and $(t_0, t_1, t_2, t_3)\mathbb{R}$ on L and set

$$(s_0, s_1, s_2, s_3)\mathbb{R} \vee (t_0, t_1, t_2, t_3)\mathbb{R} = L \mapsto (p_0, p_1, p_2, p_3, p_4, p_5)\mathbb{R}$$

with

$$p_0 = \begin{vmatrix} s_0 & s_1 \\ t_0 & t_1 \end{vmatrix}, p_1 = \begin{vmatrix} s_0 & s_2 \\ t_0 & t_2 \end{vmatrix}, p_2 = \begin{vmatrix} s_0 & s_3 \\ t_0 & t_3 \end{vmatrix}, p_3 = \begin{vmatrix} s_2 & s_3 \\ t_2 & t_3 \end{vmatrix},$$
$$p_4 = \begin{vmatrix} s_3 & s_1 \\ t_3 & t_1 \end{vmatrix}, p_5 = \begin{vmatrix} s_1 & s_2 \\ t_1 & t_2 \end{vmatrix}.$$

This defines the so-called *Plücker coordinates* for each line and we get a map

$$\lambda_0 : \mathcal{L}_3 \to P_5(\mathbb{R}).$$

The image of the map λ_0 is the quadric

$$H_5 = \{(p_0, p_1, p_2, p_3, p_4, p_5)\mathbb{R} \in P_5(\mathbb{R}) | p_0 p_3 + p_1 p_4 + p_2 p_5 = 0\}$$

which is called the *Klein quadric*. The map $\lambda_0 : \mathcal{L}_3 \to H_5$ is a bijection from the set of lines in $PG(3, \mathbb{R})$ to the points of the Klein quadric (*Plücker-Klein correspondence*).

In the present article we will work with modified Plücker coordinates instead of the ordinary ones. For this let T be the (6×6)-matrix

$$T = \begin{pmatrix} E & E \\ E & -E \end{pmatrix} \text{ with } E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We apply this matrix to the ordinary Plücker coordinates:

$$T : (p_0, p_1, p_2, p_3, p_4, p_5) \mapsto (q_0, q_1, q_2, q_3, q_4, q_5)$$

and get the modified Plücker coordinates

$$q_0 = p_0 + p_3, q_1 = p_1 + p_4, q_2 = p_2 + p_5, q_3 = p_0 - p_3, q_4 = p_1 - p_4, q_5 = p_2 - p_5 \tag{2}$$

Definition 1 The modified Plücker map is $\lambda = T\lambda_0$.

The Klein quadric is expressed in the modified Plücker coordinates by

$$H_5 = \{(q_0, q_1, q_2, q_3, q_4, q_5)\mathbb{R} \in P_5(\mathbb{R}) | q_0^2 + q_1^2 + q_2^2 - (q_3^2 + q_4^2 + q_5^2) = 0\}$$

as can be seen by inserting (2) into the formula above.

Theorem 2 *The space of oriented lines of the projective geometry* PG$(3, \mathbb{R})$ *is homeomorphic to* $\mathbb{S}^2 \times \mathbb{S}^2$. *The space of lines of* PG$(3, \mathbb{R})$ *is homeomorphic to the quotient space* $(\mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{Z}_2$, *where* $\mathbb{Z}_2$ *is generated by the map*

$$(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (-x_0, -x_1, -x_2, -x_3, -x_4, -x_5).$$

We call this space the *Sphere model* of the Klein quadric.

Proof: We split the $\mathbb{R}^6$ into $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ following the three plus and the three minus signs in the modified quadratic form. In both copies of $\mathbb{R}^3$ we select the unit sphere $\mathbb{S}^2$. Suppose a point $(q_0, q_1, q_2, q_3, q_4, q_5)\mathbb{R} \neq (0, \ldots, 0)\mathbb{R}$ is given in modified Plücker coordinates. We may also suppose $(q_0, q_1, q_2)\mathbb{R} \neq (0, 0, 0)\mathbb{R}$ otherwise we interchange left and right. We choose the positive factor $k > 0$ such that $(q_0, q_1, q_2)k \in \mathbb{S}^2$. Because of the quadratic condition for the modified Plücker coordinates also $(q_3, q_4, q_5)k \in \mathbb{S}^2$, and we get an element, say $(l, r) \in \mathbb{S}^2 \times \mathbb{S}^2$ ("left and right image"), which corresponds to the line $L = s \vee t$ we started with. We could also use the factor $-k$ and then get $(-l, -r) \in \mathbb{S}^2 \times \mathbb{S}^2$. So we have proved that the quadric is homeomorphic to the quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ modulo the map $(x, y) \mapsto (-x, -y)$. Therefore the quotient map is a simply connected twofold covering. Also the map which maps each oriented line to the corresponding unoriented line is a twofold covering. It follows that the two spaces ($\mathbb{S}^2 \times \mathbb{S}^2$ and the space of oriented lines) are homeomorphic. ∎

Each transformation (= collineation or duality) τ of $PG(3, \mathbb{R})$ induces a collineation τ_λ of $PG(5, \mathbb{R})$, which preserves H_5 and is determined by its restriction to H_5, given by:

$$\tau_\lambda|_{H_5} = \lambda \circ \tau|_{\mathcal{L}_3} \circ \lambda^{-1}. \tag{3}$$

We call τ_λ the *induced map* of τ.

Proposition 3 *If $\tau = [A] \in \mathrm{PGL}(4, \mathbb{R})$, then the induced map is given by $\tau_\lambda = [A_\lambda]$ with a matrix A_λ that may be computed in the following way:*

$\frac{1}{2}[A_\lambda] =$

	0|1	2|3	0|2	3|1	0|3	1|2	0|1	2|3	0|2	3|1	0|3	1|2
$\frac{0}{1}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{2}{3}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{0}{2}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{3}{1}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{0}{3}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{1}{2}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{0}{1}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{2}{3}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$\frac{0}{2}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{3}{1}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$\frac{0}{3}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$\frac{1}{2}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$

Each of the 36 entries is the sum of four (2×2)-subdeterminants of A. These are described by the pairs of rows and pairs of columns together with a suitable sign.

Proof: In [8] we gave for the matrix

$$[A] = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \in GL_4(\mathbb{R}) \tag{4}$$

the induced map with respect to the original (old) Plücker map λ_0, see also [13, p.368]. This is the (6×6)-matrix $A_{\lambda_0} :=$

$$\begin{pmatrix} a_{00}a_{11} - a_{01}a_{10} & a_{00}a_{12} - a_{02}a_{10} & a_{00}a_{13} - a_{03}a_{10} & a_{02}a_{13} - a_{03}a_{12} & -a_{01}a_{13} + a_{03}a_{11} & a_{01}a_{12} - a_{02}a_{11} \\ a_{00}a_{21} - a_{01}a_{20} & a_{00}a_{22} - a_{02}a_{20} & a_{00}a_{23} - a_{03}a_{20} & a_{02}a_{23} - a_{03}a_{22} & -a_{01}a_{23} + a_{03}a_{21} & a_{01}a_{22} - a_{02}a_{21} \\ a_{00}a_{31} - a_{01}a_{30} & a_{00}a_{32} - a_{02}a_{30} & a_{00}a_{33} - a_{03}a_{30} & a_{02}a_{33} - a_{03}a_{32} & -a_{01}a_{33} + a_{03}a_{31} & a_{01}a_{32} - a_{02}a_{31} \\ a_{20}a_{31} - a_{21}a_{30} & a_{20}a_{32} - a_{22}a_{30} & a_{20}a_{33} - a_{23}a_{30} & a_{22}a_{33} - a_{23}a_{32} & -a_{21}a_{33} + a_{23}a_{31} & a_{21}a_{32} - a_{22}a_{31} \\ a_{30}a_{11} - a_{31}a_{10} & a_{30}a_{12} - a_{32}a_{10} & a_{30}a_{13} - a_{33}a_{10} & a_{32}a_{13} - a_{33}a_{12} & -a_{31}a_{13} + a_{33}a_{11} & a_{31}a_{12} - a_{32}a_{11} \\ a_{10}a_{21} - a_{11}a_{20} & a_{10}a_{22} - a_{12}a_{20} & a_{10}a_{23} - a_{13}a_{20} & a_{12}a_{23} - a_{13}a_{22} & -a_{11}a_{23} + a_{13}a_{21} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}. \tag{5}$$

The entries of A_{λ_0} are (2×2)-subdeterminants of A_3 and all these subdeterminants are taken with a positive sign. The following scheme can be used as memory aid:

$$A_{\lambda_0} = \begin{array}{c|cccccc} & 0|1 & 0|2 & 0|3 & 2|3 & 3|1 & 1|2 \\ \hline \frac{0}{1} & + & + & + & + & + & + \\ \frac{0}{2} & + & + & + & + & + & + \\ \frac{0}{3} & + & + & + & + & + & + \\ \frac{2}{3} & + & + & + & + & + & + \\ \frac{3}{1} & + & + & + & + & + & + \\ \frac{1}{2} & + & + & + & + & + & + \end{array}$$

Since the transition from the ordinary Plücker coordinates to the modified ones is defined by the matrix T, we get the modified induced matrix from the old one by conjugation with T:

$$A_\lambda = TA_{\lambda_0}T^{-1}. \tag{6}$$

We put $E := \operatorname{diag}(1, 1, 1)$ and write

$$A_{\lambda_0} = \begin{pmatrix} M & N \\ C & D \end{pmatrix}, \quad T = \begin{pmatrix} E & E \\ E & -E \end{pmatrix}, \quad T^{-1} = \frac{1}{2}\begin{pmatrix} E & E \\ E & -E \end{pmatrix},$$

with the (3×3)-submatrices M, N, C, D and get

$$A_\lambda =$$

$$\begin{pmatrix} E & E \\ E & -E \end{pmatrix}\begin{pmatrix} M & N \\ C & D \end{pmatrix}\frac{1}{2}\begin{pmatrix} E & E \\ E & -E \end{pmatrix} = \frac{1}{2}\begin{pmatrix} M+N+C+D & M-N+C-D \\ M+N-C-D & M-N-C+D \end{pmatrix}.$$

This proves the proposition. ∎

If we take the homogeneous matrices $[A] = A\mathbb{R}$ and $[A_\lambda] = A_\lambda\mathbb{R}$ then we get an homomorphism

$$\varphi_\lambda = ([A] \mapsto [A_\lambda]) : PGL(4, \mathbb{R}) \to PGL(6, \mathbb{R}). \tag{7}$$

The image maps lines to lines, therefore it leaves the Klein quadric H_5 invariant.

