A class of special subsets of a BCK-algebra

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Abstract

In this paper, we consider a class of special subsets of a BCK-algebra and investigate some related properties. We describe the intersection and union of two of these subsets in a commutative BCK-lattice. We consider the intersection of all special subsets in a BCK-algebra and investigate some related properties. Finally, we introduce the connection between the class of special subsets and the set of all congruence classes induced by an ideal in a BCK-algebra.

1 Introduction

In 1966, the notion of BCK-algebra was introduced by Imai and Iséki, as a generalization of the concept of set-theoretic difference and propositional calculus (see [4]). Since then a great deal of literature has been produced on the theory of BCK-algebras. In 1975, S. Tanaka defined a notion of a special class of BCK-algebras called a commutative BCK-algebra (see [10]). The ideal theory plays an important role. In BCK-algebra, the commutative ideal is closely related to the commutative BCK-algebra. In 2009, a class of special subset connected with an order filter of a MV-algebra was defined and studied by Colin G. Bailey (see [2]). In this paper, following [2], we consider a class of special subsets of a BCK-algebra and investigate some related properties. We describe the intersection and union of two of these subsets in a commutative BCK-lattice. We consider the intersection of all special subsets in a BCK-algebra and investigate some related properties. Also, we state and prove a characterization of this intersection in a BCK-algebra satisfying condition (S). Finally, for any complement-closed

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ideal D we prove that there is a close connection between the class of special subsets of D and the set of all congruence classes induced by D in a BCK-algebra.

2 Preliminaries

We first recall some basic definitions and theorems used in this paper.

Definition 2.1. *[8] A BCK-algebra is an algebra $(X;*,0)$ of type $(2,0)$ such that, for all $x,y,z \in X$, we have:*
(i) $((x*y)*(x*z))*(z*y) = 0$;
(ii) $(x*(x*y))*y = 0$;
(iii) $x*x = 0$;
(iv) $x*y = 0$ *and* $y*x = 0$ *imply* $x = y$;
(v) $0*x = 0$.

For every BCK-algebra $X = (X;*,0)$, the order $\leq$ defined by "$x \leq y$ if and only if $x*y = 0$" is a partial order (called the BCK-ordering).

Theorem 2.2. *[8] Let $X = (X;*,0)$ be a BCK-algebra. Then the following hold: for any $x,y,z \in X$,*
$(a1)$ $(x*y)*(x*z) \leq z*y$;
$(a2)$ $(x*z)*(y*z) \leq x*y$;
$(a3)$ $0 \leq x$;
$(a4)$ $x*0 = x$;
$(a5)$ $x \leq x$;
$(a6)$ $x*y \leq x$;
$(a7)$ $(x*y)*z = (x*z)*y$;
$(a8)$ $x \leq y$ *implies* $x*z \leq y*z$ *and* $z*y \leq z*x$.

Definition 2.3. *[6, 7] Let $X = (X;*,0)$ be a BCK-algebra. Then*
(i) *X is said to be commutative if $x*(x*y) = y*(y*x)$ for all $x,y \in X$;*
(ii) *X is said to be with condition (S) if the set $A(x,y) := \{t \in X | t*x \leq y\}$ has the greatest element for all $x,y \in X$. This element is denoted by $x \oplus y$. Clearly, $x,y \leq x \oplus y$;*
(iii) *a non-empty subset I of X is called an ideal if*
(1) $0 \in I$,
(2) $y*x \in I$ *and* $x \in I$ *imply* $y \in I$.

In any BCK-algebra X, we will denote the infimum of x and y by $x \wedge y$ and the supremum of x and y by $x \vee y$ for all $x,y \in X$.

Definition 2.4. *[12] An ideal I of a BCK-algebra X is called prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for all $x,y \in X$.*

Definition 2.5. *[3] A partial ordered set P is said to be lattice if any two whose elements x,y have a g.l.b. denoted by $x \wedge y$, and a l.u.b. denoted by $x \vee y$.*

Definition 2.6. *[12] A BCK-algebra X is called a BCK-lattice if it with respect to its BCK-ordering forms a lattice.*

A BCK-algebra X is called *bounded* if it has the greatest element (denoted by 1). For any $x \in X$, we denote $1 * x$ by Nx. The next theorem gives some properties about Nx.

