q-convexity properties of locally semi-proper morphisms of complex spaces

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Abstract

We prove that if $\pi : Z \to X$ is a locally semi-proper morphism between two complex spaces and X is q-complete, then Z is $(q + r)$-complete, where r is the dimension of the fiber.

1 Introduction

According to Grauert [11] and Narasimhan [14], [15] and their solution to the Levi problem, a complex space is Stein if and only if it admits a continuous strongly plurisubharmonic exhaustion function (see Definitions 3 and 4).

In [18], Stein showed that if X and Z are two complex spaces and if $\pi : Z \to X$ is an unramified covering such that X is Stein, then Z is Stein. This result was generalized to ramified coverings by Le Barz in [13].

The notion of a Stein space was generalized by Andreotti and Grauert in [1], where they defined q-convex and q-complete complex spaces. They extended Cartan's Theorem B and proved finiteness and vanishing theorems for the cohomology of a q-convex and of a q-complete space with values in a coherent analytic sheaf.

Also, in [3], Ballico generalized Stein's result to arbitrary q-complete spaces instead of Stein spaces and in [2], he proved the same type of result for finite morphisms of complex spaces.

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In [5], Colţoiu and Vâjâitu considered locally trivial analytic fibrations $\pi : E \to B$ such that the fiber is a Stein curve and B is q-complete. In this way they improved the result of [3].

Further generalizations of the results of Ballico in [2] were obtained by Vâjâitu in [19].

The purpose of this paper is to generalize the results in [2], [3], [13], [18] and [19]. This is contained in Theorem 7.

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2 Preliminaries

All complex spaces are assumed to be reduced, finite dimensional and with countable topology.

2.1

Definition 1. *A complex space X is said to be a Stein space if the following hold:*

(a) X is holomorphically convex, i.e., for every compact set $K \subset X$ the holomorphically convex hull
$$\widehat{K}_X = \{x \in X : |f(x)| \leq \|f\|_K, \forall f \in \mathcal{O}(X)\}$$
is also compact;

(b) For every $x \in X$ there are global functions $f_1, \ldots, f_N \in \mathcal{O}(X)$ which give a local holomorphic embedding of a neighbourhood of x into $\mathbb{C}^N$;

(c) For every pair of distinct points $x \neq y$ in X there is a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

Definition 2. *Let D be an open neighbourhood of a point $z_0 \in \mathbb{C}^n$ and $f \in C^\infty(D, \mathbb{R})$. We define the Levi form $L(f, z_0)$ of f at z_0 as follows: for arbitrary $\xi, \eta \in \mathbb{C}^n$ set*

$$L(f, z_0)(\xi, \eta) := \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i \bar{\eta}_j.$$

Also we set $L(f, z_0)\xi = L(f, z_0)(\xi, \xi)$, $\xi \in \mathbb{C}^n$.

Definition 3. *1) A real valued $\mathcal{C}^2$-function $\varphi : D \to \mathbb{R}$, where D is an open set in $\mathbb{C}^n$, is said to be plurisubharmonic (respectively strongly plurisubharmonic) if and only if its Levi form is positive-semidefinite (respectively positive definite), that is for each $z_0 \in D$ and for every $\xi \in \mathbb{C}^n$ the inequality $L(f, z_0)\xi \geq 0$ (respectively > 0 on $\mathbb{C}^n \setminus \{0\}$) holds.*

2) Let X be a complex space. A function $\varphi : X \to \mathbb{R}$ is said to be (strongly) plurisubharmonic at a point $x \in X$ if there is a local chart $\iota : U \hookrightarrow \widetilde{U} \subset \mathbb{C}^n$ of X, $U \ni x$ and $\widetilde{\varphi} \in C^\infty(\widetilde{U}, \mathbb{R})$ such that:

1. $\widetilde{\varphi} \circ \iota = \varphi|_U$;

2. *The function $\widetilde{\varphi}$ is (strongly) plurisubharmonic on $\widetilde{U}$.*

The function φ is said to be (strongly) plurisubharmonic on a subset $W \subset X$ if it is (strongly) plurisubharmonic at every point of W.