Definition 4 Let $PGO_6(\mathbb{R}, 3)$ be the subgroup of $PGL_6(\mathbb{R})$ which leaves the Klein quadric H_5 invariant. It is called the group of similitudes with respect to the quadratic Plücker form.

Thus the image of φ_λ is contained in $PGO_6(\mathbb{R},3)$.

Definition 5 The orthogonal group $PO_6(\mathbb{R},3)$ is the subgroup of $PGL(6,\mathbb{R})$ which leaves the quadratic form $(q_0,q_1,q_2,q_3,q_4,q_5) \mapsto q_0^2+q_1^2+q_2^2-q_3^2-q_4^2-q_5^2$ invariant and it consists of all matrices $[A] \in PGL(6,\mathbb{R})$ for which the following equation is valid

$$A^T diag(1,1,1,-1,-1,-1)A = diag(1,1,1,-1,-1,-1)$$

In our context we need also the following group:

Definition 6 $PO_6^{\pm}(\mathbb{R},3)$ consists of all matrices $[A] \in PGL(6,\mathbb{R})$ with

$$A^T diag(1,1,1,-1,-1,-1)A = \pm diag(1,1,1,-1,-1,-1). \tag{8}$$

We have defined three groups in connection with the Klein quadric and prove

Proposition 7

$$PO_6(\mathbb{R},3) \subset PGO_6(\mathbb{R},3) = PO_6^{\pm}(\mathbb{R},3).$$

Proof: The first inclusion is obvious: a map which preserves the quadratic form also preserves the associated quadric. The second equality is also well known. A map that preserves the quadric can change the quadratic form only by a constant factor. In the projective group the factors ± 1 suffice. ■

There are the subgroups $PSO_6(\mathbb{R},3) \subset PO_6(\mathbb{R},3)$ and $PSO_6^{\pm}(\mathbb{R},3) \subset PO_6^{\pm}(\mathbb{R},3)$, both of index 2 which arise if one takes det $= 1$ in the related linear groups.
We denote by $PGL_e(4,\mathbb{R})$ the extended projective group which consists of all collineations and all dualities of $PG(3,\mathbb{R})$. It is generated by the collineation group $PGL(4,\mathbb{R})$ together with one duality. For this generating duality we choose the elliptic polarity ε of $PG(3,\mathbb{R})$ which assigns to the arbitrary point $(p_0,p_1,p_2,p_3)\mathbb{R}$ the plane with equation $p_0x_0+p_1x_1+p_2x_2+p_3x_3=0$.
Similarly we set $PSL_e(4,\mathbb{R}) = \langle SL(4,\mathbb{R}),\varepsilon\rangle$.

Proposition 8 *The induced map* $\varepsilon_\lambda \in PO_6(\mathbb{R},3)$ *has the form*

$$\varepsilon_\lambda : (q_0,q_1,q_2,q_3,q_4,q_5) \mapsto (q_0,q_1,q_2,-q_3,-q_4,-q_5)$$

and $\varepsilon_\lambda \in PO_6(\mathbb{R},3) \setminus PSO_6(\mathbb{R},3)$.

Proof. The map $(q_0,q_1,q_2,q_3,q_4,q_5) \mapsto (q_0,q_1,q_2,-q_3,-q_4,-q_5)$ leaves H_5 invariant and interchanges the two classes of maximal totally isotropic subspaces. Therefore it induces a duality of $PG(3,\mathbb{R})$. In order to show that this duality is the elliptic polarity ε we calculate the Plücker coordinates of the joining line $e_i\mathbb{R} \vee e_j\mathbb{R}$ for each pair of basis elements $\mathbf{e_i}\mathbb{R} \vee \mathbf{e_j}\mathbb{R}$. Since ε maps each pair of basis elements to the complementary pair, we can read off the induced map ε_λ in $\mathbb{R}^6$. The map ε_λ has determinant -1 and since the dimension of the vector space is 6, i.e., even, the related projective map cannot be in $PSO_6(\mathbb{R},3)$. ■

Proposition 9 *Let* $\alpha = [diag(-1,1,1,1)] \in PGL(4,\mathbb{R}) \setminus PSL(4,\mathbb{R})$, *then the induced map is* $\alpha_\lambda = [p_0, p_1, p_2, p_3, p_4, p_5] \mapsto [p_3, p_4, p_5, p_0, p_1, p_2]$, *and* $\alpha_\lambda \in PO_6^{\pm}(\mathbb{R},3) \setminus PO_6(\mathbb{R},3)$.

Proof: This can be seen by a direct calculation, using Proposition 3. ∎

Theorem 10 *The homomorphism* φ_λ *from (7) induces the following isomorphisms of groups:*

$$\begin{aligned} PGL_e(4,\mathbb{R}) &\to PO_6^{\pm}(\mathbb{R},3) \\ PSL_e(4,\mathbb{R}) &\to PO_6(\mathbb{R},3) \\ PGL(4,\mathbb{R}) &\to PSO_6^{\pm}(\mathbb{R},3) \\ PSL(4,\mathbb{R}) &\to PSO_6(\mathbb{R},3). \end{aligned} \tag{9}$$

Proof: The group $PGL_e(4,\mathbb{R})$ maps lines to lines, and since the lines in $P_3(\mathbb{R})$ correspond bijectively to the points of H_5, it follows that $\varphi_\lambda(PGL_e(4,\mathbb{R})) \subset PGO_6(\mathbb{R},3)$. To prove the surjectivity we note that the elements of $PGO_6(\mathbb{R},3) \setminus PO_6(\mathbb{R},3)$ are the orthogonal maps which interchange the two families of isotropic subspaces, and these are the φ_λ-images of the dualities of $PG_3(\mathbb{R})$. Since $PGO_6(\mathbb{R},3) = PO_6^{\pm}(\mathbb{R},3)$, see Proposition 7, we get the first of the four isomorphisms. The structure of the groups on the left hand side can be described as follows: the connected group $PSL(4,\mathbb{R})$ is extended by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Here one factor $\mathbb{Z}_2$ is generated by ε, the other factor $\mathbb{Z}_2$ is generated by α. The three coset classes are represented by ε, α, and $\varepsilon \circ \alpha$. Similarly, on the right hand side there is the connected group $PSO_6(\mathbb{R},3)$ which is extended by $\mathbb{Z}_2 \times \mathbb{Z}_2$, one factor $\mathbb{Z}_2$ generated by the map $\alpha_\lambda = (q_0, q_1, q_2, q_3, q_4, q_5)) \mapsto (q_3, q_4, q_5, q_0, q_1, q_2)$ and the other factor $\mathbb{Z}_2$ generated by $\varepsilon_\lambda = (q_0, q_1, q_2, q_3, q_4, q_5) \mapsto (q_0, q_1, q_2, -q_3, -q_4, -q_5)$. ∎

Remark 11 The last isomorphism of the theorem is one of the classical isomorphisms of Lie groups (between connected simple Lie groups of dimension 15), see for instance [9, Chap. 9, p.352].

3 The collineation group of the regular spread

We look at the regular spread in the following way: we start with the complex spread (all 1-dimensional subspaces of the complex vector space $\mathbb{C}^2$) and consider $\mathbb{C}^2$ as a 4-dimensional real vector space. Then we get a system of 2-dimensional subspaces of the real 4-dimensional vector space $\mathbb{R}^4$. We will write the homogeneous matrices with square brackets in the following.

Theorem 12 *Let* $\Gamma(\mathcal{C}) \leq PGL_3(\mathbb{R})$ *be the group of those (continuous) collineations of* $PG(3,\mathbb{R})$ *which preserve the regular spread* $\mathcal{C}$ *and let* $\Delta(\mathcal{C}) \subset \Gamma(\mathcal{C})$ *be the connected component of the identity. Then*

$$\Delta(\mathcal{C}) \cong PSL_2(\mathbb{C}) \times SO_2(\mathbb{R}) \text{ and } \Gamma(\mathcal{C}) = \langle \Delta, \kappa \rangle$$

where $\kappa = diag[1,-1,1,-1]$ *is induced by complex conjugation.*

Proof: The group of those continuous collineations of the complex affine plane that fix the origin (and hence preserve $\mathcal{C}$) is $P\Gamma L_2(\mathbb{C}) = \langle PGL_2(\mathbb{C}), \kappa\rangle$, where κ is induced by conjugation $c \mapsto \bar{c}$. In the complex case the transition $GL_2(\mathbb{C}) \mapsto PGL_2(\mathbb{C}) = PSL_2(\mathbb{C})$ is done by taking the quotient modulo $\mathbb{C}^*$. In the real situation we have to take the quotient of $GL_2(\mathbb{C})$ modulo $\mathbb{R}^*$, only. So we retain the group $\mathbb{C}^*/\mathbb{R}^* \cong SO_2(\mathbb{R})$. ∎

Next we apply the Plücker map and calculate the induced group on $\mathbb{R}^6$ resp. on $P_5(\mathbb{R})$.