Theorem 2.7. *[8] In any bounded commutative BCK-lattice X, the following hold:*
(a) $NNx = x$ *for all* $x \in X$,
(b) $Nx * Ny = y * x$ *for all* $x, y \in X$,
(c) $Nx \vee Ny = N(x \wedge y)$ *and* $Nx \wedge Ny = N(x \vee y)$ *for all* $x, y \in X$.

Theorem 2.8. *[12] Every bounded commutative BCK-algebra is a commutative BCK-lattice with* $x \wedge y = y * (y * x)$ *and* $x \vee y = N(Nx \wedge Ny)$.

Theorem 2.9. *[12] Let X be a commutative BCK-lattice. Then the following identities hold: for any* $x, y, z \in X$,
$(b1)$ $x * (y \vee z) = (x * y) \wedge (x * z)$,
$(b2)$ $x * (y \wedge z) = (x * y) \vee (x * z)$,
$(b3)$ $(x \vee y) * z = (x * z) \vee (y * z)$.

Theorem 2.10. *[8] Let I be an ideal of a BCK-algebra X. Define the relation* $\equiv_I$ *on X as follows:*

$$x \equiv_I y \text{ if and only } x * y \in I \text{ and } y * x \in I.$$

Then $\equiv_I$ *is a congruence relation on X.*

The congruence relation $\equiv_I$ defined in Theorem 2.10 is called the *ideal congruence* on X induced by the ideal I. The congruence class of element $x \in X$ is denoted by $[x]_I$, and the set of all congruence classes $\equiv_I$ on X is denoted by X/I. Hence

$$X/I = \{[x]_I : x \in X\}.$$

Theorem 2.11. *[8] Let I be an ideal of a BCK-algebra* $(X; *, 0)$ *and let* $\equiv_I$ *be the ideal congruence on X induced by I. Then* $(X/I; *, [0]_I)$ *is also a BCK-algebra where its operation is the natural operation induced from those X, i.e.,* $[x]_I * [y]_I = [x * y]_I$ *for all* $x, y \in X$.

3 Main results

In the sequence, for every subset D of X, we denote the complement of D by D^c. Now, we begin with definition of a special subset of a BCK-algebra.

Definition 3.1. *For any non-empty subset D of a BCK-algebra X and for any* $a \in X$, *we denote*

$$D_a := \begin{cases} D & \text{if } a \in D \\ \{x \in X | a * x \notin D\} & \text{if } a \in D^c. \end{cases}$$

Proposition 3.2. *If D is a non-empty subset of a BCK-algebra X, then we have*
(i) $0 \in D_a$ *for any* $a \in X$;
(ii) $0 \in D$ *if and only if* $a \notin D_a$ *for all* $0 \neq a \in D^c$.

Proof. (i) If $D = X$, then by Definition 3.1, $D_a = X$ and so $0 \in D_a$ for any $a \in X$. Now, let $D \neq X$. Then there exists $a \in X$ such that $a \notin D$ and so by Theorem 2.2$(a4)$, we get $a * 0 \notin D$. Hence by Definition 3.1, we conclude $0 \in D_a$.

(ii) Using Definition 2.1(iii), for all $0 \neq a \in D^c$ we have

$$0 \in D \text{ if and only if } a * a \in D \text{ if and only if } a \notin D_a.$$

This completes the proof. ■

Definition 3.3. *A non-empty subset D of a BCK-algebra X is called*

(i) *a down-set if for all* $x, y \in X$, $x \leq y$ *and* $y \in D$ *imply* $x \in D$;

(ii) *a prime-set if for all* $x, y \in X$, $x \wedge y \in D$ *implies* $x \in D$ *or* $y \in D$;

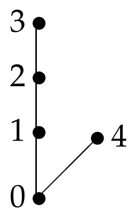
(iii) *a* $\vee$*-closed if for all* $x, y \in D$, $x \vee y \in D$.

Note that every ideal of a *BCK*-algebra is a down-set and every prime ideal is a prime-set.

The following example shows that any one of the properties down-set, prime-set and $\vee$-closed is independent of the others.

