When rings of continuous functions are weakly regular

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Abstract

We define weakly regular rings by a condition characterizing the rings $C(X)$ for weak almost P-spaces X. A Tychonoff space X is called a weak almost P-space if for every two zero-sets E and F of X with $\operatorname{int} E \subseteq \operatorname{int} F$, there is a nowhere dense zero-set H of X such that $E \subseteq F \cup H$. We show that a reduced f-ring is weakly regular if and only if every prime z-ideal in it which contains only zero-divisors is a d-ideal. Frames L for which the ring $\mathcal{R}L$ of real-valued continuous functions on L is weakly regular are characterized. We show that if the coproduct of two Lindelöf frames is of this kind, then so is each summand. Also, a continuous Lindelöf frame is of this kind if and only if its Stone-Čech compactification is of this kind.

Introduction

Throughout the paper, the term "ring" means a commutative ring with identity 1. It is well known that, for any Tychonoff space X, the ring $C(X)$ is regular in the sense of Von Neumann precisely when X is a P-space (see [16]). This result also holds in the broader context of frames [7]. Call a ring *almost regular* if each of its elements is either a zero-divisor or a unit. Every regular ring is almost regular, but not conversely. The Tychonoff spaces X for which $C(X)$ is almost regular are exactly the almost P-spaces that were introduced by Veksler in [25].

Less restricted than almost P-spaces are what Azarpanah and Karavan [1] call *weak almost P-spaces*. These are spaces X such that for every two zero-sets E and

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F of X with $\operatorname{int} E \subseteq \operatorname{int} F$, there is a nowhere dense zero-set H of X such that $E \subseteq F \cup H$. In [1] the authors characterize these spaces as precisely those X for which every singular (i.e. consisting entirely of zero-divisors) prime z-ideal of $C(X)$ is a d-ideal. We will provide another ring-theoretic characterization of these spaces.

In fact, defining a frame L to be a weak almost P-frame if it satisfies a property which is a frame-theoretic enunciation of the definition of weak almost P-spaces, we will obtain a ring-theoretic characterization of these frames which bears immediate resemblance to the frame-theoretic definition (Proposition 2.5). That characterization will then be the basis for our definition of weakly regular rings. It will turn out that an f-ring is weakly regular if and only if it satisfies the prime z-ideal condition mentioned above which characterizes the rings $C(X)$ for X a weak almost P-space.

On the frame-theoretic side of things, we show that if βL is a weak almost P-frame then so is L (Corollary 2.10); and conversely if L is a continuous Lindelöf frame (Proposition 2.12). Another result with the Lindelöf condition concerns coproducts. It says if the coproduct $L \oplus M$ of Lindelöf frames is a weak almost P-frame then so is each summand (Proposition 2.14). Applied to spaces, we deduce from it that if $X \times Y$ is a weak almost P-space where X and Y are Lindelöf with at least one of them locally compact, then X and Y are weak almost P-spaces (Corollary 2.15). The result about coproducts hinges on the fact (established in the course of the proof of the proposition) that every cozero element of the coproduct $L \oplus M$ of Lindelöf frames is a countable join of "cozero rectangles" $a \oplus b$, for a and b cozero elements in L and M respectively. It thus seemed appropriate that we end Section 2 with characterizations (Proposition 2.17) of when every cozero element of a binary coproduct of frames is a countable join of cozero rectangles.

Every P-space is an almost P-space, and every almost P-space is a weak almost P-space. For rings, regularity implies almost regularity quite easily. Less obvious is that every almost regular f-ring is weakly regular. This we show in our last result (Proposition 3.4) which also points out the position of weak regularity in relation to other variants of regularity.

We end this introduction with the following remark. In [24], Subramanian demonstrates most admirably that, as he puts it, many results in the rings $C(X)$ are due essentially to the fact that these rings are ℓ-rings. We trust that our Section 3, which is purely ring-theoretic, will bear further testimony – if such were needed – to this observation of Subramanian.

1 Preliminaries

1.1 Frames

Our references for frames are [18] and [23]. We follow, to a large extent, the notation of these texts, with minor deviations such as, for instance, denoting the frame of open sets of a topological space X by $\mathfrak{O}X$. By a *point* of L we mean a prime element, that is, an element p such that $p \neq 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. We denote the set of all points of L by $\mathrm{Pt}(L)$. The cozero part of L is denoted, as usual, by $\mathrm{Coz}\, L$. By a *quotient map* we mean a surjective frame homomorphism. A frame homomorphism $h\colon L \to M$ is *coz-onto* if for every $d \in \mathrm{Coz}\, M$ there exists some $c \in \mathrm{Coz}\, L$ such that $h(c) = d$.