Definition 4. *Let X be a complex space. An upper semi-continuous function $\varphi : X \to \mathbb{R}$ is said to be an exhaustion function on X if the sublevel sets $\{x \in X : \varphi(x) < c\}$ are relatively compact in X for any $c \in \mathbb{R}$.*

We have the following result (see [11] and [14], [15]):

Theorem 1. *A complex space X is Stein if and only if there exists $\varphi : X \to \mathbb{R}$ a continuous strongly plurisubharmonic exhaustion function on X.*

Definition 5. *Let X and Z be two complex spaces. A morphism $\pi : Z \to X$ is said to be proper if for every compact set K in X the preimage $\pi^{-1}(K)$ is compact. A morphism $\pi : Z \to X$ is said to be finite if it is proper and it has finite fibers.*

Remark 1. *Let $\pi : Z \to X$ be a finite morphism of complex spaces. Then Z is Stein if and only if X is Stein.*

We recall the following theorem of Stein [18]:

Theorem 2. *Let $\pi : Z \to X$ be an unramified covering of complex spaces. If X is Stein, then Z is Stein*

Le Barz [13] extended Stein's result to locally semi-finite morphisms of complex spaces (see Definition 6 and Theorem 3).

Definition 6. *Let X and Z be two complex spaces. We say that a morphism $\pi : Z \to X$ is*

(a) semi-finite if Z is the disjoint union of some open spaces $(W^m)_{m\in\mathbb{N}}$ such that $\pi|_{W^m} : W^m \to X$ is a finite morphism;

(b) locally semi-finite if for all $x \in X$, there exists a neighbourhood $U \ni x$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a semi-finite morphism.

Theorem 3. *Let $\pi : Z \to X$ a locally semi-finite morphism of complex spaces. If X is Stein, then Z is Stein.*

2.2 As we mentioned in the introduction, the notions of a q-convex and of a q-complete complex space were introduced in [1].

Definition 7. *1) A function $\varphi \in C^\infty(D, \mathbb{R})$, where D is an open subset of $\mathbb{C}^n$ is said to be q-convex ($q \in \mathbb{N}, 1 \le q \le n$) if its Levi form has at least $n - q + 1$ positive (> 0) eigenvalues at every point of U.*

2) Let X be a complex space. A function $\varphi : X \to \mathbb{R}$ is said to be q-convex at a point $x \in X$ if there is a local chart $\iota : U \hookrightarrow \widetilde{U} \subset \mathbb{C}^n$ of X, $U \ni x$ and $\widetilde{\varphi} \in C^\infty(\widetilde{U}, \mathbb{R})$ such that:

1. $\widetilde{\varphi} \circ \iota = \varphi|_U$;

2. *The function $\widetilde{\varphi}$ is q-convex on $\widetilde{U}$.*

The second condition can be replaced by the following:

2′. There exists a complex linear space $E \subset \mathbb{C}^n$, $\dim E \geq n - q + 1$ such that the Levi form $L(\widetilde{\varphi}, \iota(x))$ is positive definite when restricted to E.

The function φ is said to be q-convex on a subset $W \subset X$ if it is q-convex at every point of W.

Definition 8. *A complex space X is said to be q-convex, if there exists a compact subset K of X and a smooth exhaustion function $\varphi : X \to \mathbb{R}$, which is q-convex on $X \backslash K$. If we can choose $K = \emptyset$, then X is said to be q-complete.*

Remark 2. *From [14] and [15] we have that a complex space X is Stein if and only if is 1-complete.*

Ballico [3] improved Theorem 2 in another direction.

Theorem 4. *Let $\pi : Z \to X$ be an unramified covering. If X is q-complete, then Z is q-complete.*

Also, in [2], Ballico showed that if $\pi : Z \to X$ is a finite morphism between complex spaces and X is q-complete or q-convex, then Z is q-complete or q-convex, respectively.

Colţoiu and Vâjâitu [5] proved that if $\pi : E \to B$ is a locally analytic fibration of complex spaces such that the fiber is a Stein curve and B is q-complete, then E is q-complete. The case when E is a topological covering of B was already done in [3].

