Expanding maps on the circle and geodesic laminations

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Abstract

In this article we consider a family of orientation preserving expanding maps on the circle. We associate to each member of this family a geodesic lamination on the disc, endowed with a transversal measure. The map on the circle induces an expanding dynamical systems on the lamination. We explore relations between the geometry of the lamination and the symbolic dynamics of the circle map. We present a list of open problems.

1 Introduction

Expanding maps on the circle and on the interval has been studied extensively in dynamical systems, since they exhibit important dynamical and ergodic properties (*cf.* [11]). On the other hand geodesic laminations on the disc have been used in different areas of mathematics: complex dynamics [10, 16], surface automorphism dynamics [6], hyperbolic geometry [5] and symbolic dynamics [12, 13]. In [4, 9, 10, 16], and references within, geodesic laminations on the disc were associated to the angle doubling map on the circle as model of Julia sets.

In the present article we consider a family of expanding maps on the circle, these maps are orientation preserving and of degree k, with $k \geq 3$. This family of maps is defined in Section 2. In Section 3, we associate a geodesic lamination on the disc, for each map of the family. We define a transversal measure. The expanding map on the circle induces an expanding dynamical system on the lamination, see Theorem 3.1. Moreover, we show relations between the geometry of

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the lamination and the dynamics of the expanding map on the circle, in particular to its symbolic dynamics.

In [12, 14, 15] the author considered a family of fractals that can be described in term of space-filling curves, which can be modelled using geodesic laminations on the disc. These constructions are related to expanding maps on the circle. These maps belong to the family of expanding maps considered in the present article. Unlike those maps, in the present article we do not require the existence of a space-filling curve. Therefore the results of the present paper are in a more general setting than the one considered in [12, 14]. Some of the techniques used in the present paper are based in techniques considered by the author in [12, 14].

In Section 4, we consider some examples. We end the present article with Section 5, where we present a list of open problems.

2 Expanding maps on the circle

Let $I = [0, 1)$ be the unit interval, we define a map $f : I \to I$ in the following way:

1. Let $\{I_1, \ldots, I_k\}$ be a partition of I, where I_j are half-open intervals, closed to the left.
2. For $1 \leq j \leq k$, let $\phi_j : I_j \to I$ be C^1 surjective maps on the closure of I_j with $\phi_j'(t) > 1$, for all $t \in \overline{I_j}$.
3. For a fixed $\theta \in (0, 1)$, we define $R_\theta : I \to I$ as $R_\theta(t) := t + \theta \pmod 1$.
4. Let $f(t) := R_\theta \circ \phi_j(t)$, if $t \in I_j$.

We call this class of maps $f : I \to I$, *regular expanding map on* I. A special and important case of these maps is when all the ϕ_j are linear maps, in this case, we say that f is a *regular expanding linear map on* I. These maps were introduced in [14].

We remark, that if we identify $\mathbb{S}^1$ with $I = [0, 1) \pmod 1$, then f defines a continuous orientation preserving map of degree k on $\mathbb{S}^1$.

The inverse branches of the map f are $h_j : I \to I_j$, defined as $h_j := \phi_j^{-1} \circ R_\theta^{-1} = \phi_j^{-1} \circ R_{1-\theta}$, for $1 \leq j \leq k$. The collection of maps $\{h_1, \ldots, h_k\}$ is called the *system of branches of the inverse function of* f. Let $\nu : I \to \{1, \ldots, k\}$, where $\nu(t) = i$ if $t \in I_i$. The itinerary of t is the infinite sequence $a_1 a_2 \ldots$, where $a_j = \nu(f^{j-1}(t))$, for all $j \geq 1$. By definition it has the property $t \in \cap_{n=1}^{\infty} h_{a_1} \circ \cdots \circ h_{a_n}(I)$. Let us remark that the itinerary of a point is well-defined since the intervals I_i are half-open. However there might be more than one point with the same itinerary, since the maps h_j are not continuous as a map from I to I_j. The lack of continuity of the maps h_i-s plays an important role in the construction of the lamination described in Section 3.