1.2 The coreflections βL, λL and υL

We shall view βL, the Stone-Čech compactification of a completely regular frame L, as the frame of completely regular ideals of L. We write $j_L\colon \beta L \to L$ for the coreflection map, and r_L for its right adjoint, which is given by $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$. The map $j_L\colon \beta L \to L$ is coz-onto (see [5, Corollary 5]). The regular Lindelöf coreflection of L, denoted λL, is the frame of σ-ideals of $\mathrm{Coz}\, L$ (see [19]). The join map $\lambda_L\colon \lambda L \to L$ is a dense onto frame homomorphism, and is the coreflection map to L from Lindelöf frames.

Realcompact frames are coreflective in **CRegFrm** (see, for instance, [6] and [20] for details). The realcompact coreflection of L, denoted υL, is constructed in the following manner. For any $t \in L$, let $[\![t]\!] = \{x \in \mathrm{Coz}\, L \mid x \leq t\}$; so that if $c \in \mathrm{Coz}\, L$, then $[\![c]\!]$ is the principal ideal of $\mathrm{Coz}\, L$ generated by c. The map $\ell\colon \lambda L \to \lambda L$ given by

$$\ell(J) = \left[\!\!\left[\bigvee J \right]\!\!\right] \wedge \bigwedge\{P \in \mathrm{Pt}(\lambda L) \mid J \leq P\}$$

is a nucleus. The frame υL is defined to be $\mathrm{Fix}(\ell)$. The join map $\upsilon_L\colon \upsilon L \to L$ is a dense onto frame homomorphism whose right adjoint is given by $a \mapsto [\![a]\!]$. It is the coreflection map to L from realcompact frames.

1.3 Rings

A ring is said to be *reduced* if it has no nonzero nilpotent elements. A commutative f-ring A with identity element 1 has *bounded inversion* if every $a \geq 1$ is invertible. The *bounded part* of A is denoted by A^*. The *contraction* of an ideal I of A is the ideal $I^c = A^* \cap I$ of A^*. The *extension* J^e of an ideal J of A^* is the ideal of A generated by J. For any $a \in A$, we denote by $\mathfrak{M}(a)$ the set of all maximal ideals of A containing a; and we set $M(a) = \bigcap \mathfrak{M}(a)$. An ideal I of a ring A is a *z-ideal* if, for any $a, b \in A$,

$$a \in I \text{ and } \mathfrak{M}(a) = \mathfrak{M}(b) \quad \Longrightarrow \quad b \in I.$$

A systematic study of z-ideals in rings was carried out by Mason [21]. We denote by $\mathrm{Ann}(a)$ the annihilator of an element $a \in A$, and by $\mathrm{Ann}^2(a)$ the ideal $\mathrm{Ann}(\mathrm{Ann}(a))$. An ideal I of A is called a *d-ideal* if, for every $a \in I$, $\mathrm{Ann}^2(a) \subseteq I$. As shown in [22], this is equivalent to saying, for every $a, b \in A$,

$$a \in I \text{ and } \mathrm{Ann}(a) = \mathrm{Ann}(b) \quad \Longrightarrow \quad b \in I.$$

We write $\mathrm{Min}(A)$ for the set of minimal prime ideals of A, and $\mathrm{Max}(A)$ for the set of maximal ideals of A.

We refer to [3] for information regarding the f-ring $\mathcal{R}L$ of real-valued continuous functions on a frame L. For any $I \in \beta L$, the ideal $\boldsymbol{M}^I$ of $\mathcal{R}L$ is defined by

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \leq I\}.$$

The maximal ideals of $\mathcal{R}L$ are exactly the ideals $\boldsymbol{M}^I$ for $I \in \operatorname{Pt}(\beta L)$ (see [10]). For $a \in L$, the ideal $\boldsymbol{M}^{r_L(a)}$ is written as $\boldsymbol{M}_a$, and it satisfies

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

For any $\alpha \in \mathcal{R}L$, $\operatorname{Ann}(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}$. The annihilator ideals in $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}_{a^*}$, for $a \in L$ (see [11]).

2 Weak almost P-frames

We already recalled in the introduction that Azarpanah and Karavan [1] define a Tychonoff space X to be a weak almost P-space if whenever E and F are zero-sets with $\operatorname{int} E \subseteq \operatorname{int} F$, then $E \subseteq F \cup W$ for some nowhere dense zero-set W of X. For any $U \in \mathfrak{O}X$, $\operatorname{int}(X \smallsetminus U) = X \smallsetminus \operatorname{cl} U = U^*$. Consequently, the condition defining weak almost P-spaces is equivalent to saying whenever U and V are cozero-sets in X with $U^* \subseteq V^*$, then $V \cap W \subseteq U$ for some dense cozero-set W of X. We thus formulate the following definition.

Definition 2.1. A completely regular frame L is a *weak almost P-frame* if whenever a and b are cozero elements of L with $a^* \leq b^*$, then there is a dense cozero element c such that $b \wedge c \leq a$.

It is immediate that a Tychonoff space X is a weak almost P-space if and only if $\mathfrak{O}X$ is a weak almost P-frame. Here are some examples.

