On nuclearity of the algebra of adjointable operators*

Massoud Amini Mohammad B. Asadi

Abstract

We study nuclearity of the C^*-algebra $\mathbb{B}(\mathcal{E})$ of adjointable operators on a full Hilbert C^*-module $\mathcal{E}$ over a C^*-algebra $\mathcal{A}$. When $\mathcal{A}$ is a von Neumann algebra and $\mathcal{E}$ is full and self dual, we show that $\mathbb{B}(\mathcal{E})$ is nuclear if and only if $\mathcal{A}$ is nuclear and $\mathcal{E}$ is finitely generated. In particular, when $\mathcal{A}$ is a factor, then nuclearity of $\mathbb{B}(\mathcal{E})$ implies that $\mathcal{E}$, $\mathcal{A}$ and $\mathbb{B}(\mathcal{E})$ are finite dimensional.

1 Introduction

In 1976, Simon Wassermann gave a characterization of nuclear W^*-algebras by showing that a W^*-algebra $\mathcal{A}$ is nuclear if and only if it is a direct sum of finitely many type I W^*-algebras of the form $Z \otimes \mathbb{M}_n(\mathbb{C})$, with $n < \infty$ and Z an abelian W^*-algebra [12].

In this short note, we investigate the nuclearity of the C^*-algebra $\mathbb{B}(\mathcal{E})$ of adjointable operators on a Hilbert C^*-module $\mathcal{E}$ over a C^*-algebra $\mathcal{A}$. The first thing which comes to mind is to use the well known fact that nuclearity is preserved under strong Morita equivalence of C^*-algebras [2, 13]. When $\mathcal{E}$ is a full Hilbert $\mathcal{A}$ -module, $\mathcal{A}$ is strongly Morita equivalent to the C^*-algebra $\mathbb{K}(\mathcal{E})$ of compact operators on $\mathcal{E}$ (where $\mathcal{E}$ plays the role of imprimitivity bimodule [5]). In particular, nuclearity of $\mathbb{K}(\mathcal{E})$ and $\mathcal{A}$ are equivalent. On the other hand, $\mathbb{B}(\mathcal{E})$ is the multiplier algebra of $\mathbb{K}(\mathcal{E})$. However, (strong) Morita equivalence of two C^*-algebras does not pass to their multiplier algebras (neither does the nuclearity; just consider the C^*-algebras $\mathbb{C}$ and $\mathbb{K}(\mathcal{H})$ for both cases, where $\mathcal{H}$ is an

*This research was supported by grants from IPM (No. 92430215, 92470123).
Received by the editors in December 2014.
Communicated by A. Valette.
2010 *Mathematics Subject Classification* : 46L55, 18D05.
Key words and phrases : Hilbert C^*-modules, nuclearity, Morita equivalence.

infinite dimensional Hilbert space). If $\mathcal{A}$ is a von Neumann algebra and $\mathcal{E}$ is a full self dual Hilbert $\mathcal{A}$-module, then $\mathbb{B}(\mathcal{E})$ is a von Neumann algebra and $\mathbb{B}(\mathcal{E})$ and $\mathcal{A}$ are Morita equivalent as von Neumann algebras. This fact does not help the situation, as nuclearity is not preserved under Morita equivalence of von Neumann algebras (same example as above; in fact, $\mathbb{B}(\mathcal{H})$ is nuclear if and only if $\mathcal{H}$ is finite dimensional [12]).

It seems that in order to get a characterization for nuclearity of $\mathbb{B}(\mathcal{E})$ one has to use the general characterization of Wassermann. This is possible only if $\mathbb{B}(\mathcal{E})$ is a von Neumann algebra. When $\mathcal{A}$ is a von Neumann algebra and $\mathcal{E}$ is full and self dual, we show that $\mathbb{B}(\mathcal{E})$ is nuclear if and only if $\mathcal{A}$ is nuclear and $\mathcal{E}$ is finitely generated.

2 Nuclearity

In this section, we assume that $\mathcal{A}$ is a C^*-algebra and $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module. We find conditions on $\mathcal{A}$ and $\mathcal{E}$ such that $\mathbb{B}(\mathcal{E})$ is nuclear, using the following general characterization due to Simon Wassermann [12, Corollary 1.9]. In fact, Wassermann proved that a von Neumann algebra is nuclear (as C^*-algebra), if and only if it is a finite direct sum of type I von Neumann algebras of the form $Z \otimes \mathbb{M}_n(\mathbb{C})$ with $n < \infty$ and Z an abelian von Neumann algebra.