Vâjâitu [19] generalized Ballico's results from [2] and showed the following:

Theorem 5. *Let $\pi : Z \to X$ be a proper holomorphic map between finite dimensional complex spaces. If X is q-complete, then Z is $(q + r)$-complete, where r is the dimension of the fiber.*

Let X be a complex space of complex dimension n and q an integer with $1 \leq q \leq n$. For $q > 1$ the sum and the maximum of two q-convex functions on X is not q-convex as they might have different directions of positivity. It was proved in [7] and [8] that every q-convex function with corners (i.e., a function which locally is equal to the maximum of a finite family of q-convex functions) can be approximated by a $\widetilde{q}$-convex function, where $\widetilde{q} = n - \left[\frac{n}{q}\right] + 1$ (here $\left[\frac{n}{q}\right]$ denotes as usual the largest integer $\leq \frac{n}{q}$). Diederich and Fornaess also showed that this $\widetilde{q}$ is optimal. As a consequence, a finite intersection of q-convex open sets is $\widetilde{q}$-convex. The optimality of this $\widetilde{q}$ is proved by Chiriacescu, Colţoiu and Joiţa in [4] in the case of quasi-projective varieties in a cohomological context.

To overcome this type of problem, M. Peternell [16] introduced the notion of convexity with respect to a linear set $\mathcal{M}$.

As before X is a reduced, finite dimensional and with countable topology complex space. For any $x \in X$ we denote by T_xX the Zariski tangent space of X to x. Set $TX = \bigcup_{x\in X} T_xX$. Consider an arbitrary subset $\mathcal{M} \subset TX$ and for any point $x \in X$ put $\mathcal{M}_x = \mathcal{M} \cap T_xX$. Then $\mathcal{M}$ is said to be a linear set over X if $\mathcal{M}_x$ is a complex vector subspace of T_xX for any $x \in X$.

Let now $\Omega \subset X$ be an open subset. We define:

(i) $\operatorname{codim}_\Omega \mathcal{M} = \sup_{x\in\Omega} \operatorname{codim} \mathcal{M}_x$;

(ii) $\mathcal{M}|_\Omega$ as $(\mathcal{M}|_\Omega)_x = \mathcal{M}_x$ for every $x \in \Omega$.

Let $\pi : Y \to X$ be an analytic morphism of complex spaces and $\mathcal{M}$ a linear set over X. For every $y \in Y$ we have the tangent map which is a $\mathbb{C}$-linear map of complex vector spaces $\pi_{*,y} : T_yY \to T_xX$, where $x = \pi(y)$. We set

$$\pi^*\mathcal{M} := \bigcup_{y\in Y} (\pi_{*,y})^{-1}(\mathcal{M}_x).$$

We have that $\pi^*\mathcal{M}$ is a linear set over Y. Moreover, if $\operatorname{codim} \mathcal{M} \le q-1$, then $\operatorname{codim} \pi^*\mathcal{M} \le q-1$.

The following are due to M. Peternell (see [16]).

Definition 9. *Let X be a complex space, $W \subset X$ an open subset, $\mathcal{M}$ a linear set over W and $\varphi : W \to \mathbb{R}$ a smooth function.*

(a) Let $x \in W$. Then φ is said to be weakly 1-convex with respect to $\mathcal{M}_x$ if there are a local chart $\iota : U \hookrightarrow \widetilde{U}$ of X with $x \in U \subset W$ and $\widetilde{\varphi} \in \mathcal{C}^\infty(\widetilde{U}, \mathbb{R})$ such that

$$\widetilde{\varphi} \circ \iota = \varphi|_U \text{ and } L(\widetilde{\varphi}, \iota(x))\iota_*\xi \ge 0 \text{ for any } \xi \in \mathcal{M}_x.$$

Furthermore, φ is said to be weakly 1-convex with respect to $\mathcal{M}$ if φ is weakly 1-convex with respect to $\mathcal{M}_x$ for every $x \in W$.

(b) We say that φ is 1-convex with respect to $\mathcal{M}$, if for any $x \in W$ there exist an open neighbourhood $U \subset W$ of x and a 1-convex function $\psi \in \mathcal{C}^\infty(U, \mathbb{R})$ such that $\varphi|_U - \psi$ is weakly 1-convex with respect to $\mathcal{M}|_U$.