Example 2.2. Recall that a frame L is called an almost P-frame if $c = c^{**}$ for every $c \in \operatorname{Coz} L$. Every almost P-frame is a weak almost P-frame because $a^* \leq b^*$ for $a, b \in \operatorname{Coz} L$ implies $b = b^{**} \leq a^{**} = a$, so that we can take the top element of L as a witnessing dense cozero element.

Example 2.3. A frame L is called *cozero complemented* if for every $c \in \operatorname{Coz} L$ there is a $d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Let L be cozero complemented and $a^* \leq b^*$ for some $a, b \in \operatorname{Coz} L$. Pick $c \in \operatorname{Coz} L$ such that $a \wedge c = 0$ and $a \vee c$ is dense. Since $c \leq a^* \leq b^*$, so that $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = b \wedge a \leq a$, it follows that every cozero complemented frame is a weak almost P-frame.

We now seek a ring-theoretic characterization of these frames. Let us formulate the following definition, which, as the calculations that follow will show, is motivated by the frame-theoretic definition above.

Definition 2.4. A ring A is *weakly regular* if for any $a, b \in A$ with $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)$, there is a non-zero-divisor $c \in A$ such that $bc \in M(a)$.

The terminology suggests a weakening of regularity. That is indeed the case for reduced rings. To see this, recall that a reduced ring is regular if and only if $\mathrm{Min}(A) = \mathrm{Max}(A)$. Thus, if $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$, then

$$b \in \mathrm{Ann}^2(b) \subseteq \mathrm{Ann}^2(a) = \bigcap\{P \in \mathrm{Min}(A) \mid a \in P\} = \bigcap \mathfrak{M}(a) = M(a),$$

so that $c = 1$ is a non-zero-divisor with $bc \in M(a)$.

Proposition 2.5. *The following are equivalent for a completely regular frame L.*

1. *L is a weak almost P-frame.*

2. *For any $a, b \in \mathrm{Coz}\, L$ with $a^* = b^*$, there is a dense $c \in \mathrm{Coz}\, L$ such that $a \wedge c = b \wedge c$.*

3. *$\mathcal{R}L$ is a weakly regular ring.*

Proof. (1) $\Rightarrow$ (2): Suppose a and b are cozero elements with $a^* = b^*$. Then there are dense cozero elements u and v such that $b \wedge u \leq a$ and $a \wedge v \leq b$. Now $c = u \wedge v$ is a dense cozero element with $a \wedge c = b \wedge c$ because $b \wedge u \leq a$ implies $b \wedge u \wedge v \leq a \wedge u \wedge v$, and similarly for the other inequality.

(2) $\Rightarrow$ (3): Let $\alpha, \beta \in \mathcal{R}L$ be such that $\mathrm{Ann}(\alpha) \subseteq \mathrm{Ann}(\beta)$. Then $\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^*} \subseteq \boldsymbol{M}_{(\mathrm{coz}\,\beta)^*}$, which implies $(\mathrm{coz}\,\alpha)^* \leq (\mathrm{coz}\,\beta)^*$, and hence

$$(\mathrm{coz}(\alpha^2 + \beta^2))^* = (\mathrm{coz}\,\alpha \vee \mathrm{coz}\,\beta)^* = (\mathrm{coz}\,\alpha)^* \wedge (\mathrm{coz}\,\beta)^* = (\mathrm{coz}\,\alpha)^*.$$

It therefore follows from (2) that there is a positive $\gamma \in \mathcal{R}L$ such that $\mathrm{coz}\,\gamma$ is dense and $\mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\gamma = \mathrm{coz}\,\gamma \wedge \mathrm{coz}(\alpha^2 + \beta^2)$. Consequently,

$$\mathrm{coz}(\gamma\beta) = \mathrm{coz}(\gamma\beta^2) \leq \mathrm{coz}(\gamma\alpha^2 + \gamma\beta^2) = \mathrm{coz}\,\alpha \wedge \mathrm{coz}\,\gamma \leq \mathrm{coz}\,\alpha.$$

Let Q be a maximal ideal of $\mathcal{R}L$ containing α. Pick $I \in \mathrm{Pt}(\beta L)$ such that $Q = \boldsymbol{M}^I$. Then $r_L(\mathrm{coz}\,\alpha) \leq I$, which implies $r_L(\mathrm{coz}(\beta\gamma)) \leq I$, so that $\beta\gamma \in \boldsymbol{M}^I$. Consequently, $\beta\gamma \in M(\alpha)$. Now, γ is a non-zero-divisor because $\mathrm{coz}\,\gamma$ is dense, therefore $\mathcal{R}L$ is a weakly regular ring.