Example 3.4. *[12] Consider a BCK-algebra* $(X = \{0, 1, 2, 3, 4\}; *, 0)$ *in which the operation "* $*$ *" is given by the following table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0



It easy to check that:

(i) $D := \{0, 1, 4\}$ *is a down-set and prime-set but is not a* $\vee$*-closed because* $1, 4 \in D$ *and* $1 \vee 4$ *does not exist.*

(ii) $E := \{0, 1\}$ *is a down-set and* $\vee$*-closed but is not a prime-set because* $2 \wedge 4 = 0 \in E$ *does not imply* $2 \in E$ *or* $4 \in E$.

(iii) $F := \{1, 2\}$ *is a prime-set and* $\vee$*-closed but is not a down-set because* $0 \leq 1 \in F$ *and* $0 \notin F$.

Lemma 3.5. *Let D be a non-empty subset of a BCK-algebra X. If D is a down-set, then for any* $a \in X$, D_a *is a down-set, too.*

Proof. Let $x, y \in X$ such that $x \leq y$ and $y \in D_a$. Then $a * y \notin D$. Applying Theorem 2.2(a_8), it follows from $x \leq y$ that $a * y \leq a * x$. Now, if $a * x \in D$, then, since D is a down-set, we get $a * y \in D$, which a contradiction to $a * y \notin D$. Hence $a * x \notin D$ and so $x \in D_a$. Therefore D_a is a down-set. ■

Lemma 3.6. *Let D be a non-empty subset of a commutative BCK-algebra X. Then for all* $a \in D^c$ *the following hold:*

(i) If D is a $\vee$*-closed, then* D_a *is a prime-set;*

(ii) If D is a prime-set, then D_a *is a* $\vee$*-closed.*

Proof. (i) Let $x, y \in X$ such that $x \wedge y \in D_a$. Hence $a * (x \wedge y) \notin D$ and so by Theorem 2.9(b_2), we get $(a * x) \vee (a * y) \notin D$. It follows from D is a $\vee$-closed that $a * x \notin D$ or $a * y \notin D$. Therefore $x \in D_a$ or $y \in D_a$ and so D_a is a prime-set.

(ii) Let $x, y \in D_a$. Then $a * x \notin D$ and $a * y \notin D$. Since D is a prime-set, we get $(a * x) \wedge (a * y) \notin D$. Applying Theorem 2.9($b_1$), we obtain $a * (x \vee y) \notin D$. This implies that $x \vee y \in D_a$. Therefore D_a is a $\vee$-closed. ■

Proposition 3.7. *Let X be a BCK-algebra and $\emptyset \neq D \subseteq X$. Then*

(i) if D is a down-set of X, then we have

$$(\forall a, b \in D^c) \quad a \leq b \Rightarrow D_a \subseteq D_b; \tag{3.1}$$

(ii) if D is an ideal of X, then we have

$$(\forall a, b \in X) \quad a \leq b \Rightarrow D_a \subseteq D_b. \tag{3.2}$$

Proof. (i) Let D be a down-set of X, and let $a, b \in D^c$ such that $a \leq b$. Assume that $x \in D_a$. Then $a * x \notin D$. Applying Theorem 2.2(a_8), it follows from $a \leq b$ that $a * x \leq b * x$. If $b * x \in D$, then, since D is a down-set, we get $a * x \in D$, which a contradiction. Hence $b * x \notin D$ and so $x \in D_b$. Therefore $D_a \subseteq D_b$.

(ii) Let D is an ideal of X. By (i), since D is a down-set, we only need to investigate the following cases:

(1) $b \in D$. In this case, from $a \leq b$ we get $a \in D$ and so $D_a = D = D_b$.

(2) $b \notin D$ and $a \in D$. In this case, we have $D_a = D$. Now, let $x \in D_a$. If $b * x \in D$, then since D is an ideal and $x \in D$, we get $b \in D$, which a contradiction. Hence $b * x \notin D$ and so $x \in D_b$. Therefore $D_a \subseteq D_b$. ■

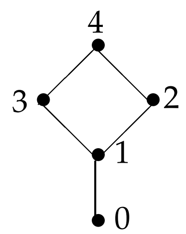
Corollary 3.8. *If D is an ideal of a BCK-algebra X, then for all $a \in X$, $D \subseteq D_a$.*

Proof. From $0 \leq a$ and $D_0 = D$ the result holds by Proposition 3.7(ii). ■

The following example shows that the condition ideal in the above Proposition part (ii) is necessary.