Definition 10. *Let X be a complex space and $\mathcal{M}$ a linear set over X. We denote by $\mathcal{B}(X, \mathcal{M})$ the set of all continuous functions $\varphi : X \to \mathbb{R}$ such that every point of X admits an open neighbourhood D on which there are functions $f_1, \dots, f_k \in \mathcal{C}^\infty(D, \mathbb{R})$ which are 1-convex with respect to $\mathcal{M}|_D$ and*

$$\varphi|_D = \max(f_1, \dots, f_k).$$

We need also the following results of M. Peternell (see [16]):

Lemma 1. *Suppose that φ is a q-convex function on a complex space X. Then there exists a linear set $\mathcal{M}$ over X of codimension $\le q-1$ such that φ is 1-convex with respect to $\mathcal{M}$.*

Lemma 2. *Let $\iota : U \hookrightarrow \widetilde{U}$ be a local chart of the complex space X and $\varphi : U \to \mathbb{R}$ a smooth function. Then φ is 1-convex with respect to some linear set $\mathcal{M}$ if and only if for every compact subset $K \subset U$ there exists $\delta > 0$ and for each $x \in K$ there exist $\widetilde{\varphi} \in C^\infty(\widetilde{U}, \mathbb{R})$ such that $\widetilde{\varphi} \circ \iota = \varphi$ and*

$$L(\widetilde{\varphi}, \iota(x))\iota_*(\xi) \geq \delta \left\| \iota_*(\xi) \right\|^2$$

for all $\xi \in \mathcal{M}_x$.

In general, an increasing union of Stein open subsets $\{X_i\}_{i\in\mathbb{N}}$ of a complex space X is not Stein, even if X is smooth (see [9] and [10]). However, if (X_{i+1}, X_i) is Runge, then $\bigcup_{i\in\mathbb{N}} X_i$ is Stein. We recall that if Y is a Stein open subset of a Stein space X, then (X, Y) is said to be a Runge pair if for any compact subset K of Y, the set $\widehat{K}_X \cap Y$ is compact. Using the approximation theorem of Oka-Weil, (X, Y) is a Runge pair if and only if Y is a Stein space and every holomorphic function on Y can be approximated uniformly on compact subset of Y by holomorphic functions on X.

The following result follows from Theorem 3 proved by Colţoiu and Vâjâ-itu in [6]; it gives us a criterion for testing the q-completeness of a complex space. The same kind of result as Theorem 3 in [6] was obtained in the q-concave case in [12].

Theorem 6. *Let X be a complex space and $\mathcal{M}$ a linear set over X. Let $\{X_i\}_{i\in\mathbb{N}}$ be an increasing sequence of open subsets of X such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and there are functions $u_i : X_i \to \mathbb{R}$, $u_i \in \mathcal{B}(X_i, \mathcal{M}|_{X_i})$ and constants $C_i, D_i \in \mathbb{R}$, $C_i < D_i$, $i \in \mathbb{N}$ with the following properties:*

(a) $\{x \in X_i : u_i(x) < D_i\} \subset\subset X_i$ *for every* $i \in \mathbb{N}$

(b) $\{x \in X_{i+1} : u_{i+1}(x) < C_i\} \subset \{x \in X_i : u_i(x) < D_i\}$ *for every* $i \in \mathbb{N}$;

(c) for every compact set $K \subset X$ there is $j = j(K) \in \mathbb{N}$ such that

$$K \subset \{x \in X_{i+1} : u_{i+1}(x) < C_i\} \text{ for every } i \geq j.$$

Then there exists an exhaustion function $v \in \mathcal{B}(X, \mathcal{M})$. In particular, if codim $\mathcal{M} \leq q - 1$, then X is q-complete.

2.3 Let X be a complex space and A an analytic subset of X. The Andreotti function will help us to get some positive eigenvalues in the "normal direction" at the

regular points of A. Denote by $\mathfrak{I}_A$ the coherent ideal sheaf of germs of holomorphic functions vanishing along A.