(3) $\Rightarrow$ (1): Suppose $a, b \in \mathrm{Coz}\, L$ are such that $a^* \leq b^*$. Pick $\alpha, \beta \in \mathcal{R}L$ with $a = \mathrm{coz}\,\alpha$ and $b = \mathrm{coz}\,\beta$. Now $a^* \leq b^*$ implies $\mathrm{Ann}(\alpha) \subseteq \mathrm{Ann}(\beta)$, and hence, by (3), there is a non-zero-divisor γ such that $\beta\gamma \in M(\alpha) = \boldsymbol{M}_{\mathrm{coz}\,\alpha}$. Thus, $c = \mathrm{coz}\,\gamma$ is a dense cozero element of L such that $b \wedge c \leq a$. Therefore L is a weak almost P-frame. ∎

Corollary 2.6. *The following are equivalent for a completely regular frame L.*

1. *L is a weak almost P-frame.*

2. *υL is a weak almost P-frame.*

3. *λL is a weak almost P-frame.*

Since a completely regular frame L is pseudocompact precisely when υL is isomorphic to βL – which also is the case for Tychonoff spaces – we deduce from the above that:

Corollary 2.7. *A pseudocompact frame L is a weak almost P-frame iff βL is a weak almost P-frame. Similarly, a pseudocompact space X is a weak almost P-space iff βX is a weak almost P-space.*

We recall from [4, Lemma 2] that if $h\colon M \to L$ is dense, then the ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is one-one. Also, recall from [2] that a quotient map $h\colon M \to L$ is a C-quotient map precisely when $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ is onto. This is however not the definition used in [2].

Corollary 2.8. *Let $h\colon M \to L$ be a dense C-quotient map. Then M is a weak almost P-frame iff L is weak almost-P.*

Interpreting this result for Tychonoff spaces we obtain the following.

Corollary 2.9. *A dense C-embedded subspace of a Tychonoff space is a weak almost P-space iff the containing space is a weak almost P-space.*

In the less restricted case we have the following corollary. Let us recall that if $h\colon M \to L$ is dense, then $h(a)^* = h(b)^*$ implies $a^* = b^*$ for very $a, b \in L$. This is so because

$$h(a^* \wedge b) = h(a^*) \wedge h(b) \leq h(a)^* \wedge h(b) = h(b)^* \wedge h(b) = 0,$$

so that $a^* \wedge b = 0$ by density, and hence $a^* \leq b^*$, whence equality follows by symmetry. Recall also that a dense onto frame homomorphism preserves pseudocomplements, hence it preserves (and reflects) dense elements.

Corollary 2.10. *Let $h\colon M \to L$ be a dense coz-onto frame homomorphism. If M is a weak almost P-frame, then L is a weak almost P-frame. Hence, if βL is a weak almost P-frame, then so is L.*

Proof. Suppose $a^* = b^*$ for some $a, b \in \operatorname{Coz} L$. Since h is coz-onto, there exist $u, v \in \operatorname{Coz} M$ such that $h(u) = a$ and $h(v) = b$. Then $h(u)^* = h(v)^*$, from which we can deduce by what we have observed above that $u^* = v^*$. Since M is weakly almost-P, there is a dense $w \in \operatorname{Coz} M$ such that $v \wedge w = u \wedge w$. Thus, $h(w)$ is a dense cozero element of L such that $b \wedge h(w) = a \wedge h(w)$. Therefore L is weakly almost-P. ∎

In spaces this result yields the following.

Corollary 2.11. *A dense z-embedded subspace of a weak almost P-space is a weak almost P-space. Hence, X is a weak almost P-space if βX is a weak almost P-space.*

We have not been able to determine if βL is always a weak almost P-frame whenever L is. We do however have a case when this happens. Recall that the "way-below" relation $\ll$ in a frame L is defined by

$$a \ll b \iff b \leq \bigvee S \text{ for some } S \subseteq L \text{ implies } a \leq \bigvee T \text{ for some finite } T \subseteq S.$$

The frame L is then called *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for every $a \in L$. In a regular continuous frame,

$$a \ll b \iff a \prec b \text{ and } {\uparrow}a^* \text{ is compact},$$

a consequence of which is that, in a continuous regular frame, $a_i \ll b_i$ for $i = 1, 2$ implies $a_1 \wedge a_2 \ll b_1 \wedge b_2$. This in general is not the case, and the frames for which it holds are called stably continuous.

In the proof that follows we shall use the fact that if $I \prec\!\!\prec J$ in βL, then $\bigvee I \in J$. For verification see for instance the paragraph preceding Example 4 in [10].