In particular, using the above fact, the algebra $\mathbb{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ is nuclear iff $\mathcal{H}$ is finite dimensional. We show that essentially the same holds for the algebra $\mathbb{B}(\mathcal{E})$ of bounded adjointable operators on full self dual Hilbert C^*-modules over factors (Corollary 2.5). More generally, when $\mathcal{E}$ is full and self dual over a von Neumann algebra $\mathcal{A}$, we show that $\mathbb{B}(\mathcal{E})$ is nuclear if and only if $\mathcal{A}$ is nuclear and $\mathcal{E}$ is finitely generated (Corollary 2.4).

Marc Rieffel introduced two notions for Morita equivalence of C^*-algebras (and one for W^*-algebras). Two C^*-algebras are called (strongly) Morita equivalent if their categories of nondegenerate *-representations are equivalent (via an imprimitivity bimodule). Morita equivalence of two C^*-algebras could only guarantee (and indeed is equivalent to) the existence of an imprimitivity bimodule for their enveloping W^*-algebras [10].

If two C^*-algebras are (strongly) Morita equivalent then the centers of their enveloping W^*-algebras (multiplier algebras) are isomorphic [2, 10]. For the case of strong Morita equivalence, the result follows from Dauns-Hoffmann Theorem [7, Theorem 4.4.8] and the fact that these C^*-algebras should have homeomorphic spectra (and so homeomorphic primitive ideal spaces). Here, we give a direct proof of this fact (and write an explicit formula for the isomorphism) when $\mathcal{A}$ is unital.

We recall that a Hilbert $\mathcal{A}$-module $\mathcal{E}$ is called full if the ideal $\langle \mathcal{E}, \mathcal{E} \rangle = Span\{\langle x, y \rangle : x, y \in \mathcal{E}\}$ is dense in $\mathcal{A}$. Also $\mathcal{E}$ is called self dual if $\mathcal{E} = \mathcal{E}'$, where $\mathcal{E}'$ is the set of all bounded linear $\mathcal{A}$-module maps from $\mathcal{E}$ to $\mathcal{A}$.

Lemma 2.1. *If $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{E}$ is a full Hilbert C^*-module then $Z(\mathcal{A}) \simeq Z(\mathbb{B}(\mathcal{E}))$.*

Proof. Given $a \in Z(\mathcal{A})$, define $t_a \in \mathbb{B}(\mathcal{E})$ by $t_a(x) = x \cdot a$, then for each $x \in \mathcal{E}$ and $T \in \mathbb{B}(\mathcal{E})$, $Tt_a(x) = T(x \cdot a) = T(x) \cdot a = t_a T(x)$, hence we have the isometry

$t : Z(\mathcal{A}) \to Z(\mathbb{B}(\mathcal{E}))$; $a \mapsto t_a$, which is clearly a $*$-homomorphism. If $T \in Z(\mathbb{B}(\mathcal{E}))$, then $T\theta_{x,y} = \theta_{x,y}T$, that is $T(x \cdot \langle y, z\rangle) = x \cdot \langle y, Tz\rangle$ for each $x, y, z \in \mathcal{E}$. Since $\mathcal{E}$ is full, there are $y_1, \cdots, y_n \in \mathcal{E}$ such that $\langle y_1, y_1\rangle + \cdots \langle y_n, y_n\rangle = 1$ [5]. Put $a = \langle y_1, Ty_1\rangle + \cdots \langle y_n, Ty_n\rangle$, then $Tx = x \cdot a$ and since T is a right $\mathcal{A}$-module map and $\mathcal{E}$ is full, $a \in Z(\mathcal{A})$. Indeed, for each $x \in \mathcal{E}$ and $b \in \mathcal{A}$, $x(ba - ab) = xba - xab = T(xb) - T(x)b = 0$, hence $ba - ab = 0$, since $\mathcal{E}$ is full. Therefore, t is surjective. ∎

In the above lemma if $\mathbb{B}(\mathcal{E}) \simeq \oplus_{k=1}^{N} Z_k \otimes \mathbb{M}_{n_k}(\mathbb{C})$ where each Z_k is a commutative C^*-algebra (this holds by [12, Corollary1.9] when $\mathcal{A}$ is a von Neumann algebra, $\mathcal{E}$ is self dual and $\mathbb{B}(\mathcal{E})$ is nuclear) then $Z(\mathcal{A}) \simeq \oplus_{k=1}^{N} Z_k$.