Choose a locally finite covering $\{U_j\}_j$ of X by relatively compact open subsets of X such that on each U_j there are functions $h_1^{(j)}, \ldots, h_{q(j)}^{(j)} \in \mathcal{O}(U_j)$ with $\mathfrak{I}_A|_{U_j} = (h_1^{(j)}, \ldots, h_{q(j)}^{(j)})$.

Let $\{\rho_j\}_j$ be a partition of unity subordinated to the covering $\{U_j\}_j$ of X.
We define the Andreotti function $f_A : X \to \mathbb{R}$ by setting:

$$f_A(x) = \sum_j \rho_j(x) \left\| h^{(j)}(x) \right\|^2, \; x \in X,$$

where

$$\left\| h^{(j)} \right\|^2 = \sum_{i=1}^{q(j)} |h_i^{(j)}|^2.$$

We remark that $f_A \geq 0$, $f_A \in C^\infty(X)$ and $A = \{f_A = 0\}$.

Suppose that $x_0 \in A$ and let $\iota : U \hookrightarrow \widetilde{U}$ be a local chart around x_0. Extend ρ_j by $\widetilde{\rho}_j \in C_0^\infty(\widetilde{U})$ and $h_i^{(j)}|_U$ by $\widetilde{h}_i^{(j)} \in \mathcal{O}(\widetilde{U})$. Locally, f_A has an extension $\widetilde{f}_A$ defined on $\widetilde{U}$ such that

1. the Levi form $L(\widetilde{f}_A, \iota(x))$ is positive semidefinite for all $x \in U \cap A$;
2. for $x \in U \cap \operatorname{Reg}(A)$ we have that the Levi form $L(\widetilde{f}_A, \iota(x))(v) = 0$ iff $v \in \iota_{*,x}(T_x A)$.

The Andreotti function is used in the next result which follows from Lemma 3 and Lemma 4 in [19]:

Lemma 3. *Let $\pi : Z \to X$ be a holomorphic map between finite dimensional reduced complex spaces and A an analytic subset of Z. Put $r = \max\{\dim \pi^{-1}(x) : x \in X\}$ and let $B \subset A$ be an analytic subset such that Sing $(A) \subseteq B$ and suppose that the restriction map $\pi|_{A \setminus B} : A \setminus B \to X$ has locally constant rank.*

Assume also that there exists a locally finite covering $\{V_l'\}_l$ of X by relatively compact open subsets and 1-convex functions $\varphi_l : V_l' \to \mathbb{R}_+$. Let $V_l \subset V_l'$ be open subsets, $\overline{V}_l \subset V_l'$ and $\bigcup V_l = X$. Denote $U_l := \pi^{-1}(V_l)$, $U_l' := \pi^{-1}(V_l')$ and put

$$\psi_l = f_A + \varphi_l \circ \pi : U_l' \to \mathbb{R}_+,$$

where f_A is the Andreotti function of the analytic subset A of Z.

Then there is an open neighbourhood Ω of $A \setminus B$ in Z and a linear set $\mathcal{M}$ over Ω, codim $\mathcal{M} \leq r$ such that $\psi_l|_{U_l \cap \Omega}$ is 1-convex with respect to $\mathcal{M}|_{U_l \cap \Omega}$ for any l.

3 The main result

Following the ideas of Le Barz [13] we give the next definition.

Definition 11. *Let X and Z be two complex spaces. We say that a morphism $\pi : Z \to X$ is*

(a) semi-proper if Z is the disjoint union of some open spaces $(W^m)_{m \in \mathbb{N}}$ such that $\pi|_{W^m} : W^m \longrightarrow X$ is proper;

(b) locally semi-proper if for all $x \in X$, there exists a neighbourhood $U \ni x$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \longrightarrow U$ is a semi-proper morphism.

Now we are ready to state the main result.