Proposition 2.12. *Let L be a continuous Lindelöf frame. Then βL is a weakly almost P-frame iff L is a weak almost P-frame.*

Proof. Only the right-to-left implication needs proof. So assume L is a weak almost P-frame, and let $U, V \in \mathrm{Coz}(\beta L)$ be such that $U^* \leq V^*$. Pick cozero elements U_n in βL such that $U_n \prec\!\!\prec U_{n+1}$ and $U = \bigvee_n U_n$. For each n, put $u_n = \bigvee U_n$ and observe that $u_n \prec\!\!\prec u_{n+1}$. Since $U_n \leq r_L(u_n) \leq U_{n+1}$, it follows that $U = \bigvee_n r_L(u_n)$. Similarly, there are cozero elements v_n in L such that $V = \bigvee_n r_L(v_n)$. Now let u and v be the cozero elements of L given by $u = \bigvee u_n$ and $v = \bigvee v_n$. Since $u = j_L(U_n)$ and $v = j_L(V_n)$, it follows from $U^* \leq V^*$ that $u^* \leq v^*$. Since L is a weak almost P-frame, there is a dense $d \in \mathrm{Coz}\, L$ such that $v \wedge d \leq u$. Since L is a continuous frame and $d \in \mathrm{Coz}\, L$, so that, by [5, Corollary 4], d is a Lindelöf element in L, there are elements d_n in L such that $d_n \ll d$ for every n and $d = \bigvee d_n$. We may assume that each $d_n \in \mathrm{Coz}\, L$ because the way-below relation interpolates in a continuous frame. Now, in view of the fact that $j_L \colon \beta L \to L$ is coz-onto, there exists, for each n, $D_n \in \mathrm{Coz}(\beta L)$ such that $j_L(D_n) = d_n$. The element $D = \bigvee_n D_n$ is a cozero element in βL. Since

$$j_L(D) = j_L\Big(\bigvee_n D_n\Big) = \bigvee_n d_n = d,$$

it follows that D is dense because d is dense. We claim that $D \wedge V \leq U$. To see this, observe first that $D_n \leq r_L(d_n)$ since $j_L(D_n) = d_n$, and hence

$$D \wedge V \leq \bigvee_n r_L(d_n) \wedge \bigvee_m r_L(v_m) = \bigvee_{n,m} r_L(d_n \wedge v_m).$$

Now, for any pair of indices (n, m),

$$d_n \wedge v_m \ll d \wedge v \leq u = \bigvee u_n,$$

which, in view of the fact that the sequence (u_n) increases, implies there is an index k such that $d_n \wedge v_m \leq u_k \prec\!\!\prec u_{k+1}$, so that $d_n \wedge v_m \in r_L(u_{k+1}) \subseteq U$. Therefore $r_L(d_n \wedge v_m) \leq U$, and hence $D \wedge V \leq U$. Thus, βL is a weak almost P-frame. ∎

Corollary 2.13. *A locally compact Lindelöf space is a weak almost P-space iff its Stone-Čech compactification is a weak almost P-space.*

In [10] it is shown that if the coproduct of two frames is an almost P-frame, then each summand is an almost P-frame. We prove a similar result for Lindelöf weak almost P-frames. Recall that if $c \in \mathrm{Coz}\, L$ and $d \in \mathrm{Coz}\, M$, then $c \oplus d \in \mathrm{Coz}(L \oplus M)$ because $c \oplus d = i_L(c) \wedge i_M(d)$ for the coproduct injections $L \xrightarrow{i_L} L \oplus M \xleftarrow{i_M} M$. It is shown in [8] that, for any $x \in L$ and $y \in M$, $(x \oplus y)^{**} = x^{**} \oplus y^{**}$. Thus, if $x \oplus y$ is dense, then both x and y are dense.

Proposition 2.14. *If the coproduct of two Lindelöf frames is a weak almost P-frame, then each summand is a weak almost P-frame.*

Proof. Let L and M be such frames. Suppose $a^* \leq b^*$ for some $a, b \in \operatorname{Coz} L$. Then $a \oplus 1$ and $b \oplus 1$ are cozero elements of $L \oplus M$. Since $b^{**} \leq a^{**}$, we have

$$(b \oplus 1)^{**} = b^{**} \oplus 1 \leq a^{**} \oplus 1 = (a \oplus 1)^{**},$$

which implies $(a \oplus 1)^* \leq (b \oplus 1)^*$. So, by hypothesis, there is a dense $U \in \operatorname{Coz}(L \oplus M)$ such that

$$(b \oplus 1) \wedge U \leq a \oplus 1. \tag{\#}$$

We claim that there are sequences (c_n) and (d_n) in $\operatorname{Coz} L$ and $\operatorname{Coz} M$ respectively such that

$$U = \bigvee_{n=1}^{\infty} (c_n \oplus d_n).$$

To show this, we write U as a join of basic elements, say $U = \bigvee_{\alpha}(a_\alpha \oplus b_\alpha)$. By complete regularity, for each α there are cozero elements $\{c_i^{(\alpha)}\}$ in L and cozero elements $\{d_j^{(\alpha)}\}$ in M such that

$$a_\alpha = \bigvee_i c_i^{(\alpha)} \quad \text{and} \quad b_\alpha = \bigvee_j d_j^{(\alpha)}.$$