Recall that every unital continuous trace C*-algebra has a compact spectrum. If a unital C*-algebra $\mathcal{A}$ is strongly Morita equivalent to a commutative C*-algebra $C_0(X)$, then $\mathcal{A}$ is continuous trace and X is compact. Therefore, a commutative C^*-algebra is unital, whenever it is strongly Morita equivalent to a unital C^*-algebra (this fails in the non commutative case; see the example in the previous section). In particular, if $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{E}$ is full Hilbert $\mathcal{A}$-module such that $\mathbb{K}(\mathcal{E})$ is abelian, then $\mathbb{K}(\mathcal{E})$ is unital.

William Paschke showed that if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{E}$ is full and self dual Hilbert $\mathcal{A}$-module, then $\mathbb{B}(\mathcal{E})$ is a von Neumann algebra [6].

Theorem 2.2. *If $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module, then the following are equivalent:*

(i) $\mathbb{B}(\mathcal{E})$ is a nuclear von Neumann algebra,

(ii) $\mathcal{A}$ is a nuclear von Neumann algebra and $\mathbb{K}(\mathcal{E})$ is unital.

Proof. $(i) \Rightarrow (ii)$. If $\mathbb{B}(\mathcal{E})$ is nuclear, then by [12, Corollary 1.9], $\mathbb{B}(\mathcal{E}) \simeq \oplus_{k=1}^{N} Z_k \otimes \mathbb{M}_{n_k}(\mathbb{C})$ where each Z_k is an abelian von Neumann algebra. Therefore, $\mathbb{K}(\mathcal{E}) \simeq \oplus_{k=1}^{N} I_k \otimes \mathbb{M}_{n_k}(\mathbb{C})$, where I_k is an ideal of Z_k such that $\mathcal{M}(I_k) = Z_k$, for each $1 \leq k \leq n$. Since $\mathbb{K}(\mathcal{E})$ and $\mathcal{A}$ are strongly Morita equivalent, and $\mathbb{K}(\mathcal{E})$ is nuclear and continuous trace, $\mathcal{A}$ is also nuclear and continuous trace. Hence the spectrum of $\mathcal{A}$ is compact, and so is the spectrum of $\mathbb{K}(\mathcal{E})$. But $\mathbb{K}(\mathcal{E})$ is strongly Morita equivalent to $\oplus_{k=1}^{N} I_k$, hence the spectrum of the latter (which is the disjoint union of the spectra of I_k's) is also compact. Thus each I_k is unital and so is $\mathbb{K}(\mathcal{E})$. Therefore $\mathbb{K}(\mathcal{E}) = \mathbb{B}(\mathcal{E})$ is a von Neumann algebra. Finally, since $\mathcal{A}$ is unital and strongly Morita equivalent to a von Neumann algebra, it is a von Neumann algebra by [1, Theorem 3.3].

$(i) \Rightarrow (ii)$. This is evident. ∎

It is well known that being a C^*-algebra of compact operators is preserved under strong Morita equivalence of C^*-algebras (c.f. [4]; in particular, being finite dimensional is preserved under strong Morita equivalence for unital C^*-algebras). Indeed, $\mathbb{K}(\mathcal{E})$ is unital if and only if $\mathcal{E}$ is finitely generated and self dual [11]. It is easy to see that if $\mathcal{E}$ is a finitely generated Hilbert C^*-module over a finite dimensional C^*-algebra $\mathcal{A}$, then $\mathcal{E}$ and $\mathbb{K}(\mathcal{E})$ are finite dimensional. On the other hand, if $\mathcal{A}$ is unital, $\mathbb{K}(\mathcal{E})$ is finite dimensional if and only if $\mathcal{E}$ is finite dimensional, if and only if $\mathcal{E}$ and $\mathcal{A}$ are both finite dimensional. Therefore, we have