Example 3.9. *[12] Consider a BCK-algebra $(X = \{0, 1, 2, 3, 4\}; *, 0)$ in which the operation " $*$ " is given by the following table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	3	0



We can see that $D := \{0, 1\}$ is a down-set but is not an ideal of X. Simply calculation, we obtain $D_2 = \{0\}$ and $0 \leq 2$ but $D_0 = D \not\subseteq D_2$.

Proposition 3.10. *Let X be a bounded commutative BCK-lattice and let $\emptyset \neq D \subseteq X$ is a $\vee$-closed. Then* *(i) if D is a down-set, then*

$$D_a \cup D_b = D_{a\vee b} \text{ for all } a, b \in D^c \text{ or } a, b \in D;$$

(ii) if D is an ideal, then

$$D_a \cup D_b = D_{a\vee b} \text{ for all } a, b \in X.$$

Proof. (i) Assume that D is a down-set and investigate the following cases:

(1) $a, b \in D^c$. Since D is a down-set, it follows from $a \leq a \vee b$ and $a \in D^c$ that $a \vee b \in D^c$. Hence by Proposition 3.7(i), we get $D_a \subseteq D_{a\vee b}$. Similarly, we have $D_b \subseteq D_{a\vee b}$ and so $D_a \cup D_b \subseteq D_{a\vee b}$. In order to show the inverse inclusion, assume that $x \in D_{a\vee b}$. Then $(a \vee b) * x \notin D$ and so by Theorem 2.9(b_3), we get $(a * x) \vee (b * x) \notin D$. It follows from D is a $\vee$-closed that $a * x \notin D$ or $b * x \notin D$. Thus $x \in D_a$ or $x \in D_b$ and so $x \in D_a \cup D_b$. Therefore $D_a \cup D_b = D_{a\vee b}$.

(2) $a, b \in D$. Since D is a $\vee$-closed, we have $a \vee b \in D$ and so by Definition 3.1, we have $D_a \cup D_b = D = D_{a\vee b}$.

(ii) Assume that D is an ideal. By (i) and the commutativity of $\cup$ and $\vee$, since D is a down-set, we only need to investigate the case $a \in D$ and $b \in D^c$. In this case, using Proposition 3.7(ii), from $a, b \leq a \vee b$, we obtain $D_a \cup D_b \subseteq D_{a\vee b}$. In order to show the inverse inclusion, let $x \in D_{a\vee b}$. Similar to the argument of (i), we obtain $a * x \notin D$ or $b * x \notin D$. Hence $x \in D_b$ and so $D_{a\vee b} \subseteq D_a \cup D_b$. Therefore $D_a \cup D_b = D_{a\vee b}$. ■

In order to describe the intersection, similar to the above proposition, we need the following lemma.

Lemma 3.11. *Let X be a bounded commutative BCK-lattice. Then, we have*

$$(\forall a, b, x \in X) \quad (a \wedge b) * x = (a * x) \wedge (b * x).$$

Proof. Using Theorems 2.7 and 2.9, we get

$$\begin{aligned} (a \wedge b) * x &= Nx * N(a \wedge b) \\ &= Nx * (Na \vee Nb) \\ &= (Nx * Na) \wedge (Nx * Nb) \\ &= (a * x) \wedge (b * x). \end{aligned}$$ ■

Proposition 3.12. *Let X be a commutative BCK-lattice and let $\emptyset \neq D \subseteq X$. Then*

(i) if D is a down-set and prime-set, then

$$D_a \cap D_b = D_{a\wedge b} \text{ for all } a, b \in D^c \text{ or } a, b \in D;$$

(ii) if D is a prime ideal, then

$$D_a \cap D_b = D_{a\wedge b} \text{ for all } a, b \in X.$$

Proof. (i) Assume that D is a down-set and prime-set and investigate the following cases:

(1) $a, b \in D^c$. In this case, since D is a prime-set, it follows from $a, b \in D^c$ that $a \wedge b \in D^c$. Using Proposition 3.7(i) and the fact that $a \wedge b \leq a, b$, we conclude $D_{a\wedge b} \subseteq D_a \cap D_b$. Now, let $x \in D_a \cap D_b$. Then $a * x \in D^c$ and $b * x \in D^c$ and so it follows from D is a prime-set that $(a * x) \wedge (b * x) \in D^c$. Hence by Lemma 3.11, we obtain $(a \wedge b) * x \in D^c$. This implies $x \in D_{a\wedge b}$. Therefore $D_a \cap D_b = D_{a\wedge b}$.