We say that a point t is *periodic* or f*-periodic* if there exists a positive integer m, such that $f^m(t) = t$. In this case the itinerary of t is of the form: $a_1 \ldots a_m a_1 \ldots a_m$ $a_1 \ldots a_m \ldots$. We denote this itinerary by $\overline{a_1 \cdots a_m}$. We say that t is *pre-periodic* or f*-pre-periodic*, if t is not periodic and there exists a positive integer l such that

$f^l(t)$ is periodic. In this case the itinerary of t is of the form $a_1 \dots a_l \overline{a_{l+1} \cdots a_{l+m}}$, for some positive integer m.

Let the intervals I_j be of the form $[t_j, t_{j+1})$, and $\alpha_j = t_{j+1} - t_j$, i.e. its Lebesgue measure. By the definition of the map f, $h_j(\theta) = t_j$, i.e. $f(t_j) = \theta$, for $1 \le j \le k$.

Throughout the article we shall denote $I_{a_1 \dots a_n} := h_{a_1} \circ \cdots \circ h_{a_n}(I)$ and $h_{a_1 \dots a_n} := h_{a_1} \circ \cdots \circ h_{a_n}$. We call the sets $I_{a_1 \dots a_n}$ *cylinders* of I. These sets consist of a finite union of half-open intervals, by the definition of the maps h_i. The total Lebesgue measure of $I_{a_1 \dots a_n}$ goes to zero as n growths, in particular, if the the maps ϕ_j-s are linear the Lebesgue measure of $I_{a_1 \dots a_n}$ is $\alpha_{a_1} \cdots \alpha_{a_n}$. We denote the collection of all cylinders by $\mathcal{I}$. i.e.

$$\mathcal{I} := \{I_{a_1 \dots a_n} \, : \, n \ge 1, \, 1 \le a_i \le k\}.$$

The *extreme points* or *extremities* of a cylinder are the boundary points of the intervals that form the cylinder. We call *neighbouring extreme points* of a cylinder J, a pair x_1, x_2 of extreme points of J such that $x_1 \in J$ and $x_2 \in \overline{J} \setminus J$; and one the of the circle intervals, considered with the standard orientation, (x_2, x_1), (x_1, x_2) is disjoint of J. For example, if the cylinder J is of the form

$$J = [y_1, y_2) \cup [y_3, y_4) \cup [y_5, y_6);$$

the pairs of extreme points are y_1, y_6; y_3, y_2 and y_5, y_4. Neighbouring extreme points come from the discontinuity point, of the maps h_i's, i.e. the point θ.

Proposition 2.1. *Let $I_{a_1 \dots a_n}$ be a cylinder and the pair x_1, x_2 be neighbouring extreme points of the cylinder, such that $x_1 \in I_{a_1 \dots a_n}$ and $x_2 \in \overline{I_{a_1 \dots a_n}} \setminus I_{a_1 \dots a_n}$. Then $x_1 = t_i$ and $x_2 = t_{i+1}$ or $x_1 = h_{a_1 \dots a_{n'}}(t_i)$ and $x_2 = \lim_{t \to t_i^-} h_{a_1 \dots a_{n'}}(t) = h_{a_1 \dots a_{n'}}(t_{i+1})$, for some $i \in \{1, \dots, k\}$, $1 \le n' < n$, here $i + 1$ is taken mod k.*

Proof. We use induction on n. If $n = 1$, then the cylinder is $I_{a_1} = [t_{a_1}, t_{a_1+1})$, so its neighbouring extreme points are t_{a_1} and t_{a_1+1}. We suppose that the statement is true for $n = m$. Let $n = m + 1$. The cylinder $I_{a_1 \dots a_m}$ is decomposed into sub-cylinders as follows:

$$I_{a_1 \dots a_m} = \bigcup_{i=i}^{k} I_{a_1 \dots a_m i}.$$

We shall check that the statement is true for $I_{a_1 \dots a_m i}$. Since $I_{a_1 \dots a_m i} = h_{a_1 \dots a_m i}(I)$, the points $h_{a_1 \dots a_m}(t_i)$, $\lim_{t \to t_i^-} h_{a_1 \dots a_m}(t)$ are neighbouring extreme points of the cylinder $I_{a_1 \dots a_m i}$. If this cylinder has more than one connected component, the other neighbouring extreme points are neighbouring extreme points of the parent cylinder, i.e. $I_{a_1 \dots a_m}$, since the only discontinuity point of the map $h_{a_1 \dots a_m i}$ is θ. Therefore the statement is true for $I_{a_1 \dots a_m i}$. ∎

3 Geodesic laminations

Let $\mathbb{D}^2$ be the closed unit disk in the plane, and $\mathbb{S}^1$ its boundary. We identify $\mathbb{S}^1$ with $I = [0, 1)$, since the map f is well defined on $\mathbb{S}^1$. We think this map as acting on the boundary of the disk.