Theorem 7. *Let X and Z be two complex spaces and $\pi : Z \to X$ a locally semi-proper morphism and $r = \max\{\dim \pi^{-1}(x) : x \in X\}$. If X is q-complete, then Z is $(q+r)$-complete.*

Proof. Since X is q-complete there exists a smooth q-convex exhaustion function $\varphi : X \to \mathbb{R}$ on X. Due to Lemma 1 there exists a linear set $\mathcal{M}$ over X of codimension $\leq q-1$ such that φ is 1-convex with respect to $\mathcal{M}$. The idea is to use the q-completeness criterion provided by Theorem 6.

Now we need the following result from [19]:

Proposition 1. *Let $\pi : Z \to X$ be a holomorphic map. Then there exists a decreasing chain of $p+1$ analytic subsets A_k of Z, where $p \leq \dim Z$, $Z = A_p \supset A_{p-1} \supset \cdots \supset A_1 \supset A_0 = \emptyset$ such that for every $k \in \{1, 2, \ldots, p\}$ we have $\dim A_{k-1} < \dim A_k$, Sing $(A_k) \subset A_{k-1}$ and*

$$\pi|_{A_k \backslash A_{k-1}} : A_k \backslash A_{k-1} \to X$$

has locally constant rank.

The above decomposition of Z with respect to π is called the singular filtration of π (see also [17]).

So, consider $A_1 \supset A_2 \supset \cdots \supset A_p$ the analytic subsets of Z given by Proposition 1 and the corresponding Andreotti functions f_{A_k}, $k = \overline{1,p}$.

The next ingredient that we need is a lemma. This lemma was proved by Le Barz [13] in the case of 0-dimensional fibers, but the proof in the general case (the dimension of the fiber is > 0) goes exactly the same way.

Lemma 4. *Let X and Z be two complex spaces and $\pi : Z \to X$ a locally semi-proper morphism. Then there exists a locally finite covering $\{U_j\}_j$ of Z and a locally finite covering $\{V_l\}_l$ of X such that the following conditions hold:*

1. *for all j, there exists a positive integer m_j and a local chart $\iota_j : U_j \hookrightarrow \widetilde{U}_j$, where $\widetilde{U}_j$ is an open subset of $\mathbb{C}^{m_j}$;*
2. *for all l, there exists a positive integer n_l and a local chart $\tau_l : V_l \hookrightarrow \widetilde{V}_l$, where $\widetilde{V}_l$ is an open subset of $\mathbb{C}^{n_l}$;*
3. *for all j, there exists $l(j)$ such that we have $\pi(U_j) \subset V_{l(j)}$ and $\pi|_{U_j}$ extends to a holomorphic map $\widetilde{\pi} : \widetilde{U}_j \to \widetilde{V}_{l(j)}$;*

Also there exists a C^∞ function $f : Z \to \mathbb{R}$ such that:

- $\{z \in Z : f(z) < c_1\} \cap \{z \in Z : (\varphi \circ \pi)(z) < c_2\} \subset\subset Z, \forall c_1, c_2 \in \mathbb{R}$;
- *for all j, there exists a map $g_j : V_{l(j)} \to \mathbb{R}$ such that $f|_{U_j} = g_j \circ \pi|_{U_j}$;*
- *g_j has a C^∞ extension, $\widetilde{g}_j : \widetilde{V}_{l(j)} \to \mathbb{R}$;*

- *for all compact sets $K \subset X$,*

$$\sup_{j\in\mathbb{N}} \left\{ \left| \frac{\partial^2 \widetilde{g}_j}{\partial z_r^{(l(j))} \partial \bar{z}_s^{(l(j))}} \right|_{|_{\tau_{l(j)}(V_{l(j)}\cap K)}} : V_{l(j)} \cap K \neq \emptyset, r,s = \overline{1, n_{l(j)}} \right\} < \infty.$$

We choose $\{W_j^1\}_j$ a locally finite covering of Z and $\{W_k^2\}_k$ a locally finite covering of X such that the conditions in Lemma 4 hold. We denote by $\widetilde{\varphi} : \widetilde{W}^2_{k(j)} \to \mathbb{R}$ a q-convex extension of $\varphi|_{W^2_{k(j)}}$. Also consider the function $f : Z \to \mathbb{R}$, the function $g_j : W^2_{k(j)} \to \mathbb{R}$ and its extension $\widetilde{g}_j$ given by Lemma 4.