(2) $a, b \in D$. Since D is a down-set, from $a \wedge b \leq a, b$, we conclude $a \wedge b \in D$ and so by Corollary 3.8 we have $D_a \cap D_b = D = D_{a\wedge b}$.

(ii) Assume that D is a prime ideal. By (i), since D is a down-set, we only need to investigate the case $a \in D$ and $b \in D^c$. In this case, by Proposition 3.7(ii), from $a \wedge b \leq a, b$ we obtain $D_{a\wedge b} \subseteq D_a \cap D_b$. For the reverse inclusion, note that $a \in D$ gives $D_a = D$ and $a \wedge b \leq a$ in D implies $a \wedge b \in D$ so $D_{a\wedge b} = D$. Thus, $D_a \cap D_b = D \cap D_b \subseteq D = D_{a\wedge b}$. Therefore $D_a \cap D_b = D_{a\wedge b}$. ∎

Proposition 3.13. *Let D be a non-empty subset of a bounded commutative BCK-algebra X. If D is a down-set and prime-set, then*

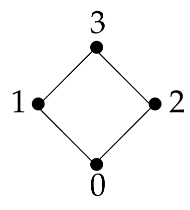
$$(\forall a \in X) \quad (D_a)_a = D.$$

Proof. If $a \in D$, then, since $D_a = D$, we get $(D_a)_a = D$. Now, assume that $a \in D^c$. Since D is a non-empty down-set, $0 \in D$. It follows from $a * a = 0 \in D$ that $a \in (D_a)^c$. Let $x \in (D_a)_a$. Then $a * x \in (D_a)^c$. From this follows that $a * (a * x) \in D$. Then by commutativity of X, we get $a \wedge x \in D$. Hence, since D is a prime-set and $a \notin D$, we conclude $x \in D$. Therefore $(D_a)_a \subseteq D$. In order to show the inverse inclusion, let $x \in D$. Since $a * (a * x) = a \wedge x \leq x$ and D is a down-set, we obtain $a * (a * x) \in D$. From this follows that $a * x \in (D_a)^c$, that is, $x \in (D_a)_a$. Therefore $D \subseteq (D_a)_a$ and so $(D_a)_a = D$. ∎

We next give an example of a bounded commutative BCK-lattice satisfying the Propositions 3.12 and 3.13

Example 3.14. *[12] Let $(X = \{0, 1, 2, 3\}; *, 0)$ be a bounded commutative BCK-lattice in which the operation " $*$ " is given by the following table:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0



We can see that $D := \{0, 1\}$ is a prime ideal satisfying the property $\vee$-closed and $D_2 = D_3 = D$. It is clear that

$$(\forall a, b \in X) \quad D_a \cup D_b = D_{a\vee b}\,, D_a \cap D_b = D_{a\wedge b} \text{ and } (D_a)_a = D$$

Definition 3.15. *For any non-empty subset D of a BCK-algebra X, we denote*

$$\Gamma(D) := \{x \in X | a * x \in D^c, \forall a \in D^c\}.$$

Proposition 3.16. *Let D be a non-empty sunset of a BCK-algebra X. Then the following statements hold:*

(i) $\Gamma(D) = \bigcap\limits_{a\in D^c} D_a$.

(ii) $0 \in D$ *if and only if* $\Gamma(D) \subseteq D$.

Proof. (i) The proof is clear by Definition 3.15.

(ii) Let $0 \in D$. Assume to the contrary that $\Gamma(D) \nsubseteq D$. Then there exists $x \in \Gamma(D)$ such that $x \in D^c$. Hence it follows from $x \in \Gamma(D)$ that $x \in D_x$, that is, $x * x \notin D$. Thus $0 \notin D$, which a contradiction. Therefore $\Gamma(D) \subseteq D$.

Conversely, let $\Gamma(D) \subseteq D$. Then by Theorem 3.2(i), the proof is straightforward. ∎

In the following theorem, we give a condition for a subset of a BCK-algebra to be an ideal.