Proposition 13 *The lines of $\mathcal{C}$ correspond to the following points of the Klein Quadric: $Q = \lambda(\mathcal{C}) = \{[1,0,0,x_3,x_4,x_5] | x_3^2 + x_4^2 + x_5^2 = 1\}$.*

Proof: The lines of the regular spread are mapped by λ_0 and λ as follows:

$$\begin{aligned}[1,0,a,b] \vee [0,1,-b,a] &\mapsto [1,-b,a,a^2+b^2,b,-a]\\ &\mapsto [1+a^2+b^2,0,0,1-(a^2+b^2),-2b,2a]\end{aligned}$$

and

$$[0,0,1,0] \vee [0,0,0,1] \mapsto [0,0,0,1,0,0] \text{ rsp. } [1,0,0,-1,0,0].$$

Therefore the λ-image of any line of the regular spread is contained in the space $\mathbf{e_0}\mathbb{R} \vee \mathbf{e_3}\mathbb{R} \vee \mathbf{e_4}\mathbb{R} \vee \mathbf{e_5}\mathbb{R}$. ∎

As a consequence, the group $PSL_2(\mathbb{C})$ is mapped by φ_λ to a subgroup of $PGL_4(\langle \mathbf{e_0}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}\rangle)$. Since on $\langle \mathbf{e_0}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}\rangle$ we have the bilinear form $(x_0, x_3, x_4, x_5) \mapsto x_0^2 - x_3^2 - x_4^2 - x_5^2$, the related group on this space is the group $PO_4(\mathbb{R}, 1)$. This leads to

Proposition 14 *There is a Lie group isomorphism*

$$\omega : PSL_2(\mathbb{C}) \to PSO_4(\mathbb{R}, 1)$$

between 6-dimensional connected simple Lie groups.

Proof: The map $\varphi : PGL_4(\mathbb{R}) \to PSO_6^{\pm}(\mathbb{R}, 3)$ is a group isomorphism by Theorem 10. The restriction ω to $PSL_2(\mathbb{C})$ has its image in the stabilizer of the space $\mathbf{e_0}\mathbb{R} \vee \mathbf{e_3}\mathbb{R} \vee \mathbf{e_4}\mathbb{R} \vee \mathbf{e_5}\mathbb{R}$. Therefore the image is a subgroup of $PO_4(\mathbb{R}, 1)$. Since the image is connected and 6-dimensional, it is the group $PSO_4(\mathbb{R}, 1)$. ∎

Remark 15 This is one of the well known "classical" isomorphisms. Here, in the Plücker-Klein context we get a proof for the isomorphism in a natural way.

Using Proposition 3 we find

Proposition 16 *The φ-image of the central group*

$$SO_2(\mathbb{R}) = \left\{ \alpha_r = \begin{pmatrix} \cos r & \sin r & 0 & 0 \\ -\sin r & \cos r & 0 & 0 \\ 0 & 0 & \cos r & \sin r \\ 0 & 0 & -\sin r & \cos r \end{pmatrix} | r \in \mathbb{R} \right\}$$

is the group $\varphi(SO_2(\mathbb{R})) = \{\varphi(\alpha_r) | r \in \mathbb{R}\}$ with

$$\varphi(\alpha_r) = diag[1, \begin{pmatrix} \cos 2r & \sin 2r \\ -\sin 2r & \cos 2r \end{pmatrix}, 1, 1, 1].$$ ∎

Combining propositions 14 and 16 we get the φ-image of the connected component of the collineation group of $\mathcal{C}$

$$\varphi : PSL_2(\mathbb{C}) \times SO_2(\mathbb{R}) \to PSO_4(\mathbb{R},1) \times SO_2(\mathbb{R}).$$

Set $\mathbb{R}^4_{0,3,4,5} = \mathbf{e_0}\mathbb{R} \vee \mathbf{e_3}\mathbb{R} \vee \mathbf{e_4}\mathbb{R} \vee \mathbf{e_5}\mathbb{R}$ and $\mathbb{R}^2_{1,2} = \mathbf{e_1}\mathbb{R} \vee \mathbf{e_2}\mathbb{R}$ then the groups $PO_4(\mathbb{R},1)$ and $SO_2(\mathbb{R})$ act on the projective spaces $P\mathbb{R}^4_{0,3,4,5}$ and $P\mathbb{R}^2_{1,2}$, respectively.

We can now describe the full group of transformations (collineations and dualities) of the regular spread:

Theorem 17 *The full group $Aut_e(\lambda(\mathcal{C}))$ of automorphic transformations of the regular spread is $PO_4(\mathbb{R},1) \times O_2(\mathbb{R})$.*

Proof: We calculate the induced map of complex conjugation:

$$\kappa = [1,-1,1,-1] \overset{\varphi}{\mapsto} diag[1,1,-1,1,1,-1].$$

Furthermore, using Proposition 3 we calculate

$$\rho = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \overset{\varphi}{\mapsto} diag[1,1,1,-1,-1,1],$$

where $\rho \in SL_2(\mathbb{C})$ and the elliptic polarity ε of $PG(3,\mathbb{R})$ is mapped as

$$\varepsilon \overset{\varphi}{\mapsto} diag[1,1,1,-1,-1,-1].$$

It follows that $\varphi(\rho\varepsilon) = diag[1,1,1,1,1,-1] \in PO_4(\mathbb{R},1) \setminus PSO_4(\mathbb{R},1)$ and $\varphi(\rho\varepsilon\kappa) = diag[1,1,-1,1,1,1] \in O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$. ∎

Remark: The kernel of the spread $\mathcal{C}$ is not only $SO_2(\mathbb{R})$ but the group $O_2(\mathbb{R})$. On $PG(3,\mathbb{R})$ the duality $\rho\varepsilon\kappa$ fixes each line of the spread.

The space of lines in $P_5(\mathbb{R})$ which are disjoint to the Klein quadric (0-secants) is 8-dimensional. To see this, we choose a point in the 5-dimensional complement of the Klein quadric and a line in its 4-dimensional pencil of 0-secants.[1] This gives a 9-dimensional space of flags. Since each line defines a one-parameter family of flags, we have to substract 1 and get the result.

Proposition 18 *The automorphic transformation group $Aut_e(\lambda(\mathcal{C}))$ of the regular spread $\mathcal{C}$ coincides with the stabilizer of $PO_6^{\pm}(\mathbb{R},3)$ on a 0-secant .*

Proof. By Theorem 10 the group $PO_6^{\pm}(\mathbb{R},3)$ is the isomorphic image of $PGL_e(4,\mathbb{R})$ under φ_λ. From Proposition 13 we know that $\lambda(\mathcal{C}) = H_5 \cap \langle e_0, e_3, e_4, e_5 \rangle$. Therefore the subspace $U = \langle e_0, e_3, e_4, e_5 \rangle$ is invariant under collineations and dualities if and only if $\lambda(\mathcal{C})$ is invariant. Let π_5 be the polarity associated with the Klein quadric H_5, then $\pi_5(U)$ is the 0-secant $\langle e_1, e_2 \rangle$. Since π_5 is associated with H_5, the invariance of U is equivalent to the invariance of $\pi_5(U) = \langle e_1, e_2 \rangle$. ∎

[1]By an n-secant, $n \in \{0,1,2\}$, of a quadric Q we mean a line of span Q which has exactly n common points with Q.

4 The structure of 3-dimensional parallelisms

In the last section we studied the complex spread $\mathcal{C}$. This spread has the property of being regular i.e., every triple of lines in $\mathcal{C}$ generates a regulus $\mathcal{R} \subset \mathcal{C}$. We will now consider arbitrary regular spreads $\mathcal{C}$. Note that each regular spread is isomorphic to the complex spread [11, 17.4].

Proposition 19 *Let* $\mathbf{C}$ *be the set of all regular spreads of* $PG_3(\mathbb{R})$ *and let* $\mathfrak{Z}$ *be the set of all* 0*-secants of the Klein quadric* H_5*, then*

$$\gamma : \mathbf{C} \to \mathfrak{Z};\ \mathcal{C} \mapsto \pi_5\Big(\operatorname{span}\lambda(\mathcal{C})\Big) =: \gamma(\mathcal{C}) \tag{10}$$

is a bijection.

Proof. If $\mathcal{C} \subset \mathcal{L}_3$ is a regular spread, then $\lambda(\mathcal{C})$ is an elliptic subquadric of H_5 and $\operatorname{span}\lambda(\mathcal{C}) \subset \mathcal{P}_5$ is a 3-space whose polar line $\pi_5(\operatorname{span}\lambda(\mathcal{C})) \subset \mathcal{P}_5$ has empty intersection with H_5, see the proof of Proposition 18. If $\mathcal{C}_1 \neq \mathcal{C}_2$ then $\operatorname{span}\lambda(\mathcal{C}_1) \neq \operatorname{span}\lambda(\mathcal{C}_2)$ and also $\gamma(\mathcal{C}_1) \neq \gamma(\mathcal{C}_2)$. Therefore γ is an injective map. Conversely, let $L \subset P_5(\mathbb{R})$ a line with $L \cap H_5 = \emptyset$, then $\pi_5(L) \cap H_5$ is an elliptic subquadric of H_5 and $\lambda^{-1}(\pi_5(L) \cap H_5)$ is a regular spread. ■

This mapping γ is extremely helpful when one deals with sets of regular spreads, especially regular parallelisms.

Definition 20 Let γ be the bijection from (10) then for a regular parallelism $\mathbf{P}$ of $\mathrm{PG}(3,\mathbb{R})$ we put

$$\gamma(\mathbf{P}) := \{\gamma(\mathcal{C}) \mid \mathcal{C} \in \mathbf{P}\} \ \text{ and } \ \dim\Big(\operatorname{span}\gamma(\mathbf{P})\Big) =: d_{\mathbf{P}}. \tag{11}$$

We call $d_{\mathbf{P}}$ the *dimension of* $\mathbf{P}$ and shortly speak of a $d_{\mathbf{P}}$*-parallelism*.

Since the notion of dimension is only defined for regular parallelisms, we will consider exclusively regular parallelisms in the following.

We recall from [4, Lemma 2.7]:

Theorem 21 *Clifford and (regular)* 2*-parallelisms coincide.*

In this case $\gamma(\mathbf{P})$ is a plane of lines and the plane has empty intersection with H_5, cf. [6, Def.1.10 and Rem.1.11]. Since the Clifford parallelism is well studied, see for instance [6], we may assume that $d_{\mathbf{P}} \geq 3$. Moreover, in [8] some parallelisms with $4 \leq d_{\mathbf{P}} \leq 5$ are investigated, therefore we shall concentrate on $d_{\mathbf{P}} = 3$ in the following.
We recall the definition of a gl-star and the construction process of a parallelism using it, see [1]. Let Q be an elliptic quadric of $PG(3,\mathbb{R})$, up to isomorphism the unit 2-sphere $\mathbb{S}^2$. A 2-secant is a line of $PG_3(\mathbb{R})$ which intersects Q in two points.