Corollary 2.3. *If $\mathcal{A}$ is a von Neumann algebra and $\mathcal{E}$ is full self dual Hilbert $\mathcal{A}$-module, then the following are equivalent:*
(i) $\mathbb{B}(\mathcal{E})$ *is nuclear,*
(ii) $\mathcal{A}$ *is nuclear and* $\mathbb{K}(\mathcal{E})$ *is unital.*
(iii) $\mathcal{A}$ *is nuclear and* $\mathcal{E}$ *is finitely generated.*

Corollary 2.4. *If $\mathcal{A}$ is a factor and $\mathcal{E}$ is a full self dual Hilbert $\mathcal{A}$-module, then the following are equivalent:*
(i) $\mathbb{B}(\mathcal{E})$ *is nuclear,*
(ii) *both* $\mathcal{E}$ *and* $\mathcal{A}$ *are finite dimensional,*
(iii) $\mathbb{B}(\mathcal{E})$ *is finite dimensional.*

The above corollary also holds when $\mathcal{A}$ is a unital C^*-algebra with trivial center and $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module such that $\mathbb{B}(\mathcal{E})$ is a von Neumann algebra.

Finally, we remark that, since strongly Morita equivalent unital C^*-algebras have isomorphic centers, if any of the equivalent conditions in Corollary 2.3 hold then there are an integer N and some abelian von Neumann algebras $Z_1, \cdots, Z_N$ such that simultaneously $\mathcal{A} \simeq \oplus_{k=1}^N Z_k \otimes \mathbb{M}_{n_k}(\mathbb{C})$ and $\mathbb{B}(\mathcal{E}) \simeq \oplus_{k=1}^N Z_k \otimes \mathbb{M}_{n'_k}(\mathbb{C})$, for some integer numbers $n_1, \cdots, n_N$ and $n'_1, \cdots, n'_N$.

In particular, if $\mathcal{A}$ is a unital continuous trace C^*-algebra with vanishing Dixmier-Douady class, then $\mathcal{A}$ is a von Neumann algebra if and only if its spectrum is a hyper-Stonean space. In this case, $\mathcal{A}$ has a form $\oplus_{k=1}^N Z_k \otimes \mathbb{M}_{n_k}(\mathbb{C})$, where each Z_k is an abelian von Neumann algebra. Consequently, a unital continuous trace C^*-algebra with vanishing Dixmier-Douady class is a factor if and only if it is finite dimensional.

References

[1] C.A. Akemman, M. Amini, and M.B. Asadi, *Which multiplier algebras are W^*-algebras?*, arXiv:1304.7453 [math.OA].

[2] W. Beer, *On Morita equivalence of nuclear C^*-algebras*, J. Pure Appl. Algebra **26** (1982), 249–267.

[3] M. Frank and V.I. Paulsen, *Injective envelopes of C*-algebras as operator modules*, Pacific J. Math. **212** (2003), 57–69.

[4] A. an Huef, I. Raeburn, and D.P. Williams, *Properties preserved under Morita equivalence of C^*-algebras*, Proc. Amer. Math. Soc. **135** (2007), 1495–1503.

[5] V.M. Manuilov and E.V. Troitsky, *Hilbert C^*-modules*, Translations Math. Monographs, vol. **226**, Amer. Math. Soc., Providence, RI, 2005.

[6] William L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[7] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, New York, 1979.

[8] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Amer. Math. Soc., Providence, 1998.

[9] Marc A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. **13** (1974), 176–257.

[10] Marc A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Appl. Alg. **5** (1974), 51–96.

[11] Marc A. Rieffel, *Morita equivalence for operator algebras*, Proc. Symp. Pure Math. **38** (1982), 285–298.

[12] S. Wassermann, *On tensor products of certain group C^*-algebras*, J. Functional Analysis **23** (1976), 239–254.

[13] H.H. Zettl, *Strong Morita equivalence of C^*-algebras preserves nuclearity*, Arch. Math. **38** (1982), 448–452.

Department of Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University, P.O.Box 14115-134, Tehran, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O.Box: 19395-5746, Tehran, Iran
email:mamini@modares.ac.ir, mamini@ipm.ir

School of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran, Tehran, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O.Box: 19395-5746, Tehran, Iran
email:mb.asadi@ut.ac.ir