Theorem 3.17. *Let D be a subset of a BCK-algebra X. Then the following are equivalent:*

(i) *D is an ideal of X;*

(ii) $0 \in D$ *and* $D \subseteq \Gamma(D)$*;*

(iii) $0 \in D$ *and* $D = \Gamma(D)$.

Proof. $(i) \Rightarrow (ii)$ Let D be an ideal of X and $x \in D$. If $x \notin D_a$ for some $a \in D^c$, then $a * x \in D$. Thus, since D is an ideal and $x \in D$, we get $a \in D$, which a contradiction. Therefore $x \in D_a$ and so $D \subseteq D_a$ for any $a \in D^c$. Therefore $D \subseteq \Gamma(D)$.

$(ii) \Rightarrow (iii)$ By hypothesis, it suffices to show that $\Gamma(D) \subseteq D$. If not, then there exists $b \in \Gamma(D)$ such that $b \in D^c$. It follows from $b \in \Gamma(D)$ that $b \in D_b$ and so $b * b \in D^c$, that is, $0 \notin D$, which a contradiction. Therefore $D = \Gamma(D)$.

$(iii) \Rightarrow (i)$ Assume to the contrary that D is not an ideal of X. Then there exist $x, y \in X$ such that $y * x \in D$ and $x \in D$ but $y \notin D$. Since $x \in D \subseteq \Gamma(D)$, we get $x \in D_y$. This implies $y * x \in D^c$, which a contradiction. Therefore D is an ideal of X. ∎

The next theorem gives a characterization of $\Gamma(D)$ in a BCK-algebra with condition (S).

Theorem 3.18. *Let X be a BCK-algebra with condition (S) and D be a non-empty downset of X. Then*

$$\Gamma(D) = \{x \in X | x \oplus d \in D, \forall d \in D\}.$$

Proof. Using Proposition 3.16(i) and condition (S), we have

$$\begin{aligned}
x \in \Gamma(D) &\Leftrightarrow x \in \bigcap_{a\in D^c} D_a \\
&\Leftrightarrow (\forall a \in D^c)\ x \in D_a \\
&\Leftrightarrow (\forall a \in D^c)\ a * x \notin D \\
&\Leftrightarrow (\forall a \in D^c)\ (\forall d \in D)\ a * x \nleq d \\
&\Leftrightarrow (\forall a \in D^c)\ (\forall d \in D)\ a \nleq x \oplus d \\
&\Leftrightarrow (\forall d \in D)\ x \oplus d \in D \\
&\Leftrightarrow x \in \{x \in X | x \oplus d \in D, \forall d \in D\}.
\end{aligned}$$

Therefore $\Gamma(D) = \{x \in X | x \oplus d \in D, \forall d \in D\}$. ■

We next describe the connection between the special subsets and the congruence classes of a *BCK*-algebra. For this purpose, we need the following definition.

Definition 3.19. *A non-empty subset D of a BCK-algebra X is called a complement-closed if*

$$(\forall a \in D^c)\ (\exists x \in D^c)\ a * x \in D^c.$$

Lemma 3.20. *For any ideal D of a BCK-algebra X, the following are equivalent:*
(i) *D is a complement-closed;*
(ii) *For any $a \in D^c$, $D_a \neq D$.*

Proof. $(i) \Rightarrow (ii)$ Let $a \in D^c$. Then by Definition 3.19, there exists $x \in D^c$ such that $a * x \in D^c$. It follows from $a * x \in D^c$ that $x \in D_a$. But $x \notin D$, hence $D_a \neq D$.

$(ii) \Rightarrow (i)$ Let $a \in D^c$. Then by (ii), we have $D_a \neq D$. Then it follows from Corollary 3.8 that $D \subsetneq D_a$. Hence there exists $x \in D_a$ such that $x \in D^c$. Thus from $x \in D_a$, we conclude $a * x \in D^c$. Therefore D is a complement-closed. ■

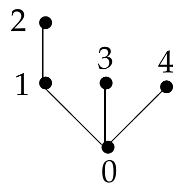
Corollary 3.21. *Let D be a complement-closed ideal of a BCK-algebra X. Then*

$$(\forall a, b \in X) \quad D_a = D_b \Rightarrow a, b \in D \text{ or } a, b \in D^c.$$