Definition 22 By a *a generalized line star* with respect to an elliptic quadric Q (abbreviated gl-star) we mean a set $\mathcal{A}$ of 2-secants of Q such that each non-interior point p of Q is incident with exactly one line L_p of $\mathcal{A}$. The gl-star $\mathcal{A}$ is called *topological* if the map $p \mapsto L_p$ is continuous.

From each (not necessarily topological) gl-star $\mathcal{A}$ a parallelism can be constructed. Rather than presenting the original construction in [1] we use a shortcut version given in [1, Remark 21], which is all we need here. We embed the elliptic quadric Q into $\widehat{Q} = P\mathbb{R}^4_{0,3,4,5}$ by setting $x_0^2 = x_3^2 + x_4^2 + x_5^2$ and $x_1 = x_2 = 0$. So $Q = \widehat{Q} \cap H_5$, and $\lambda^{-1}(Q)$ is the complex spread. We denote by π_3 the polarity of $\widehat{Q}$ associated with Q, i.e. which is defined by the bilinear form $\langle x, y \rangle = -x_0y_0 + x_3y_3 + x_4y_4 + x_5y_5$ on $\widehat{Q}$. For each 2-secant $A \in \mathcal{A}$, we obtain a well-defined spread by taking

$$\mathcal{C}_{\mathcal{A}} = \gamma^{-1}(\pi_3(A)).$$

Using the definition of γ, we can make this more explicit:

$$\lambda(\mathcal{C}_{\mathcal{A}}) = H_5 \cap \pi_5 \circ \pi_3(A).$$

According to [1, Remark 21], we obtain a regular parallelism by setting

$$\mathbf{P}(\mathcal{A}) = \{\mathcal{C}_A \mid A \in \mathcal{A}\}.$$

Theorem 23 *A regular parallelism* $\mathbf{P}$ *of* $PG(3,\mathbb{R})$ *is 3-dimensional if and only if it can be constructed from some non-ordinary gl-star* $\mathcal{A}$.

Proof: Let $\mathbf{P}(\mathcal{A}) = \{\mathcal{C}_A, A \in \mathcal{A}\}$ be the parallelism constructed from the gl-star $\mathcal{A}$. Then by the construction process for all spreads $\mathcal{C}_A$, $A \in \mathcal{A}$, the γ-related 0-secant has the form $\pi_3(A)$ and is a subspace of $\langle Q \rangle$. The space generated by these 0-secants is a subspace of $\langle Q \rangle$ and therefore $\dim \mathbf{P} \leq 3$. If $\dim \mathbf{P} = 2$ then the parallelism is a Clifford parallelism and the gl-star is ordinary, a contradiction to the assumption.
Conversely, let $\mathbf{P}$ be a 3-dimensional regular parallelism. Then $\Sigma = \langle \{\gamma(S) | S \in \mathbf{P}\} \rangle$ is 3-dimensional. Let S_1 and S_2 be two spreads of $\mathbf{P}$, $S_1 \neq S_2$, i. e., also $S_1 \cap S_2 = \emptyset$. Then the lines $\gamma(S_k) := L_k$ are 0-secants with respect to H_5 for $k = 1,2$. From $L_k \subset \Sigma$, $k = 1,2$, it follows that $\pi_5(\Sigma) \subset \pi_5(L_1) \cap \pi_5(L_2)$ and therefore $\pi_5(\Sigma) \cap H_5 \subset \pi_5(L_1) \cap \pi_5(L_2) \cap H_5 = \operatorname{span} \lambda(S_1) \cap H_5 \cap \operatorname{span} \lambda(S_2) \cap H_5 = \lambda(S_1) \cap \lambda(S_2) = \emptyset$. This means that the line $L = \pi_5(\Sigma)$ is a 0-secant of H_5. Up to isomorphism there is only one type of 0-secants of the Klein quadric and $\Sigma = \pi_5(L)$ intersects H_5 in an elliptic quadric Q with $\operatorname{span} Q = \Sigma$. Since all lines of $\gamma(\mathbf{P})$ are 0-secants, they lie in $\operatorname{span} Q \setminus Q$. Applying π_3 we get a set $\mathcal{A}$ of 2-secants of Q which generates the parallelism see the construction process after Definition 22. Therefore the given 3-parallelism is constructed from a gl-star. This gl-star is not ordinary, otherwise we would get $\dim \Sigma = 2$. ■

Another (indeed the most general) construction process for regular parallelisms is the following: since the 0-secants of the Klein quadric correspond bijectively to the regular spreads of $PG_3(\mathbb{R})$, one can try to find conditions ensuring that a given set of 0-secants corresponds to a parallelism.

Theorem 24 *A set* $\mathcal{H}$ *of 0-secants defines a regular parallelism* $\gamma^{-1}(\mathcal{H})$ *if and only if each tangential hyperplane of* H_5 *contains exactly one line of* $\mathcal{H}$.

This was proved in [4, Lemma 2.3]. We call a line set with this property a *hyperflock determining line set* (hfd-line set). By the definition of the dimension of a regular spread the dimension of the parallelism and the dimension of span$\mathcal{H}$ coincide, more precisely, we prove

Theorem 25 *Let* $\mathbf{P}$ *be a regular parallelism of* $PG(3,\mathbb{R})$ *and* $\mathcal{H} = \gamma(\mathbf{P})$ *its describing hfd-line set. Then* $\mathbf{P}$ *is 3-dimensional if and only if it is non Clifford and* $\mathcal{H} \subset \operatorname{span} Q$ *for some elliptic quadric* Q *of* H_5.

Proof: If $\mathcal{H} \subset \operatorname{span} Q$, then also $\Sigma = \operatorname{span}\mathcal{H} \subset \operatorname{span} Q$ and $\mathbf{P}$ is at most 3-dimensional. Since $\mathbf{P}$ is not a Clifford parallelism, we get $\dim \mathbf{P} = 3$. Conversely, if $\mathbf{P}$ is 3-dimensional, then by Theorem 23 the parallelism $\mathbf{P}$ is generated by a non-ordinary gl-star $\mathcal{A}$. Let Q be the defining elliptic quadric, then $\mathcal{H} \subset \operatorname{span} Q$. ∎

Definition 26 We define the *set of latent lines of a regular parallelism* $\mathbf{P}$ of $\mathrm{PG}(3,\mathbb{R})$ to be the line set

$$\lambda^{-1}\Big(\operatorname{span}\gamma(\mathbf{P}) \cap H_5\Big) =: \Lambda_{\mathbf{P}} \subseteq \mathcal{L}_3. \tag{12}$$

Remark 27 We recall from [4]: In the case of a 2-parallelism (i.e. a Clifford parallelism) $\gamma(\mathbf{P})$ is a plane of lines and this plane has empty intersection with H_5, that is $\Lambda_{\mathbf{P}} = \emptyset$.

All non-Clifford parallelisms exhibited in [1], [2], and [3] are 3-dimensional, and we will show below that their latent line set is a regular spread. More explicitly, we give the characterization

Theorem 28 *A regular parallelism* $\mathbf{P}$ *of* $PG(3,\mathbb{R})$ *is 3-dimensional if and only if the latent line set* $\Lambda_{\mathbf{P}}$ *is a regular spread. This spread is not a member of* $\mathbf{P}$.

Proof: Let $\Sigma := span\gamma(\mathbf{P})$ and suppose that $\Lambda_{\mathbf{P}}$ is a regular spread. Then $\lambda(\Lambda_{\mathbf{P}}) = Q$ is an elliptic quadric. Since $\Sigma \cap H_5 = Q$, it follows that Σ is 3-dimensional.
If the parallelism is 3-dimensional, i. e., $\dim \Sigma = 3$, then by Theorem 23 the set $\Sigma \cap H_5 = Q$ is an elliptic quadric and $\Lambda_{\mathbf{P}} = \lambda^{-1}(Q)$ is a regular spread.
Now by the definition of γ, the π_5-polar line of the 3-space span$\gamma(\mathbf{P})$ is $\gamma(\Lambda_{\mathbf{P}})$. As $\gamma(\Lambda_{\mathbf{P}})$ and span$\gamma(\mathbf{P})$ are complementary subspaces of $\mathrm{PG}(5,\mathbb{R})$, so $\Lambda_{\mathbf{P}}$ is different from any member of $\mathbf{P}$. ∎

5 The automorphisms of 3-parallelisms

Each 3-dimensional regular parallelism $\mathbf{P}$ can be constructed from a generalized line star $\mathcal{A}$ by Theorem 23. Let $\mathcal{A}$ be a gl-star with respect to the elliptic quadric $Q = P\mathbb{R}^4_{0,3,4,5} \cap H_5$ of $\mathrm{PG}(5,\mathbb{R})$ and denote the polarity of $\operatorname{span} Q =: \widehat{Q}$ which is associated with Q by π_3. Then $\mathbf{P}(\mathcal{A}) = \{(\gamma^{-1} \circ \pi_3)(A) \mid A \in \mathcal{A}\}$ is a regular parallelism of $\mathrm{PG}(3,\mathbb{R})$; see the construction process after Definition 22.

Lemma 29 *Assume that the non-Clifford regular parallelism* $\mathbf{P}(\mathcal{A})$ *is constructed from the (non-ordinary) gl-star* $\mathcal{A}$. *A collineation or duality* τ *of* $\mathrm{PG}(3,\mathbb{R})$ *onto itself leaves the parallelism* $\mathbf{P}(\mathcal{A})$ *invariant if, and only if, the induced collineation* τ_λ *of* $\mathrm{PG}(5,\mathbb{R})$ *onto itself leaves the gl-star* $\mathcal{A}$ *invariant.*

Furthermore, $\tau_\lambda(Q) = Q$ and the regular spread $\lambda^{-1}(Q)$ is also invariant under τ and $\lambda^{-1}(Q)$ does not belong to $\mathbf{P}(\mathcal{A})$.