Consequently,

$$a_\alpha \oplus b_\alpha = \bigvee_i c_i^{(\alpha)} \oplus \bigvee_j d_j^{(\alpha)} = \bigvee_{i,j} (c_i^{(\alpha)} \oplus d_j^{(\alpha)}),$$

so that

$$U = \bigvee_{\alpha,i,j} (c_i^{(\alpha)} \oplus d_j^{(\alpha)}).$$

Since U is a cozero element in a Lindelöf frame, it is a Lindelöf element, and hence we can find countably many $c_n \in \operatorname{Coz} L$ and countably many $d_n \in \operatorname{Coz} M$ such that $U = \bigvee_{n=1}^{\infty} (c_n \oplus d_n)$. Now

$$\begin{aligned}(b \oplus 1) \wedge \bigvee_{n=1}^{\infty} (c_n \oplus d_n) &= \bigvee_{n=1}^{\infty} \big((b \oplus 1) \wedge (c_n \oplus d_n)\big) \\ &= \bigvee_{n=1}^{\infty} \big((b \wedge c_n) \oplus d_n\big).\end{aligned}$$

Thus, the inequality in (#) implies $\bigvee_{n=1}^{\infty} \big((b \wedge c_n) \oplus d_n\big) \leq a \oplus 1$, whence $(b \wedge c_n) \oplus d_n \leq a \oplus 1$ for every n, and hence $b \wedge c_n \leq a$ for every n, which implies $b \wedge \bigvee_n c_n \leq$

a. To finish the proof we show that the cozero element $\bigvee_n c_n$ of L is dense. Since U is dense and

$$U = \bigvee_n (c_n \oplus d_n) \leq \Big(\bigvee_n c_n\Big) \oplus \Big(\bigvee_n d_n\Big),$$

it follows that $\Big(\bigvee_n c_n\Big) \oplus \Big(\bigvee_n d_n\Big)$ is dense, whence $\bigvee_n c_n$ is dense. Therefore L is a weak almost P-frame. Similarly, M is a weak almost P-frame. ∎

Corollary 2.15. *Let X and Y be Lindelöf spaces with one of them locally compact. If $X \times Y$ is a weak almost P-space, then both X and Y are weakly almost P-spaces.*

Proof. By [18, Proposition II 13], $\mathfrak{O}(X \times Y) \cong \mathfrak{O}X \oplus \mathfrak{O}Y$. Therefore $\mathfrak{O}X \oplus \mathfrak{O}Y$ is a weak almost P-frame, and so $\mathfrak{O}X$ and $\mathfrak{O}Y$ are weak almost P-frames, which implies X and Y are weak almost P-spaces. ∎

Remark 2.16. The fact that $L \oplus M$ is Lindelöf was used only to enable us to write the cozero element U as a join of countable many "cozero rectangles" $c_n \oplus d_n$. The result therefore is true for any pair (L, M) of frames for which every cozero element of $L \oplus M$ is a join of countably many cozero rectangles. We end the section with a digression from our main train of thought, and give characterizations of such pairs.

To start, we mention that the term "cozero rectangle" is borrowed from [9], and the pointed analogues of the characterizations that follow are in that paper, excluding, of course, the one about the Lindelöf coreflections.

Recall that if, for $i = 1, 2$, $h_i \colon M_i \to L_i$ are frame homomorphisms, then the induced frame homomorphism $h_1 \oplus h_2 \colon M_1 \oplus M_2 \to L_1 \oplus L_2$ is given by

$$(h_1 \oplus h_2)\Big(\bigvee_\alpha (x_\alpha \oplus y_\alpha)\Big) = \bigvee_\alpha \big(h_1(x_\alpha) \oplus h_2(y_\alpha)\big).$$

Proposition 2.17. *The following are equivalent for frames L and M.*

1. *Every cozero element of $L \oplus M$ is a countable join of cozero rectangles.*
2. *For any coz-onto homomorphisms $h \colon K \to L$ and $g \colon N \to M$, the homomorphism $h \oplus g \colon K \oplus N \to L \oplus M$ is coz-onto.*
3. *$j_L \oplus j_M \colon \beta L \oplus \beta M \to L \oplus M$ is coz-onto.*
4. *$\lambda_L \oplus \lambda_M \colon \lambda L \oplus \lambda M \to L \oplus M$ is coz-onto.*

Proof. (1) ⇒ (2): Given a cozero element $U = \bigvee_n (a_n \oplus b_n)$ in $L \oplus M$, take, for each n, cozero elements u_n in K and cozero elements v_n in N such that $h(u_n) = a_n$ and $g(v_n) = b_n$. Then $\bigvee_n (u_n \oplus v_n)$ is a cozero element of $K \oplus N$ mapped to U by $h \oplus g$.

(2) ⇒ (3): This is trivial because the Stone-Čech maps are coz-onto.