Using the boundedness condition for the second derivatives of the function f, on every compact set, there exists a convex and strictly increasing function χ so that $(\chi \circ \widetilde{\varphi} + \widetilde{g}_j)|_{\widetilde{W}^2_{k(j)}}$ is q-convex for all j.

Because $\{z \in Z : f(z) < c_1\} \cap \{z \in Z : (\varphi \circ \pi)(z) < c_2\} \subset\subset Z, \forall c_1, c_2 \in \mathbb{R}$ we get that $\chi \circ \varphi \circ \pi + f$ is an exhaustion function. We denote by Z_i the sublevel sets $\{\chi \circ \varphi \circ \pi + f < i\}$ which are relatively compact in Z. This increasing sequence of open sets that cover Z will be the one that is needed in Theorem 6.

Now we have to build the functions u_i in Theorem 6. For this we will make use of a lemma which is based upon Lemma 3. For details one should consult the Main Lemma of [19] and the Remark that follows.

Lemma 5. *Let $\pi : Z \to X$ be a holomorphic map between reduced complex spaces with $r = \max\{\dim \pi^{-1}(x) : x \in X\}$. Then there exists $\mathcal{N}$ a linear set of codimension $\leq r$ over Z such that for any relatively compact open subset U of Z, there exists a finite covering $\{V_l\}_l$ of $\overline{\pi(U)}$ by relatively compact open subsets and smooth functions $\psi_l : U_l \to \mathbb{R}_+$ such that ψ_l is 1-convex with respect to $\mathcal{N}$ over $U_l \cap U$, where $U_l = \pi^{-1}(V_l)$.*

Now we go back to the proof. Since $Z_i \subset\subset Z$, there exists a linear set $\mathcal{N}$ of codimension $\leq r$ over Z, a finite covering $\{V_l^i\}_l$ of $\overline{\pi(Z_i)}$ by relatively compact open subsets and smooth functions $\psi_l^i : U_l^i \to \mathbb{R}_+$ such that ψ_l^i is 1-convex with respect to $\mathcal{N}$ over $U_l^i \cap Z_i$, where $U_l^i = \pi^{-1}(V_l^i)$. The functions ψ_l^i may be taken > 0.

Let $\{\rho_l^i\}_l$ be a partition of unity subordinated to the covering $\{V_l^i\}_l$ and we define a smooth function u_i on Z_i as follows:

$$u_i = \chi \circ \varphi \circ \pi + f + \sum_l \epsilon_l^i \cdot (\rho_l^i \circ \pi)^2 \cdot \psi_l^i,$$

where $\epsilon_l^i > 0$ are sufficiently small constants to be chosen later in the proof. Since the above sum is > 0, there exists $\delta_i > 0$ such that for all $z \in Z_i$ we have $\sum \epsilon_l^i \cdot (\rho_l^i \circ \pi)^2 \cdot \psi_l^i \geq \delta_i$. By choosing the constants ϵ_l^i to be sufficiently small we may assume that $\sum \epsilon_l^i \cdot (\rho_l^i \circ \pi)^2 \cdot \psi_l^i < 1$.

First we show that the functions u_i satisfy the conditions (a), (b) and (c) from Theorem 6. We define $C_i := i - 1$ and $D_i := i$. For simplicity we denote $\sum \epsilon_l^i \cdot (\rho_l^i \circ \pi)^2 \cdot \psi_l^i$ by $\sum^i$. Since $\sum^i \geq \delta_i$, we have that $\{u_i < i\} \subset\subset \{\chi \circ \varphi \circ \pi + f <$

$i\}$, so this proves (a). For the second condition, let $z \in Z_{i+1}$ such that $u_{i+1}(z) < i-1$. We get that $\chi \circ \varphi \circ \pi + f + \sum^{i+1} < i-1$, thus $\chi \circ \varphi \circ \pi + f < i-1$ and $z \in Z_i$. Adding $\sum^i$ to the last inequality, we obtain $\chi \circ \varphi \circ \pi + f + \sum^i < i-1+\sum^i < i$, since $\sum^i < 1$. Now for condition (c), since $\bigcup\{\chi \circ \varphi \circ \pi + f < i-2\} = Z$, it is enough to prove that $\{z \in Z_{i+1} : \chi \circ \varphi \circ \pi + f < i-2\} \subset \{z \in Z_{i+1} : u_{i+1} < i-1\}$. Adding $\sum^i$ to $\chi \circ \varphi \circ \pi + f$ and using the fact that $\sum^i < 1$, we easily get the claim.