Proof. Put $\mathcal{Y} := \{\pi_3(A) \mid A \in \mathcal{A}\}$, then $\mathbf{P} = \gamma^{-1}(\mathcal{Y})$ by the construction process [1, remark 21] (recalled after 22), and $\mathcal{Y}$ is a hfd-lineset, see the definition of hdf-line sets given after Theorem 24. By the Main Theorem 1.1. of [8] we infer:

$$\tau(\mathbf{P}) = \mathbf{P} \quad \Leftrightarrow \quad \tau_\lambda(\mathcal{Y}) = \mathcal{Y}. \tag{13}$$

As $\mathbf{P}$ is non-Clifford, so $\dim(\operatorname{span}\mathcal{Y}) > 2$; also $\mathcal{Y} \subset \widehat{Q}$, hence $\operatorname{span}\mathcal{Y}$ coincides with the 3-space $\widehat{Q}$. From $\tau_\lambda(\mathcal{Y}) = \mathcal{Y}$ follows $\tau_\lambda(\widehat{Q}) = \widehat{Q}$. This, $\tau_\lambda(H_5) = H_5$ and $Q \subset H_5$ together imply $\tau_\lambda(Q) = Q$. Thus we have $\tau_\lambda|_{\widehat{Q}} \circ \pi_3 = \pi_3 \circ \tau_\lambda|_{\widehat{Q}}$, from which we easily derive:

$$\tau_\lambda(\mathcal{Y}) = \mathcal{Y} \quad \Leftrightarrow \quad \tau_\lambda(\mathcal{A}) = \mathcal{A}. \tag{14}$$

Now (13) and (14) guarantee the validity of the first assertion.

Since $\tau_\lambda(Q) = Q$, each automorphic transformation of $\mathbf{P}(\mathcal{A})$ leaves $\widehat{Q} = P\mathbb{R}^4_{0,3,4,5}$ invariant and also the space $\pi_5(\widehat{Q}) = P\mathbb{R}^2_{1,2}$. The spread $\lambda^{-1}(Q)$ is the latent line set and does not belong to $\mathbf{P}$ by Theorem 28. ∎

Definition 30 We denote by $\Lambda(\mathcal{A})$ the collineation group of $\mathcal{A}$, i.e., the subgroup of $PO_4(\mathbb{R},1)$ which leaves the quadric $Q \subset P\mathbb{R}^4_{0,3,4,5}$ and the set of 2-secants $\{A \mid A \in \mathcal{A}\}$ invariant.

Theorem 31 *If the non-Clifford regular topological parallelism $\mathbf{P}$ of $PG(3,\mathbb{R})$ is constructed from the (non-ordinary) gl-star $\mathcal{A}$, then for the group of automorphic transformations of $\mathbf{P}$ holds*

$$Aut_e(\mathbf{P}) \cong \Lambda(\mathcal{A}) \times O_2(\mathbb{R}).$$

Proof: We apply the isomorphism φ_λ to $Aut_e(\mathbf{P})$ and study the image in $PGO_6(\mathbb{R},3)$. By Lemma 29 the group $\varphi_\lambda(Aut_e(\mathbf{P}))$ is the subgroup of $PGO_6(\mathbb{R},3)$ which leaves the space $P\mathbb{R}^4_{0,3,4,5}$ and the gl-star $\mathcal{A}$ invariant. With the space $P\mathbb{R}^4_{0,3,4,5}$ also the space $\pi_5(P\mathbb{R}^4_{0,3,4,5}) = P\mathbb{R}^2_{1,2}$ is invariant, and the subgroup of $PGO_6(\mathbb{R},3)$ fixing these two spaces is the group $PO_4(\mathbb{R},1) \times O_2(\mathbb{R})$. Since the gl-star is fixed we get in the first factor the subgroup $\Lambda(\mathcal{A})$. ∎

The group $\Lambda(\mathcal{A})$ permutes the 2-secants of the gl-star, and since these correspond to the spreads of the parallelism, $\varphi_\lambda^{-1}(\Lambda(\mathcal{A}))$ permutes the spreads of the parallelism. The group $O_2(\mathbb{R})$ fixes $P\mathbb{R}^4_{0,3,4,5} = \hat{Q}$ and therefore Q elementwise, and it follows that $O_2(\mathbb{R})$ fixes $\mathcal{A}$ elementwise. Hence $O_2(\mathbb{R})$ leaves each spread of the parallelism invariant. This group exists for each gl-star $\mathcal{A}$, therefore we need only determine the group $\Lambda(\mathcal{A})$, which acts faithfully on Q and add the second factor $O_2(\mathbb{R})$ afterwards. Because of $\dim O_2(\mathbb{R}) = 1$ we have

$$\dim Aut_e(\mathbf{P}) = \dim \Lambda(\mathcal{A}) + 1.$$

We denote by Λ^1 the connected component of the identity of $\Lambda(\mathcal{A})$.

Proposition 32 *a) If* $\dim \Lambda^1 = 3$ *then* Λ^1 *is either isomorphic to* $SO_3(\mathbb{R})$ *or to* $PSL_2(\mathbb{R})$ *or* Λ^1 *fixes a point* $p \in Q$.
b) If $\dim \Lambda^1 = 1$ *or* 2 *then* Λ^1 *fixes a point* p *of* Q.

Proof: We will use the isomorphism $PSO_4(\mathbb{R},1) \cong PSL_2(\mathbb{C})$ in the following. The first group $PSO_4(\mathbb{R},1)$ acts on $P_3(\mathbb{R})$ leaving the 2-sphere $Q \cong \mathbb{S}^2$ invariant, and the second group $PSL_2(\mathbb{C})$ is the projective group of the complex projective line $P_1(\mathbb{C}) \cong \mathbb{S}^2$. It follows that each element $\alpha \in PSO_4(\mathbb{R},1)$ fixes at least one point $p \in Q$ (apply the isomorphism and use that $\mathbb{C}$ is algebraically closed). Each element $\alpha \in PSO_4(\mathbb{R},1)$ which fixes three points of Q is the identity (take the isomorphism and use the linearity of $GL_2(\mathbb{C})$). It follows that each element $\alpha \in PSO_4(\mathbb{R},1)$ fixes one or two points of Q.
a) Suppose first that Λ^1 is simple. Then either Λ^1 is isomorphic to $SO_3(\mathbb{R})$, i.e., the stabilizer of $PSO_4(\mathbb{R},1)$ on an interior point of Q - this is the maximal compact subgroup, or Λ^1 is isomorphic to $PSL_2(\mathbb{R})$, the group which stabilizes an exterior point of Q. If the group Λ^1 is not simple but solvable then from the classification of transitive actions on $Q \cong \mathbb{S}^2$ according to [12] one knows that Λ^1 cannot be transitive on Q. Assume there is no fixed point, then Λ^1 has only one-dimensional orbits on Q. An element which fixes three points of Q is the identity (see the beginning of the proof). Therefore, all actions are transitive effective transformation groups of a 3-dimensional solvable Lie group on an 1-dimensional orbit. Because of the classification of transitive actions on 1-manifolds (cf. [14, 69.30]) this is impossible. This contradiction implies that Λ^1 has a fixed point $p \in Q$.
b) As shown in the beginning of the proof each element $\alpha \in \Lambda^1$, α distinct from the identity, fixes exactly one or two points of Q. If it fixes 3 points, then it is the identity.
If $\dim \Lambda^1 = 1$, then we take some element $\alpha \in \Lambda^1$ α not the identity. This element fixes one or two points of Q. Since Λ^1 is commutative, also Λ^1 fixes these points. Now suppose $\dim \Lambda^1 = 2$ and Λ^1 not commutative. Then there is a normal subgroup $\mathbb{R} \lhd \Lambda^1$, and this subgroup fixes one or two points. If it fixes one point, then the normalizer of $\mathbb{R}$, i.e. Λ^1, also fixes this point. If there are two fixed points, then the normalizer either fixes both points or interchanges them. The last case is not possible since Λ^1 is connected. ∎

Theorem 33 *Let* $\mathcal{A}$ *be a non-ordinary topological gl-star and let* Λ^1 *be the connected component of* $\Lambda(\mathcal{A})$. *Then* $\dim \Lambda^1 \geq 3$ *is not possible. If* $\dim \Lambda^1 = 2$ *or* 1, *then* Λ^1 *fixes a 2-secant* $A \in \mathcal{A}$.

Proof: Since $\mathcal{A}$ is not ordinary, the generated parallelism is not Clifford. By [7] the parallelism has transformation group with dimension $g \leq 4$, only. Since one dimension is for the group $O_2(\mathbb{R})$ which fixes each spread of the parallelism, we have $\dim \Lambda^1 \leq 3$.
Suppose first $\dim \Lambda^1 = 3$ and $\Lambda^1 \cong SO_3(\mathbb{R})$, then $SO_3(\mathbb{R})$ is the group which fixes an interior point of Q, up to conjugation the origin 0. If we fix a point $p \in Q$, then also the opposite point $(p \vee 0) \cap Q$ is fixed. The rotations about the axis $p \vee 0$ have to fix the unique line $L \in \mathcal{A}$ containing p, hence $L = p \vee 0$ belongs to $\mathcal{A}$. Therefore the gl-star is the ordinary star with the origin 0 as vertex, a contradiction.