Proof. Using Lemma 3.20 and Definition 3.1, the proof is straightforward. ■

Example 3.22. *Let $(X = \{0, 1, 2, 3, 4\}; *, 0)$ be a BCK-algebra in which the operation "$*$" is given by the following table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0



Consider the ideals $D := \{0, 4\}$ and $E := \{0, 3, 4\}$. We can see that:
(i) $D_1 = \{0, 3, 4\}$, $D_2 = \{0, 1, 3, 4\}$ *and* $D_3 = \{0, 1, 2, 4\}$. *Hence D is a complement-closed;*
(ii) $E_1 = E$ *and so E is not a complement-closed.*

Theorem 3.23. *If D be an ideal of a BCK-algebra X, then we have*

$$(\forall a, b \in D^c) \quad D_a \subseteq D_b \ \Leftrightarrow a * b \in D.$$

Proof. $(\Rightarrow)$ Let $D_a \subseteq D_b$ for some $a, b \in D^c$. Clearly $b \neq 0$. By Proposition 3.2(ii), we have $b \notin D_b$. From this follows that $b \notin D_a$. Therefore $a * b \in D$.

$(\Leftarrow)$ Let $a * b \in D$ for some $a, b \in D^c$. Suppose that $x \in D_a$. Then $a * x \notin D$. Assume to the contrary that $b * x \in D$. By Theorem 2.2(a_2), we have $(a * x) * (b * x) \leq a * b$. Since D is an ideal, it follows from $a * b \in D$ that $(a * x) * (b * x) \in D$. Then from $b * x \in D$, we get $a * x \in D$, which a contradiction. Hence $b * x \notin D$ and so $x \in D_b$. Therefore $D_a \subseteq D_b$. ■

Corollary 3.24. *For any ideal D of a BCK-algebra X, we have*

$$(\forall a, b \in D^c) \quad D_a = D_b \; \Leftrightarrow a * b \in D \text{ and } b * a \in D.$$

Proof. This is an immediate consequence from Theorem 3.23. ∎

Notation 3.25. *For any non-empty subset D of a BCK-algebra X, we denote*

$$L(D) := \{D_a : a \in X\}.$$

It is clear that $D \in L(D)$. Now, we give a characterization of $L(D)$.

Theorem 3.26. *Let D be a complement-closed ideal of a BCK-algebra X. Define the operation " $*$ " on $L(D)$ by*

$$(\forall a, b \in X) \quad D_a * D_b = D_{a*b}. \tag{3.3}$$

Then $(L(D); *, D)$ is a BCK-algebra.

Proof. We first show that the operation " $*$ " is well-defined. For this purpose, we assume that $D_a = D_c$ and $D_b = D_d$ for some $a, b, c, d \in X$. We note that by Corollary 3.21, $D_a = D_c$ implies that $a, c \in D^c$ or $a, c \in D$. Similarly, we have $b, d \in D^c$ or $b, d \in D$. Hence we only need to investigate the following cases:

(i) $a, c \in D$ and $b, d \in D$. In this case, since D is an ideal, we get $a * b \in D$ and $c * d \in D$ by Theorem 2.2($a6$). Therefore we have

$$D_a * D_b = D_{a*b} = D = D_{c*d} = D_c * D_d.$$

(ii) $a, c \in D$ and $b, d \in D^c$. In this case, from $a * b \leq a$ and $c * d \leq c$, we get $a * b \in D$ and $c * d \in D$ and so similar to (i), the result holds.

(iii) $a, c \in D^c$ and $b, d \in D$. In this case, since $b \in D$ and $a \notin D$, it follows from D is an ideal that $a * b \in D^c$. Similarly, we can show that $c * d \in D^c$. By Corollary 3.24, from $a, c \in D^c$ and $D_a = D_c$, we obtain

$$a * c \in D \text{ and } c * a \in D. \tag{3.4}$$

Moreover, it follows from $b, d \in D$ that

$$b * d \in D \text{ and } d * b \in D. \tag{3.5}$$

Now, we have

$$\begin{aligned} ((a * b) * (c * d)) * (d * b) &= ((a * b) * (d * b)) * (c * d) \text{ by Theorem 2.2}(a7) \\ &\leq (a * d) * (c * d) \text{ by Theorem 2.2}(a2) \text{ and } (a8) \\ &\leq a * c \text{ by Theorem 2.2(a2)}. \end{aligned}$$

Then, since $a * c \in D$, we get $((a * b) * (c * d)) * (d * b) \in D$ and so from $d * b \in D$, we obtain

$$(a * b) * (c * d) \in D. \tag{3.6}$$

By the similar argument, we can show that

$$(c * d) * (a * b) \in D. \tag{3.7}$$

Applying Corollary 3.24, from (3.6) and (3.7), we obtain $D_{a*b} = D_{c*d}$, that is, $D_a * D_b = D_c * D_d$.