A geodesic in $\mathbb{D}^2$ is an arc of circle that meets the boundary of $\mathbb{D}^2$ perpendicularly.

Definition 3.1. *A* geodesic lamination *on* $\mathbb{D}^2$ *is a non-empty closed set of geodesics of the disk and that any two of these geodesics do not intersect except at their end points.*

The construction of the geodesic lamination Λ is as follows: We consider the extremities of the intervals I_i's. And we join pairwise consecutive extremities by geodesics in $\mathbb{D}^2$. Let $a_1 \dots a_n$ be a word in the alphabet $\{1, \dots, k\}$. We join by geodesics the points $h_{a_1} \cdots h_{a_n}(t_j)$ and $h_{a_1} \cdots h_{a_n}(t_{j+1})$ for $j = 0, \dots, k$, where $j+1$ is taken mod k. We do this for all possible finite words in this alphabet and later we take the closure in the topology given by the Hausdorff metric of $\mathbb{D}^2$. We denote the set obtained in this way by Λ. The elements obtained in this way are either geodesics of $\mathbb{D}^2$ or points of $\mathbb{S}^1$; in the latter case the points are called *degenerate geodesics*. Due to Proposition 2.1, the set Λ is formed by the closure in the Hausdorff metric of $\mathbb{D}^2$ of the set of geodesics that join neighbouring extreme points of all elements of $\mathcal{I}$. Figure 1 shows the set Λ, corresponding to Example 1 of Section 4. The set Λ form a geodesic lamination, the proof is the same as in the case of a regular expanding linear map on I, done in [14]. For completeness sake, we present it here.

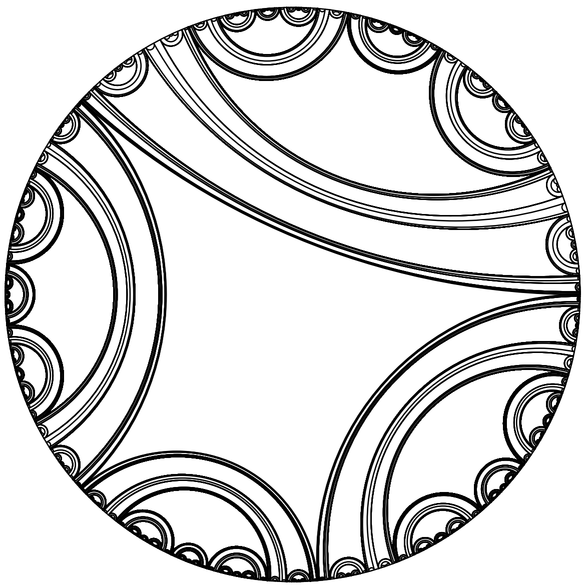

Figure 1: Geodesic lamination of Example 1, with $\alpha = (\sqrt{5}-1)/2$ and $\beta = \alpha^2$.

Proposition 3.1. *The set* Λ *is a geodesic lamination on* $\mathbb{D}^2$*.*

Proof. Let $J = I_{a_1 \cdots a_n}$ be a cylinder and suppose that it consists of more than one connected component. Let x_1 and x_2 be neighbouring end points of this cylinder belonging to different connected components, see Figure 2. Due to the proof of Proposition 2.1 we can suppose, without loss of generality, that $x_2 = h_{a_1 \cdots a_n}(\theta)$ and $x_1 = h_{a_1 \cdots a_n}(\theta^-) := \lim_{t \to t^{*-}} h_{a_1 \cdots a_n}(t)$. So $x_1 = h_{a_1 \cdots a_{n-1}}(t_{a_n+1})$ and $x_2 = h_{a_1 \cdots a_{n-1}}(t_{a_n})$ where the sum $a_n + 1$ is taken mod k.