Now suppose $\dim \Lambda^1 = 3$ and $\Lambda^1 \cong PSL_2(\mathbb{R})$. Then we may assume that the point $(0,0,0,1)\mathbb{R} \in P\mathbb{R}^4_{0,3,4,5}$ outside the 2-sphere $Q = \mathbb{S}^2$ is fixed. Then fixing a point of the upper hemisphere $\{(1,x_3,x_4,x_5)\mathbb{R} \mid x_3^2+x_4^2+x_5^2 = 1, x_5 > 0\}$, also the point $(1,x_3,x_4,-x_5)\mathbb{R}$ of the lower hemisphere is fixed, i.e., the upper points are paired with the lower points. Therefore the points of the equator $K = \{(1,x_2,x_3,0)\mathbb{R} \mid x_2^2+x_3^2 = 1\}$ are paired among themselves. Let $p \in K$ and let Λ^1_p be the subgroup of Λ^1 fixing p then $\dim \Lambda^1_p = 2$. But with p also the second intersection point q of the unique 2-secant passing through p is fixed, and we have $\dim \Lambda^1_{p,q} = 1$, a contradiction.
Now let the 3-dimensional group Λ^1 be different from $SO_3(\mathbb{R})$ and from $PSL_2(\mathbb{R})$, then by Proposition 32, Λ^1 fixes a point $p \in \mathbb{S}^2$. Since Λ^1 fixes also the second intersection point q of the 2-secant through p, the group must be contained in $PSO_4(\mathbb{R},1)_{p,q}$, but this group is only 2-dimensional, a contradiction. Thus the case $\dim \Lambda^1 = 3$ cannot happen.
If $\dim \Lambda^1 = 2$ or 1, then by Proposition 32 b Λ^1 fixes a point $p \in \mathbb{S}^2$ and therefore also the second intersection point $q \in \mathbb{S}^2$ of the uniquely determined 2-secant. It follows that the 2-secant $A = p \vee q \in \mathcal{A}$ is fixed under the group Λ^1. ■

Up to conjugation we may choose the two fixed points $p = n = (1,0,0,1)\mathbb{R}$ (north pole) and $q = s = (1,0,0,-1)\mathbb{R}$ (south pole). The fixed 2-secant is then the line $A = n \vee s = \{(x_0,0,0,x_5) \mid x_0^2 + x_5^2 > 0\}$ (the north-south axis). It follows that $\Lambda^1 \leq PSO_4(\mathbb{R},1)_{n,s}$.
The group $PSO_4(\mathbb{R},1)_{n,s}$ acts on $\mathbb{R}^4_{0,3,4,5}$ in the following way

$$\left\{ \begin{bmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ \sinh t & 0 & 0 & \cosh t \end{bmatrix} \mid t \in \mathbb{R}, \varphi \in \mathbb{R}\ mod\ 2\pi \right\}.$$

This is a group isomorphic to a cylinder group $SO_2(\mathbb{R}) \times \mathbb{R}$. The first factor $SO_2(\mathbb{R})$ is the rotation group about the north-south axis, and the second factor $\mathbb{R}$ is the so called hyperbolic group. Its orbits on the 2-sphere $\mathbb{S}^2 = \{(1,x_3,x_4,x_5)\mathbb{R} \mid x_3^2+x_4^2+x_5^2 = 1\}$ are the points n, s and all meridians between n and s. Each element of the hyperbolic group acts on all meridians with the same orientation (from n in direction of s or from s in direction of n).

Lemma 34 *Let $\mathcal{A}$ be a topological generalized gl-star with respect to $\mathbb{S}^2$, and $A \in \mathcal{A}$ a 2-secant. Then there exists at least one other 2-secant in $\mathcal{A}$ which intersects A in the interior of $\mathbb{S}^2$.*

Proof: We assume again that the gl-star is defined in $\mathbf{e_0}\mathbb{R} \vee \mathbf{e_3}\mathbb{R} \vee \mathbf{e_4}\mathbb{R} \vee \mathbf{e_5}\mathbb{R}$ with $\mathbb{S}^2 = \{(x_0,x_3,x_4,x_5) \mid x_0 = 1, x_3^2+x_4^2+x_5^2 = 1\}$, and the first 2-secant is the north south axis $n \vee s = \{(x_0,x_3,x_4,x_5) \mid x_3 = x_4 = 0$, which intersects the interior of $\mathbb{S}^2$ in $\{(x_0,x_3,x_4,x_5) \mid x_0 = 1, x_3 = x_4 = 0, -1 < x_5 < 1\}$. Let $E = \{(x_0,x_3,x_4,x_5) \mid x_5 = 0\}$ be the "horizontal" plane and let $L_\infty = \{(x_0,x_3,x_4,x_5) \mid x_0 = x_5 = 0\}$ be the "line at infinity" of E. Let $\mathbb{S}^1$ be the circle of oriented lines in E passing through $(1,0,0,0)$, then we define a map

$$\sigma : L_\infty \to \mathbb{S}^1$$

in the following way: for a point $p \in L_\infty$ we denote by L_p the uniquely determined line of the gl-star $\mathcal{A}$ incident with p. Let $\beta : (1, x_3, x_4, x_5) \mapsto (1, x_3, x_4, 0)$ be the orthogonal projection onto E, then we set $M_p = \beta(L_p)$. If we suppose by contradiction, that $n \vee s$ is not met by a second 2-secant, then M_p is not incident with the origin $(1, 0, 0, 0)$ and there is a unique line N_p in E orthogonal to M_p and incident with $(1, 0, 0, 0)$. We define as $\sigma(p)$ this line N_p with orientation from $(1, 0, 0, 0)$ to $N_p \cap M_p$. All steps in the definition of σ are continuous, therefore σ is a continuous map. If $p_1 \neq p_2$ are different points on L_∞ then the lines N_{p_1} and N_{p_2} are different, therefore the map σ is injective. Now we define a map

$$\alpha : \mathbb{S}^1 \to L_\infty$$

as follows: for each oriented line N through $(1, 0, 0, 0)$ in E we define $M = N^\perp$ through $(1, 0, 0, 0)$ in E and set $\alpha(N) = M \cap L_\infty$. Then α is a 2-fold covering $\alpha : \mathbb{S}^1 \to L_\infty = P_1(\mathbb{R})$. The map $\sigma : L_\infty \to \mathbb{S}^1$ just constructed is a section for this covering, a contradiction. ∎

Proposition 35 *The case* $\dim \Lambda^1 = 2$ *is not possible. If* $\dim \Lambda^1 = 1$*, then* Λ^1 *is the rotation group with respect to the north-south axis.*

Proof: We use the structure of $PSO_4(\mathbb{R}, 1)_{n,s}$ described after Theorem 33. By Lemma 34 we find a 2-secant L that intersects the north-south axis in a point s, say $s = (1, 0, 0, h), -1 < h < 1$, and up to conjugation we may assume that the plane $F = \langle n \vee s, L\rangle$ is the plane $\{(x_0, x_3, 0, x_5)\mathbb{R} \mid x_0, x_3, x_5 \in \mathbb{R}\}$. The intersection of this plane with $\mathbb{S}^2$ is the circle $\{(1, x_3, 0, x_5) \mid x_3^2 + x_5^2 = 1\}$ and the 2-secant L intersects this circle in two points, one point $a = (1, x_3, 0, x_5), x_3 < 0$ on the left meridian and one point $b = (1, x_3', 0, x_5'), x_3' > 0$ on the right meridian. If $\dim \Lambda^1 = 2$ then we can find an element $\tau \neq id$ in the hyperbolic subgroup. Since τ acts on all meridians with the same orientation it follows that $L = a \vee b$ and $L^\tau = a^\tau \vee b^\tau$ intersect outside of $\mathbb{S}^2$, a contradiction.
If $\dim \Lambda^1 = 1$, then there are three possibilities: $\Lambda^1 = SO_2(\mathbb{R})$ (first factor), $\Lambda^1 = \mathbb{R}$ (second factor), or Λ^1 is a spiral subgroup of $PSO_4(\mathbb{R}, 1)_{n,s}$. If Λ^1 is the second factor, we get the same contradiction as above. If Λ^1 is a spiral subgroup, then we take an element $\eta = (\varphi, t)$ with $\varphi = 2\pi$ and $t \neq 0$. Then the lines L and L^η are in the same plane and intersect outside of $\mathbb{S}^2$, a contradiction. It follows that in the case of $\dim \Lambda^1 = 1$ the group Λ^1 is (up to conjugation) the rotation group about the north-south axis. ∎

We summarize in

Theorem 36 *Let* $\mathcal{A}$ *be a topological gl-star, then the connected component* $\Lambda^1(\mathcal{A})$ *is either the identity or it is the rotation group* $SO_2(\mathbb{R})$ *about some axis. Therefore* $\dim \Lambda^1(\mathcal{A}) = 0$ *or* 1.

Using Theorem 31 we get the main result

Corollary 37 *Any topological 3-dimensional (regular) parallelism of* $PG(3, \mathbb{R})$ *is of group dimension* 1 *or* 2. ∎

Remark 38 All topological 3-dimensional parallelisms constructed in [2] are of group dimension 2, since the corresponding gl-stars admit rotations about an axis; as appropriate gl-stars we presented rotated n-pencils, $n \in \{3, 4, 5, ...\}$ and the rotated tangent sets of $\{2k + 1\}$-cuspoids, $k \in \{1, 2, 3, ...\}$. All these gl-stars are axial, which means that they belong to a special linear complex of lines (German: "Gebüsch").

6 Topological rotational 3-dimensional parallelisms

We consider 3-dimensional parallelisms which admit a 2-dimensional automorphism group. By Theorem 23 and Proposition 35 they are generated by a generalized line star $\mathcal{A}$ which admits a rotation group about some axis $A \in \mathcal{A}$.

Definition 39 A 3-dimensional parallelism $\mathbf{P}$ and its generating gl-star $\mathcal{A}$ is called *rotational*, if $\mathcal{A}$ is invariant under some rotation group $SO_2(\mathbb{R})$ about some axis $A \in \mathcal{A}$.

Mostly we will work in the affine part, hence we use inhomogeneous coordinates $x = x_3/x_0, y = x_4/x_0, z = x_5/x_0$. Then the gl-star is defined with respect to the 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and the rotation axis A is the north-south axis $A = n \vee s, n = (0, 0, 1), s = (0, 0, -1)$. Note that $A \in \mathcal{A}$, because the rotation group fixes the unique element of $\mathcal{A}$ containing n.

