A Schur multiplier characterization of coarse embeddability

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Abstract

We give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces. Consequently, all locally compact groups whose weak Haagerup constant is 1 embed coarsely into Hilbert spaces, and hence the Baum-Connes assembly map with coefficients is split-injective for such groups.

In this note we study coarse embeddability of locally compact groups into Hilbert spaces. An important application of this concept in [16], [13] and [5] is that the Baum-Connes assembly map with coefficients is split-injective for all locally compact groups that embed coarsely into a Hilbert space (see [2] and [15] for more information about the Baum-Connes assembly map). Here, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see also [6, Theorem 5.3] for the discrete case), and this characterization can be regarded as an answer to the non-equivariant version of [12, Question 1.5]. As a result, any locally compact group with weak Haagerup constant 1 embeds coarsely into a Hilbert space and hence the Baum-Connes assembly map with coefficients is split-injective for all these groups.

Let G be a σ-compact, locally compact group. A *(left) tube* in $G \times G$ is a subset of $G \times G$ contained in a set of the form

$$\mathrm{Tube}(K) = \{(x,y) \in G \times G \mid x^{-1}y \in K\}$$

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where K is any compact subset of G. Following [1, Definition 3.6], we say that a map u from G into a Hilbert space H is a *coarse embedding* if u satisfies the following two conditions:

- for every compact subset K of G there exists $R > 0$ such that
$$(s,t) \in \mathrm{Tube}(K) \implies \|u(s) - u(t)\| \leq R;$$
- for every $R > 0$ there exists a compact subset K of G such that
$$\|u(s) - u(t)\| \leq R \implies (s,t) \in \mathrm{Tube}(K).$$

We say that a group G *embeds coarsely into a Hilbert space* or *admits a coarse embedding into a Hilbert space* if there exist a Hilbert space H and a coarse embedding $u : G \to H$. Note that a coarse embedding need not be injective, and we also do not require it to be continuous.

Every second countable, locally compact group G admits a proper left-invariant metric d, which is unique up to coarse equivalence (see [14] and [9]). So the preceding definition is equivalent to Gromov's notion of coarse embeddability of the metric space (G, d) into Hilbert spaces. We refer to [5, Section 3] for more on coarse embeddability into Hilbert spaces for locally compact groups.

A kernel $\varphi\colon G \times G \to \mathbb{C}$ is a *Schur multiplier* if for every bounded operator $A = (a_{x,y})_{x,y\in G} \in B(\ell^2(G))$, the matrix $(\varphi(x,y)a_{x,y})_{x,y\in G}$ again defines a bounded operator, denoted $M_\varphi A$, on $\ell^2(G)$. In this case, it follows from the closed graph theorem that M_φ in fact defines a *bounded* operator $B(\ell^2(G)) \to B(\ell^2(G))$, and the *Schur norm* $\|\varphi\|_S$ of φ is defined to be the operator norm of M_φ.

A kernel $\varphi\colon G \times G \to \mathbb{C}$ *tends to zero off tubes*, if for any $\varepsilon > 0$ there is a tube $T \subseteq G \times G$ such that $|\varphi(x,y)| < \varepsilon$ whenever $(x,y) \notin T$. Note that if $\varphi\colon G \to \mathbb{C}$ is a function, then φ vanishes at infinity (written $\varphi \in C_0(G)$), if and only if the associated kernel $\widehat{\varphi}\colon G \times G \to \mathbb{C}$ defined by $\widehat{\varphi}(x,y) = \varphi(x^{-1}y)$ tends to zero off tubes.

Theorem 1. *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *G embeds coarsely into a Hilbert space.*
2. *There exists a sequence of Schur multipliers $\varphi_n\colon G \times G \to \mathbb{C}$ such that*
    - $\|\varphi_n\|_S \leq 1$ *for every natural number n;*
    - *each φ_n tends to zero off tubes;*
    - $\varphi_n \to 1$ *uniformly on tubes.*

If any of these conditions holds, one can moreover arrange that the coarse embedding is continuous and that each φ_n is continuous.

It is well-known that the notion of coarse embeddability into Hilbert spaces can be characterized by positive definite kernels (see [8, Theorem 2.3] for the discrete case and [4, Theorem 1.5] for the locally compact case).