(3) ⇒ (1): Let $U \in \mathrm{Coz}(L \oplus M)$. By (3), there is a $V \in \mathrm{Coz}(\beta L \oplus \beta M)$ such that $(j_L \oplus j_M)(V) = U$. As we observed in the proof of Proposition 2.14, there

are cozero elements $c_n \in \mathrm{Coz}(\beta L)$ and $d_n \in \mathrm{Coz}(\beta M)$ such that $V = \bigvee_n (c_n \oplus d_n)$ because $\beta L \oplus \beta M$ is Lindelöf. Thus,

$$U = (j_L \oplus j_M)\Big(\bigvee_n (c_n \oplus d_n)\Big) = \bigvee_n \big(j_L(c_n) \oplus j_M(d_n)\big),$$

which is a countable join of cozero rectangles.

(1) $\Leftrightarrow$ (4): The same line of argument as the foregoing one shows this since $\lambda L \oplus \lambda M$ is Lindelöf. ∎

3 Characterizing weakly regular f-rings

The characterization in statement (2) in Proposition 2.5 suggests an analogous characterization for weakly regular rings. For f-rings we indeed do have such. Recall that a *radical ideal* is one which whenever it contains a power of an element, then it already contains the element. For any a in a ring, $M(a)$ is a radical ideal. Observe that if $\mathfrak{M}(x) \subseteq \mathfrak{M}(y)$, then $\mathfrak{M}(xy) = \mathfrak{M}(y)$. The last implication in the following proof is modelled on that of [1, Theorem 4.2(ii)].

Proposition 3.1. *The following properties of a reduced f-ring A are equivalent.*

1. *A is weakly regular.*
2. *For any $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ implies there is a non-zero-divisor c such that $ac \in M(b)$ and $bc \in M(a)$.*
3. *Every singular prime z-ideal in A is a d-ideal.*

Proof. (1) $\Rightarrow$ (2): Assume (1) and suppose that $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ for some $a, b \in A$. Then there are non-zero-divisors u and v such that $au \in M(b)$ and $bv \in M(a)$. Hence $c = uv$ is a non-zero-divisor with $ac \in M(b)$ and $bc \in M(a)$.

(2) $\Rightarrow$ (3): Let P be a singular prime z-ideal in A. Suppose that, for some $a, b \in A$, $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $a \in P$. We must show that $b \in P$. By (2), there is a non-zero-divisor c such that $bc \in M(a)$. Thus, $\mathfrak{M}(a) \subseteq \mathfrak{M}(bc)$, which implies $\mathfrak{M}(bc) = \mathfrak{M}(abc)$. Since $abc \in P$ and P is a z-ideal, it follows that $bc \in P$, and hence $b \in P$ because P is prime and $c \notin P$.

(3) $\Rightarrow$ (1): Let $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ and suppose, by way of contradiction, that for any non-zero-divisor c, $bc \notin M(a)$. Define the set $S \subseteq A$ by

$$S = \{b^n c \mid c \text{ is a non-zero-divisor and } n = 0, 1, 2 \ldots\},$$

and note that S is multiplicatively closed. Also, $M(a) \cap S = \emptyset$ because if $b^n c \in M(a)$ for some n and some non-zero-divisor c, then $(bc)^n \in M(a)$, so that $bc \in M(a)$ since $M(a)$ is a radical ideal. Let P be a prime ideal minimal over $M(a)$ and disjoint from S. Since $M(a)$ is a z-ideal, it follows from [22, Theorem 1.1] that P is a z-ideal. Since S contains all non-zero-divisors, P is singular, and hence, by hypothesis, P is a d-ideal, and therefore $\mathrm{Ann}^2(a) \subseteq P$ as $a \in P$. Consequently, the relations $b \in \mathrm{Ann}^2(b) \subseteq \mathrm{Ann}^2(a) \subseteq P$ imply $b \in P$, which is a contradiction because $b \in S$. Therefore A is weakly regular. ∎

Below we use the fact that if A is a reduced f-ring with bounded inversion and

$$S = \{a \in A^* \mid a \text{ is invertible in A}\},$$

then $A = A^*[S^{-1}]$ (see [12, Lemma 3.4]). That is, A is the ring of fractions of its bounded part relative to the set S. A consequence of this is that ideals of A are exactly the ideals

$$I^e = \{s^{-1}u \mid s \in S \text{ and } u \in I\}$$

for I any ideal of A^*.

Corollary 3.2. *Let A be a reduced f-ring with bounded inversion.*

(a) If A is weakly regular and every prime singular z-ideal of A^* extends to a z-ideal in A, then A^* is weakly regular.

(b) If A^* is weakly regular and every prime singular z-ideal of A contracts to a z-ideal in A^*, then A is weakly regular.

Proof. (a) Let I be a singular prime z-ideal in A^*. Then of course I^e is a prime ideal in A, and it consists entirely of zero-divisors. By hypothesis, I^e is a z-ideal, and hence it is a d-ideal since A is weakly regular. By [12, Lemma 3.8], I^{ec} is a d-ideal in A^*, and by [12, Proposition 3.0], $I = I^{ec}$. Therefore A^* is weakly regular.