Lemma 40 *In a topological rotational gl-star $\mathcal{A}$, there exists a 2-secant L_0 such that the two intersection points with $\mathbb{S}^2$ have the same height (z-value).*

Proof: We assume that for all 2-secants the two endpoints have different height. Since we only study topological parallelisms (and topological gl-stars), for each endpoint of a 2-secant with smaller height than the other endpoint there is an open neighborhood of endpoints with smaller height. Therefore the set $A = \{z \in [-1, 1], \text{z is the smaller height of an endpoint of a 2-secant}\}$ is an open subset of $[-1, 1]$. Similarly the subset B of higher endpoints is open. This gives a disjoint union $A \cup B = [-1, 1]$, a contradiction to the connectedness of $[-1, 1]$. ∎

Proposition 41 *Let $\mathcal{A}$ be a topological rotational gl-star with rotation axis A. The set $\mathcal{A} \setminus \{A\}$ splits into two disjoint subsets: $\mathcal{A} \setminus \{A\} = \mathcal{H} \cup \mathcal{B}$, where $\mathcal{H}$ consists of all lines contained in some horizontal plane E and meeting A, and the 2-secants $B \in \mathcal{B}$ having end points separated by E. Up to isomorphism, we may take E to be the equatorial plane.*

Proof: By Lemma 40 there is a 2-secant L_0 which lies in a horizontal plane of height z_0, $-1 < z_0 < 1$. We apply the rotation group about the axis $n \vee s$: if L_0 does not intersect the axis, then rotation about an angle of measure π gives a parallel to L_0 and these two lines intersect in the exterior of $\mathbb{S}^2$, a contradiction. Therefore L_0 and the SO_2-images intersect A and we get an ordinary pencil of lines in the horizontal plane. Up to conjugation in the group $PSO_4(\mathbb{R}, 1)_{n,s}$ we may assume that $z = 0$. A second horizontal plane cannot occur, since this would lead again to parallel lines which intersect in the exterior of $\mathbb{S}^2$. ∎

Next we present a construction method for 3-dimensional topological rotational parallelisms. Since there exists the rotation group about the axis $(0,0,1) \vee (0,0,-1)$, it suffices to define the 2-secants with upper endpoint on $\{(+\sqrt{1-t^2},0,t) \mid 0 \le t \le 1\}$; all other 2-secants then follow by rotation.

Theorem 42 (Construction of topological rotational parallelisms):
We choose the following set of 2-secants:

$$\{L_t = (+\sqrt{1-t^2},0,t) \vee (g(t), -\sqrt{1-(f(t)^2+g(t)^2)}, -f(t)) \mid 0 \le t \le 1\}$$

where $t \mapsto f(t) : [0,1] \to [0,1]$ is a continuous strictly monotonic function and $t \mapsto g(t) : [0,1] \to [-1,0]$ is a continuous monotonic function with $g(0) = -1,\ g(1) = 0$ and $-\sqrt{1-f(t)^2} \le g(t) \le 0$. Then the $SO_2(\mathbb{R})$-image of these 2-secants together with the rotation axis and the ordinary pencil in the equatorial plane defines a topological rotational gl-star $\mathcal{A}$ and also a topological rotational parallelism **P**.

Proof.
a) The assumptions imply that any two 2-secants L_1, L_2 defined by $0 \le t_1 < t_2 \le 1$ have orthogonal projections to the (x,z)-plane which intersect in a point $(x_0,z_0) \in \{(x,z) \mid x^2+z^2 < 1\}$. The preimage of (x_0,z_0) on L_1 is a point (x_0,y_1,z_0) which lies on the inner interval of $L_1 \cap \mathbb{S}^2$ and therefore in the interior of $\mathbb{S}^2$. Hence, if the 2-secants L_1 and L_2 intersect, they have an intersection point in the interior of the 2-sphere.
b) We determine for each line L_t the intersection point with the plane E_∞ at infinity. For this we calculate its inclination $m(t)$ from the triangle

$$\{A = (+\sqrt{1-t^2},0,t), B = (g(t), -\sqrt{1-(f(t)^2+g(t)^2)}, -f(t)),$$
$$C = (+\sqrt{1-t^2},0,-f(t))\} :$$

$$m(t) = \frac{\overline{AC}}{\overline{BC}} = \frac{t+f(t)}{\sqrt{(\sqrt{1-t^2}-g(t))^2+1-(f(t)^2+g(t)^2)}} := \frac{a(t)}{b(t)}.$$

From the assumption on f it follows that $a(t)$ is strictly increasing in t with $a(t) \rightsquigarrow 2$ for $t \rightsquigarrow 1$.
Now $b(t)$ can be written as

$$b(t) = \sqrt{1-t^2-2g(t)\sqrt{1-t^2}+1-f(t)^2}.$$

Since g is a monotonic function, it follows that $b(t)$ is decreasing. Using the assumption $-\sqrt{1-f(t)^2} \le g(t)$ we obtain
$b(t) \le \sqrt{1-t^2+2\sqrt{1-f(t)^2}\sqrt{1-t^2}+1-f(t)^2} = \sqrt{1-t^2}+\sqrt{1-f(t)^2}$.
Therefore $b(t) \rightsquigarrow 0$ for $t \rightsquigarrow 1$. It follows that $m(t) = \frac{a(t)}{b(t)} \rightsquigarrow \infty$ for $t \rightsquigarrow 1$. More exactly, $m(t)$ is strictly increasing from 0 to ∞ if t varies from 0 to 1. Applying the rotation group $SO_2(\mathbb{R})$ about the north-south axis, we get a bijection $\mu : \mathcal{A} \to E_\infty$. The space E_∞ is homeomorphic to the real projective plane, and using μ^{-1} we transfer this topology to $\mathcal{A}$.

c) The 2-secants of $\mathcal{A}$ cover uniquely the points of $\mathbb{S}^2$ and the points of E_∞, therefore it remains to show that each proper point outside of $\mathbb{S}^2$ is covered by a unique 2-secant of $\mathcal{A}$. We denote the intersection point of the 2-secant L_t with the plane E_∞ at infinity by c_t. Then each 2-secant $L_t = a_t \vee b_t$ with $a_t = (\sqrt{1-t^2}, 0, t)$, $b_t = (g(t), -\sqrt{1-(f(t)^2+g(t)^2}, -f(t))$ is the union of three closed intervals: the interval $[a_t, b_t]$ with (a_t, b_t) in the interior of $\mathbb{S}^2$, the interval $A_t = [a_t, c_t]$ of the points with $z \geq 0$ and the interval $B_t = [c_t, b_t]$ of all points with $z \leq 0$. We define $F^+ = \bigcup\{A_t \mid t \in [0,1]\}$ and $F^- = \bigcup\{B_t \mid t \in [0,1]\}$. Then by (a) F^+ and F^- are homeomorphic to the point set of a rectangle. We define the plane $G = \{(x,y,z) \in \mathbb{R}^3 \mid y = 0\}$ and extend it by the points at infinity to the real projective plane $\bar{G}$. Let G^+ and G^- be the subs ets defined by $x^2+z^2 \geq 1, x, z \geq 0$ and $x^2+z^2 \geq 1, x, z \leq 0$, respectively.
We define a map

$$\sigma : F^+ \to G^+$$

in the following way: Let p be a point in F^+, then p lies on exactly one orbit of the rotation group $SO_2(\mathbb{R})$ about the north-south axis. This orbit intersects G^+ in exactly one point, and we take this point as the image $\sigma(p)$ of p.
We will now prove that $\sigma : F^+ \to G^+$ is a bijection. The rectangle F^+ has the following boundary: $\partial F^+ = \{a_t \mid 0 \leq t \leq 1\} \cup A_0 \cup A_1 \cup \{c_t \mid 0 \leq t \leq 1\}$. This boundary is mapped by σ to the set $\partial G^+ = \{a_t \mid 0 \leq t \leq 1\} \cup A_0 \cup A_1 \cup \sigma(\{c_t \mid 0 \leq t \leq 1\})$ which is the boundary of the rectangle G^+. It follows that the image $\sigma(F^+)$ has a subset ∂G^+ which is homeomorphic to a circle $\mathbb{S}^1$, and if σ is not surjective the fundamental group $\pi_1(\sigma(F^+))$ has the group $\mathbb{Z}$ as a subgroup. Since σ is continuous, it induces a homomorphism $\sigma^* : \pi_1(F^+) \to \pi_1(G^+)$ and $\sigma^*(\pi_1(F^+)) = \pi_1(\sigma(F^+))$. Since $\pi_1(F^+) = 0$, it follows that $\sigma^*(\pi_1(F^+)) = 0 = \pi_1(\sigma(F^+))$, a contradiction. Similarly we may define a map $\tau : F^- \to G^-$ and we can show that τ is surjective.
In order to prove the injectivity of σ, we observe that each 2-secant $L_t, 0 < t < 1$ defines a regulus after rotation about the north-south axis. This regulus lies on a rotational hyperboloid which intersects the plane $\bar{G}$ in a hyperbola H_t. This hyperbola is incident with the points $(\sqrt{1-t^2}, t), (-\sqrt{1-t^2}, t), (-\sqrt{1-f(t)^2}, -f(t)), (\sqrt{1-f(t)^2}, -f(t))$, and it has the two asymptotes with slope $m(t)$ and $-m(t)$. Let H_{t_1} and H_{t_2} be two hyperbolas with $0 < t_1 < t_2 < 0$ and let their branches in G^+ be described by the functions $z = h_1(x), x \geq \sqrt{1-t_1^2}$ and $z = h_2(x), x \geq \sqrt{1-t_2^2}$. Now assume that the two branches intersect. Then there is a value x_0 with $h_1(x_0) > h_2(x_0)$. Since the asymptotic values satisfy $m_1 < m_2$ there exists some $x_1 > x_0$ with $h_1(x_1) < h_2(x_1)$. It follows that the two branches intersect in two points and therefore the two hyperbolas H_1 and H_2 intersect in 8 points, a contradiction. If the two branches touch in one point, then the two hyperbolas H_1 and H_2 touch each other in 4 points. This is only possible if the two hyperbolas coincide, i. e., if $t_1 = t_2$, a contradiction. We have thus proved that the map $\sigma : F^+ \to G^+$ is injective. Similarly, also the map $\tau : F^- \to G^-$ is injective, and by rotation it follows that every pair of 2-secants L_{t_1} and L_{t_2}, $0 \leq t_1 < t_2 \leq 1$ does not intersect on $\{(x,y,z) \mid x^2+y^2+z^2 \geq 1\} \cup E_\infty$. It follows that the continuous maps f and g define a gl-star $\mathcal{A}$. This gl-star is a so-called

topological gl-star which has the topology given by $E_\infty \cong P_2(\mathbb{R})$. Following up the steps for the construction of a parallelism from a gl-star, we see that for each topological gl-star the related parallelism is also topological. It follows that the constructed rotational parallelisms are topological. ∎

In order to determine the full group of transformations, two special cases have to be considered: $f(t) = t$ and $g(t) = -\sqrt{1 - f(t)^2}$.