Now we prove that $u_i \in \mathcal{B}(Z_i, \mathcal{P}|_{Z_i})$, where $\mathcal{P} := \pi^*\mathcal{M} \cap \mathcal{N}$. We have that $\mathcal{P}$ is a linear set over Z and codim $\mathcal{P} \leq q+r-1$. It is enough to show that every point $z \in Z_i$ admits an open neighbourhood D such that u_i is 1-convex with respect to $\mathcal{P}|_D$. Using Lemma 2, this is equivalent to proving that for every compact $K \subset Z_i$ there exists $\delta > 0$ and for all $z \in K$ there exists an extension $\widetilde{u}_i$ of u_i such that

$$L(\widetilde{u}_i, \iota(z))\iota_*(\xi) \geq \delta \left\|\iota_*(\xi)\right\|^2$$

for all $\xi \in \mathcal{P}_z$.

This is a local statement. So, without any loss of generality, we may suppose that there are local charts $\iota : U \hookrightarrow \widetilde{U} \subset \mathbb{C}^m$, $z \in U \subset Z_i$, $K \subset U$ and $\tau : V \hookrightarrow \widetilde{V} \subset \mathbb{C}^n$, $x := \pi(z) \in V \subset X$, $\pi(K) \subset V$ such that:

(i) $\pi(U) \subset V$ and there exists an extension $\widetilde{\pi} : \widetilde{U} \to \widetilde{V}$, $\widetilde{\pi} \circ \iota = \tau \circ (\pi|_U)$;

(ii) there exists $A > 0$ and smooth extensions $\widetilde{\varphi} : \widetilde{V} \to \mathbb{R}_+$ and $\widetilde{g} : \widetilde{V} \to \mathbb{R}_+$ such that

$$L(\chi \circ \widetilde{\varphi} + \widetilde{g}, \tau(x))\tau_*(\zeta) \geq A \left\|\tau_*(\zeta)\right\|^2$$

for all $\zeta \in \mathcal{M}_x$ and $x \in \pi(K)$ (this is true due to Lemma 4);

(iii) there are constants $a_l > 0$ and smooth extensions $\widetilde{\psi}_l : \widetilde{U} \to \mathbb{R}_+$ of ψ_l such that

$$L(\widetilde{\psi}_l, \iota(z))\iota_*(\xi) \geq a_l \left\|\iota_*(\xi)\right\|^2$$

for all $\xi \in \mathcal{N}_z$ and $z \in K$ (this is true due to Lemma 5).

Let $\widetilde{\rho}_l$ be smooth extensions of ρ_l to $\widetilde{V}$.

So we get an extension $\widetilde{u}_i : \widetilde{U} \to \mathbb{R}_+$ of $u_i|_U$ given as follows:

$$\widetilde{u}_i = \chi \circ \widetilde{\varphi} \circ \widetilde{\pi} + \widetilde{g} \circ \widetilde{\pi} + \sum_l \epsilon_l \cdot (\widetilde{\rho}_l \circ \widetilde{\pi})^2 \cdot \widetilde{\psi}_l.$$

Now, using the same computations as in [19] (see Theorem A, pages 231-232), we get, for a sufficiently small positive ϵ, that for any choice of the constants ϵ_l, with $0 < \epsilon_l \leq \epsilon$, the Levi form of $\widetilde{u}_i$ at $\iota(z)$ in direction $\iota_*(\xi)$ is strictly positive for $\xi \in \mathcal{P}_z$. This means that there exists $\delta > 0$ such that $L(\widetilde{u}_i, \iota(z))\iota_*(\xi) \geq \delta \left\|\iota_*(\xi)\right\|^2$ for all $\xi \in \mathcal{P}_z$. ∎

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