(b) The proof goes along the lines of that of (a). ∎

Remark 3.3. In [17, Corollary 2.6.1], Ighedo shows that every z-ideal of any ring $\mathcal{R}L$ contracts to a z-ideal of $\mathcal{R}^*L$. Thus, the (b) part of the foregoing corollary gives another reaffirmation of the fact that if βL is a weak almost P-frame then so is L since $\mathcal{R}(\beta L)$ is isomorphic to $\mathcal{R}^*L$.

We conclude by examining the position of weak regularity for f-rings vis-à-vis other weaker variants of regularity. First let us recall some terminology. Endo [14] calls a ring *quasi-regular* if its classical ring of quotients is regular. In [15, Theorem 2.2], Evans characterizes quasi-regular rings internally. He shows that A is quasi-regular if and only if for every $a \in A$ there exists $b \in A$ such that $\mathrm{Ann}^2(a) = \mathrm{Ann}(b)$ if and only if for every $a \in A$ there exists a non-zero-divisor $d \in A$ such that $a^2 = ad$. At the beginning of the paper we agreed to say a ring is almost regular if every non-zero-divisor in it is invertible.

The proposition that we shall prove shortly is motivated by what happens in function rings $\mathcal{R}L$. Recall that in frames we have the following irreversible implications. We use the abbreviations AP, WAP and CC for "almost P", "weak almost P", and "cozero complemented".

$$\text{P} \implies \text{AP} \implies \text{WAP} \qquad \text{and} \qquad \text{P} \implies \text{CC} \implies \text{WAP}.$$

Furthermore,

$$\text{CC} + \text{AP} \implies \text{P}.$$

Frames L with any of these properties have ring-theoretic characterizations. We list them below, giving reference where each characterization first appeared.

1. L is a P-frame iff $\mathcal{R}L$ is a regular ring [7].
2. L is an almost P-frame iff $\mathcal{R}L$ is an almost regular ring [10].
3. L is cozero complemented iff $\mathcal{R}L$ is a quasi-regular ring [13].

We now show that the ring analogues of the implications above hold for reduced f-rings.

Proposition 3.4. *For reduced f-rings the following implications hold.*

1. *Regularity $\Longrightarrow$ almost regularity $\Longrightarrow$ weak regularity.*
2. *Regularity $\Longrightarrow$ quasi-regularity $\Longrightarrow$ weak regularity.*
3. *Quasi-regularity $+$ almost regularity $\Longrightarrow$ regularity.*

Proof. (1) To show the first implication, suppose A is a regular ring, and let $a \in A$ be a non-zero-divisor. Pick $b \in A$ such that $a = a^2b$. Then $a(1 - ab) = 0$, and hence $ab = 1$ since a is not a divisor of zero. Therefore a is invertible, and hence A is almost regular.

For the second implication, assume A is almost regular, and suppose $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ for some $a, b \in A$. Let M be a maximal ideal of A containing a. Since M consists entirely of zero-divisors, $\mathrm{Ann}^2(u)$ is a proper ideal of A for every $u \in M$. Indeed, if 1 were in $\mathrm{Ann}^2(u)$ we would have $\mathrm{Ann}(u) = 0$, whence u would be a non-zero-divisor. Let $u, v \in M$, and suppose $w \in \mathrm{Ann}(u^2 + v^2)$. Then $(wu)^2 + (wv)^2 = 0$, and hence $(wu)^2 = 0$, which implies $wu = 0$ since A is reduced. Thus, $\mathrm{Ann}(u^2 + v^2) \subseteq \mathrm{Ann}(u)$, which implies $\mathrm{Ann}^2(u) \subseteq \mathrm{Ann}^2(u^2 + v^2)$. Therefore the set

$$K = \bigcup \{\mathrm{Ann}^2(x) \mid x \in M\}$$

is a directed union of proper d-ideals of A, and is therefore a proper d-ideal with $M \subseteq K$, and therefore $M = K$. Thus, M is a d-ideal. Since $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$, we have $\mathrm{Ann}(a) = \mathrm{Ann}(a^2 + b^2)$, which implies $a^2 + b^2 \in M$ because $a \in M$ and M is a d-ideal. This implies $b^2 \in M$, and hence $b \in M$. Because M is an arbitrary maximal ideal containing a, it follows that $b \in M(a)$. So choosing $c = 1$, we have that c is a non-zero-divisor with $bc \in M(a)$. Therefore A is weakly regular.

(2) Only the second implication need be shown. Assume that A is quasi-regular, and suppose $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ for some $a, b \in A$. By [15, Theorem 2.2], there is a non-zero-divisor c such that $a^2 = ac$. Then $a(a - c) = 0$, so that $b(a - c) = 0$ since $a - c \in \mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$. So, $bc = ba$, and hence bc is in every ideal that contains a. Therefore $bc \in M(a)$, showing that A is weakly regular.

(3) Let $a \in A$. Again by Evans' result, $a^2 = ac$ for some non-zero-divisor c, which is then invertible since A is almost regular. Thus $a = a^2c^{-1}$, and therefore A is regular. ∎

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