Proposition 43 *If $f(t) = t$ in Theorem 42 but not $g(t) = -\sqrt{1-f(t)^2}$, then in addition to the connected component $SO_2(\mathbb{R})$ the collineation group of the gl-star admits the group $\mathbb{Z}_2$ generated by the map $(x,y,z) \mapsto (-x,y,-z)$. The full automorphism group of the related parallelism is*

$$(SO_2(\mathbb{R}) \rtimes \mathbb{Z}_2) \times O_2(\mathbb{R}).$$

Proof: For each $t \in [0,1]$ there is a unique rotation with an angle $\varphi(t)$ such that the image secant $L_t^{\varphi(t)}$ has its lower intersection point on $\{(x,y,z) \mid x \leq 0, y = 0, z = -\sqrt{1-x^2}\}$. In this way the generating family of 2-secants

$$\{L_t = (\sqrt{1-t^2}, 0, t) \vee (g(t), -\sqrt{1-(t^2+g(t)^2}, -t) \mid 0 \leq t \leq 1\}$$

is mapped to the family

$$\{L_t^{\varphi(t)} = (-g(t), -\sqrt{1-(t^2+g(t)^2}, t) \vee (-\sqrt{1-t^2}, 0, -t) \mid 0 \leq t \leq 1\}.$$

These two families are interchanged by the map $(x,y,z) \mapsto (-x,y,-z)$. Since each automorphism fixing the plane $\{(x,0,z) \mid x,z \in \mathbb{R}\}$ fixes the x-axis and the z-axis, a further automorphism could only be $(x,0,z) \mapsto (-x,0,z)$ or $(x,0,z) \mapsto (x,0,-z)$. A direct calculation shows that these maps are not possible. Therefore the full automorphism group of the gl-star is $SO_2(\mathbb{R}) \rtimes \mathbb{Z}_2$ and the theorem follows by Theorem 31. ∎

Proposition 44 *If $g(t) = -\sqrt{1-f(t)^2}$ in Theorem 42 but not $f(t) = t$, then besides the connected component $SO_2(\mathbb{R})$ the collineation group of the gl-star has the group $\mathbb{Z}_2$ generated by the map $(x,y,z) \mapsto (-x,y,z)$. The full automorphism group of the related parallelism is*

$$(SO_2(\mathbb{R}) \rtimes \mathbb{Z}_2) \times O_2(\mathbb{R}).$$

Proof: From $g(t) = -\sqrt{1-f(t)^2}$ it follows that the generating family of 2-secants is

$$\{L_t = (\sqrt{1-t^2}, 0, t) \vee (g(t), 0, -f(t) \mid 0 \leq t \leq 1\}.$$

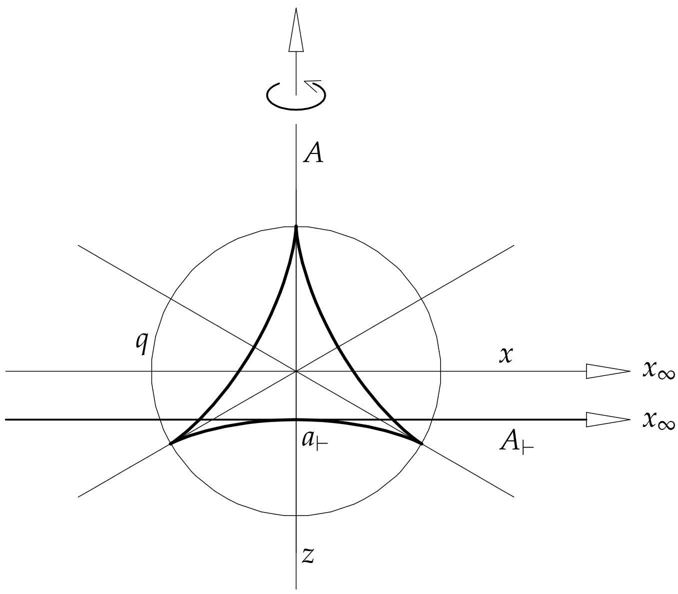
After rotation through the angle $\varphi = \pi$ about the north-south axis, we obtain the family

$$\{L_t^\pi = (-\sqrt{1-t^2}, 0, t) \vee (-g(t), 0, -t) \mid 0 \leq t \leq 1\}.$$

These two families of 2-secants define a so-called gl-pencil in the plane $\{(x,0,z) \mid x,z \in \mathbb{R}\} \cup L_\infty$, i.e., a family of 2-secants of $\mathbb{S}^1 = \{(x,0,z) \mid x^2+z^2 = 1\}$ such that each point $(x,0,y)$ with $x^2+z^2 \geq 1$ (and each point on the line L_∞ at infinity) is on exactly one 2-secant. This gl-pencil is then rotated about the north-south axis to give a rotational gl-star. In order to determine the full group Λ of the gl-star, it suffices to study the subgroup which fixes the (x,z)-plane. Here we see the group $\mathbb{Z}_2$ which is generated by $(x,0,z) \mapsto (-x,0,z)$. But since we assumed $f(t) \neq t$, there is no collineation which interchanges the upper half plane $(z>0)$ with the lower half plane $(z<0)$. It follows that the full group of the gl-star is $SO_2(\mathbb{R}) \rtimes \mathbb{Z}_2$ and using Theorem 31 we get the full transformation group of the related parallelism. ∎

Remark 45 The parallelisms of the previous theorem were already studied in [2].

We recall here a special example of this class, using Steiner's 3-cuspoid, see [2, Example 22]:



Steiner's 3-cusped hypocycloid

The figure displays a gl-pencil and the (plane) 2-secants can be seen as tangent lines to the hypocycloid. This gl-pencil is rotated about the axis A and defines a rotational gl-star. Note that in this model the horizontal regular pencil is not the equatorial plane, but has height $z<0$.

Proposition 46 *If both conditions are not true: neither $f(t) = t$ nor $g(t) = -\sqrt{1-f(t)^2}$, then the full group of the gl-star is the rotation group $SO_2(\mathbb{R})$. The group of all transformations of the related 3-parallelism is $SO_2(\mathbb{R}) \times O_2(\mathbb{R})$.*

Proof: The subgroup of $PO_4(\mathbb{R},1)$ which leaves the axis $A = n \vee s$ and the equatorial plane invariant is $SO_2(\mathbb{R}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Since both conditions are not true, we obtain as full group of the gl-star only the group $SO_2(\mathbb{R})$. By Theorem 31 the full transformation group of the related parallelism is $SO_2(\mathbb{R}) \times O_2(\mathbb{R})$. ∎

Remark 47 If both conditions are fulfilled: $f(t) = t$ and $g(t) = -\sqrt{1-f(t)^2}$, then we get an ordinary pencil in the (x,z)-plane and after rotating, the ordinary gl-star. The related parallelism is therefore the Clifford parallelism which has dimension 2. So, under the assumption of dimension 3, this case cannot happen.

References

[1] D. Betten and R. Riesinger, Topological parallelisms of the real projective 3-space, Results in Math. 47, pp. 226–241 (2005).

[2] D. Betten and R. Riesinger, Constructing topological parallelisms of $\mathrm{PG}(3,\mathbb{R})$ via rotation of generalized line pencils, Adv. Geom. Vol. 8, pp. 11–32 (2008).

[3] D. Betten and R. Riesinger, Generalized line stars and topological parallelisms of the real projective 3-space, J. Geom, 91 1–20 (2008).

[4] D. Betten and R. Riesinger, Hyperflock determining line sets and totally regular parallelisms of $\mathrm{PG}(3,\mathbb{R})$, Mh. Math, 161, 43–58 (2010).

[5] D. Betten and R. Riesinger, Parallelisms of $\mathrm{PG}(3,\mathbb{R})$ composed of non-regular spreads, Aequat. Math, 81, 227–250 (2011).

[6] D. Betten and R. Riesinger, Clifford parallelism: old and new definitions, and their use, J. Geom, 103, 31–73 (2012).

[7] D. Betten and R. Riesinger, Collineation groups of topological parallelisms, Adv. Geom. 14, 175-189 (2014).

[8] D. Betten and R. Riesinger, Automorphisms of some topological regular parallelisms of $PG(3,\mathbb{R})$, Results in Math. 66, 291-326 (2014).

[9] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York and London 1962.

[10] N. Jacobson, Lie algebras, Interscience, New York, 1962.

[11] N. Knarr, Translation planes, Springer 1995.

[12] Mostow, G. D., The extensibility of local Lie groups of transformations and groups on surfaces, Ann. of Math. 52, 606-636 (1950).

[13] G. Pickert, Analytische Geometrie, Akad. Verlagsges. Geest & Portig, 7. Aufl., Leipzig, 1976.

[14] H. Salzmann - D. Betten - T. Grundhöfer - H. Hähl -R. Löwen - M. Stroppel, Compact Projective Planes, de Gruyter, Berlin, 1995.

[15] O. Veblen and J.W. Young, Projective geometry I, Blaisdell Publishing company, New York, Toronto, London, 1946.

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