Following [11], G has the *weak Haagerup property with constant 1*, if there is a sequence of continuous functions $\varphi_n \in C_0(G)$ converging uniformly to 1 on compact subsets of G and such that the associated kernels $\widehat{\varphi}_n \colon G \times G \to \mathbb{C}$ are Schur multipliers with $\|\widehat{\varphi}_n\|_S \leq 1$.

From Theorem 1 together with [5, Theorem 3.5] we immediately obtain the following.

Corollary 2. *If G is a σ-compact, locally compact group with the weak Haagerup property with constant 1, then G embeds coarsely into a Hilbert space. If G is moreover second countable, then in particular the Baum-Connes assembly map with coefficients is split-injective.*

We now turn to the proof of Theorem 1. It is not hard to see that the countability assumption in [10, Proposition 4.3] is superfluous. We thus record the following (slightly more general) version of [10, Proposition 4.3].

Lemma 3. *Let G be a group with a symmetric kernel $k \colon G \times G \to [0, \infty)$. The following are equivalent.*

1. *For every $t > 0$ one has $\|e^{-tk}\|_S \leq 1$.*
2. *There exist a real Hilbert space $\mathcal{H}$ and maps $R, S \colon G \to \mathcal{H}$ such that*
$$k(x,y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

Recall that a kernel $k \colon G \times G \to \mathbb{R}$ is *conditionally negative definite* if k is symmetric ($k(x,y) = k(y,x)$), vanishes on the diagonal ($k(x,x) = 0$) and

$$\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) \leq 0$$

for any finite sequences $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i = 0$. It is well-known that k is conditionally negative definite if and only if there is a function u from G to a real Hilbert space such that $k(x,y) = \|u(x) - u(y)\|^2$.

A kernel $k \colon G \times G \to \mathbb{C}$ is called *proper*, if $\{(x,y) \in G \times G \mid |k(x,y)| \leq R\}$ is a tube for every $R > 0$.

Theorem 1 is contained in Theorem 4 below, which extends both [6, Theorem 5.3] and [4, Theorem 1.5] in different directions. An important ingredient in the proof of Theorem 4 is the following result (which generalizes without change from the second countable case to the σ-compact case).

Theorem ([5, Theorem 3.4])**.** *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *The group G embeds coarsely into a Hilbert space.*
2. *There is a continuous conditionally negative definite kernel $h \colon G \times G \to \mathbb{R}$ which is proper and bounded on tubes.*

Theorem 4. *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *The group G embeds coarsely into a Hilbert space.*
2. *There exists a sequence of (not necessarily continuous) Schur multipliers $\varphi_n\colon G \times G \to \mathbb{C}$ such that*
   - $\|\varphi_n\|_S \leq 1$ *for every natural number n;*
   - *each φ_n tends to zero off tubes;*
   - $\varphi_n \to 1$ *uniformly on tubes.*
3. *There exists a (not necessarily continuous) symmetric kernel $k\colon G \times G \to [0,\infty)$ which is proper, bounded on tubes and satisfies $\|e^{-tk}\|_S \leq 1$ for all $t > 0$.*
4. *There exists a (not necessarily continuous) conditionally negative definite kernel $h\colon G \times G \to \mathbb{R}$ which is proper and bounded on tubes.*

Moreover, if any of these conditions holds, one can arrange that the coarse embedding in (1)*, each Schur multiplier φ_n in* (2)*, the symmetric kernel k in* (3) *and the conditionally negative definite kernel h in* (4) *are continuous.*

Proof. We show (1) $\iff$ (4) $\iff$ (3) $\iff$ (2).

That (1) implies (4) with h continuous follows directly from [5, Theorem 3.4].

Suppose (4) holds. By the GNS construction there are a real Hilbert space $\mathcal{H}$ and a map $u\colon G \to \mathcal{H}$ such that

$$h(x,y) = \|u(x) - u(y)\|^2.$$

It is easy to check that the assumptions on h imply that u is a coarse embedding. Thus (1) holds.

That (4) implies (3) follows with $k = h$ using Schoenberg's Theorem and the fact that normalized positive definite kernels are Schur multipliers of norm 1. Note also that conditionally negative definite kernels are symmetric and take only non-negative values.