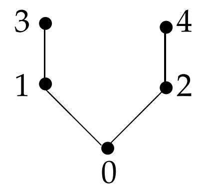
(iv) $a, c \in D^c$ and $b, d \in D^c$. The proof of this case is similar to the proof of (iii) by some modification.

Summarizing the above, we conclude that the operation " $*$ " on $L(D)$ is well-defined. Now, we define the mapping $\varphi : L(D) \to X/D$ by $\varphi(D_a) = [a]_D$, where X/D is the quotient BCK-algebra induced by ideal D. In order to show that φ is well-defined, assume that $D_a = D_b$ for some $a, b \in X$. Similar to the above argument, we get $a, b \in D$ or $a, b \in D^c$. If $a, b \in D$, then clearly, $[a]_D = [b]_D$ and so $\varphi(D_a) = \varphi(D_b)$. If $a, b \in D^c$, then by Corollary 3.24, we have $a * b \in D$ and $b * a \in D$, which implies that $[a]_D = [b]_D$ and so $\varphi(D_a) = \varphi(D_b)$. Hence φ is well-defined. Now, let $\varphi(D_a) = \varphi(D_b)$. Then $[a]_D = [b]_D$ and so by the property of congruence class, we obtain $a * b \in D$ and $b * a \in D$. Hence by Corollary 3.24, we get $D_a = D_b$. Therefore φ is injective. Obviously, φ is onto. Therefore φ is a bijective function which preserving the operation " $*$ " on $L(D)$. Then, since X/D is a BCK-algebra, φ induces a BCK-algebra structure on $(L(D); *, D)$. Therefore $(L(D); *, D)$ is a BCK-algebra. ■

We now give an example to illustrate the above theorem.

Example 3.27. *[12] Let $(X = \{0, 1, 2, 3, 4\}; *, 0)$ be a BCK-algebra in which the operation " $*$ " is given by the following table:*

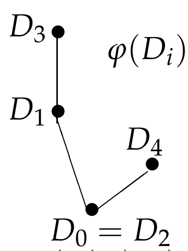
$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0



Consider the ideal $D := \{0, 2\}$. We can check that D is a complement-closed. Simply calculation we obtain:

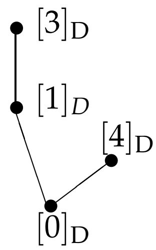
$$L(D) = \{D_0 = D_2, D_1 = \{0, 2, 4\}, D_3 = \{0, 1, 2, 4\}, D_4 = \{0, 1, 2, 3\}\};$$

$$X/D = \{[0]_D, [1]_D, [3]_D, [4]_D\}; (L(D); *) \cong (X/D; *)$$



$(L(D); *)$

$\varphi(D_i) = [i]_D; i = 0, 1, 3, 4$



$(X/D; *)$

In the following example, we show that the condition complement- closed in Theorem 3.26 is necessary.

Example 3.28. *Let $X = \{0,1,2,3,4\}$ be a BCK-algebra as in Example 3.22. Consider the ideal $E = \{0,3,4\}$. E is not a complement-closed. Simplifying calculation we obtain:*

$$\mid L(E) \mid = \mid \{E = E_1, E_2 = \{0,1,3,4\}\} \mid = 2, \quad \mid X/E \mid = \mid \{[0]_D, [1]_D, [2]_D\} \mid = 3.$$

From this follows that $L(D) \ncong D/X$.

Conclusion and Future Work

In this work, we investigate the properties of class of special subsets in a *BCK*-algebra. Defining an operation on this class, we proved that it is isomorphic to the set of all congruence classes induced by an ideal.

Our future works are to study the class of special subsets in some logic algebraic structures such as *BCI*-algebra, *BL*-algebra, *MV*-algebra, residuated lattice, etc.

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