Suppose (3) holds. We show that (4) holds. From Lemma 3 we see that there are a real Hilbert space $\mathcal{H}$ and maps $R, S\colon G \to \mathcal{H}$ such that

$$k(x,y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

As k is bounded on tubes, the map S is bounded. If we let

$$h(x,y) = \|R(x) - R(y)\|^2,$$

then it is easily checked that h is proper and bounded on tubes, since k has these properties and S is bounded. It is also clear that h is conditionally negative definite. Thus (4) holds.

If (3) holds, we set $\varphi_n = e^{-k/n}$ when $n \in \mathbb{N}$. It is easy to check that the sequence φ_n has the desired properties so that (2) holds.

Finally, suppose (2) holds. We verify (3). Essentially, we use the same standard argument as in the proof of [11, Proposition 4.4] and [3, Theorem 2.1.1].

Since G is locally compact and σ-compact, it is the union of an increasing sequence $(U_n)_{n=1}^{\infty}$ of open sets such that the closure K_n of U_n is compact and contained in U_{n+1} (see [7, Proposition 4.39]). Fix an increasing, unbounded sequence (α_n) of positive real numbers and a decreasing sequence (ε_n) tending to zero such that $\sum_n \alpha_n \varepsilon_n$ converges. By assumption, for every n we can find a Schur multiplier φ_n tending to zero off tubes and such that $\|\varphi_n\|_S \le 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \le \varepsilon_n/2.$$

Upon replacing φ_n by $|\varphi_n|^2$ one can arrange that $0 \le \varphi_n \le 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \le \varepsilon_n.$$

Define kernels $\psi_i : G \times G \to [0,\infty[$ and $\psi : G \times G \to [0,\infty[$ by

$$\psi_i(x,y) = \sum_{n=1}^{i} \alpha_n(1 - \varphi_n(x,y)), \qquad \psi(x,y) = \sum_{n=1}^{\infty} \alpha_n(1 - \varphi_n(x,y)).$$

It is easy to see that ψ is well-defined, bounded on tubes and $\psi_i \to \psi$ pointwise (even uniformly on tubes, but we do not need that).

To see that ψ is proper, let $R > 0$ be given. Choose n large enough such that $\alpha_n \ge 2R$. As φ_n tends to zero off tubes, there is a compact set $K \subseteq G$ such that $|\varphi_n(x,y)| < 1/2$ whenever $(x,y) \notin \mathrm{Tube}(K)$. Now if $\psi(x,y) \le R$, then $\psi(x,y) \le \alpha_n/2$, and in particular $\alpha_n(1 - \varphi_n(x,y)) \le \alpha_n/2$, which implies that $1 - \varphi_n(x,y) \le 1/2$. We have thus shown that

$$\{(x,y) \in G\times G \mid \psi(x,y) \le R\} \subseteq \{(x,y) \in G\times G \mid 1 - \varphi_n(x,y) \le 1/2\} \subseteq \mathrm{Tube}(K),$$

and ψ is proper.

We now show that $\|e^{-t\psi}\|_S \le 1$ for every $t > 0$. Since ψ_i converges pointwise to ψ, it will suffice to prove that $\|e^{-t\psi_i}\|_S \le 1$, because the set of Schur multipliers of norm at most 1 is closed under pointwise limits. Since

$$e^{-t\psi_i} = \prod_{n=1}^{i} e^{-t\alpha_n(1-\varphi_n)},$$

it is enough to show that $e^{-t\alpha_n(1-\varphi_n)}$ has Schur norm at most 1 for each n. And this is clear:

$$\|e^{-t\alpha_n(1-\varphi_n)}\|_S = e^{-t\alpha_n}\|e^{t\alpha_n\varphi_n}\|_S \le e^{-t\alpha_n}e^{t\alpha_n\|\varphi_n\|_S} \le 1.$$

The only thing missing is that ψ need not be symmetric. Put $k = \psi + \check{\psi}$ where $\check{\psi}(x,y) = \psi(y,x)$. Clearly, k is symmetric, bounded on tubes and proper. Finally, for every $t > 0$

$$\|e^{-tk}\|_S \le \|e^{-t\psi}\|_S \|e^{-t\check{\psi}}\|_S \le 1,$$

since $\|\check{\varphi}\|_S = \|\varphi\|_S$ for every Schur multiplier φ.

Finally, the statements about continuity follow from [5, Theorem 3.4] and the explicit constructions used in our proof of (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2). ∎

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