Let us consider the cylinder $J' = I_{a_1 \cdots a_{n-1} a_n + 1}$ whose left end point is x_1. We shall show that J' is contained in the gap of J formed by $[x_1, x_2)$. Let us suppose that the cylinder J' consists of more than one connected component and one of these components is outside the gap $[x_1, x_2)$, say to the right of x_2. So there exists $1 \leq j \leq k$ such that $h_{a_1 \cdots a_{n-1} a_n + 1}(t_j)$ is the discontinuity point for this cylinder, i.e.

$$x_1 < y_1 = h_{a_1 \cdots a_{n-1} a_n + 1}(t_j^-) < x_2 < y_2 = h_{a_1 \cdots a_{n-1} a_n + 1}(t_j).$$

So

$$\begin{array}{c} h_{a_1 \cdots a_{n-1}}(t_{a_n + 1}) < h_{a_1 \cdots a_{n-1} a_n + 1}(t_{j+1}) < h_{a_1 \cdots a_{n-1}}(t_{a_n}) < h_{a_1 \cdots a_{n-1} a_n + 1}(t_j) \\ h_{a_1 \cdots a_{n-1} a_n + 1}(\theta) < h_{a_1 \cdots a_{n-1} a_n + 1}(t_{j+1}) < h_{a_1 \cdots a_{n-1} a_n}(\theta) < h_{a_1 \cdots a_{n-1} a_n + 1}(t_j). \end{array}$$

Since the maps h_i preserve the cyclic order of the images of $\{t_1, \ldots, t_k\}$. We have that the points $h_{a_n+1}(\theta)$, $h_{a_n+1}(t_{j+1})$, $h_{a_n}(\theta)$ and $h_{a_n+1}(t_j)$ are in this cyclic order in $\mathbb{S}^1$. However the first two points and the last belong to the interval I_{a_n+1} and the third point to I_{a_n} that contradicts the fact that these are disjoint intervals. ∎

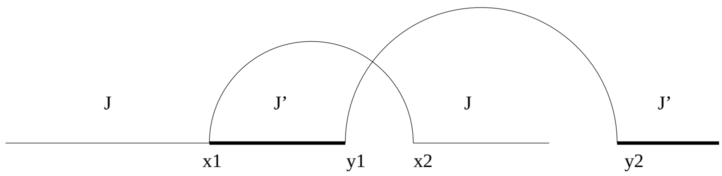

Figure 2: Figure relative to the proof of Proposition 3.1.

Proposition 3.2. *Every point in I is an end point of a geodesic in Λ.*

Proof. If the point t belongs to the backward f-orbit of θ, i.e. $f^m(t) = \theta$ for some positive integer m. Then point t is an extreme point of a cylinder. So by definition there is a geodesic in the lamination having t as an end point.

If t does not belong to the backward f-orbit of θ. Let $a_1 a_2 \ldots$ be its itinerary, i.e. $f^{i-1}(t) \in I_{a_i}$, for $i \geq 1$. Let

$$\mathcal{L} := \bigcup_{n \geq 1} \{\lambda \in \Lambda \ : \ \lambda \ \text{ geodesic that joins neighbouring end points of } \ I_{a_1 \ldots a_n}\}.$$

By the definition of the geodesic lamination Λ, there exists a sequence of elements of the set $\mathcal{L}$: $\{\lambda_l\}_{l \geq 1}$ such that t is in the closure of the end points of λ_l. Since Λ is closed then the closure of $\mathcal{L}$ is contained in Λ. Hence t is an end point of an element of the closure of $\mathcal{L}$. ∎

Proposition 3.3. *Let $a_1 a_2 \ldots$ be the itinerary of θ. If*

(a) $a_m \neq a_1$, for all $m > 1$, or

(b) θ is preperiodic and its itinerary is of the form: $a_1 \ldots a_{l-1} \overline{a_l \ldots a_{m+l}}$ for some m and l with $m > l > 0$ and $a_1 \ldots a_l \neq a_j \ldots a_{j+l-1}$, for $l \leq j \leq l + m$.

Then θ is a degenerate geodesic of the lamination Λ.

Proof. The conditions (a) and (b) on the itinerary θ imply that θ is not f-periodic in either case.

First, we shall prove that the point θ is in the interior of the cylinders $I_{a_1\cdots a_n}$, for all $n > 0$. If θ is not in the interior of $I_{a_1\cdots a_n}$, for some n; then by Proposition 2.1, $\theta = h_{a_1\ldots a_{n'}}(t_i)$, for some $i \in \{1,\ldots,k\}$, and $n' < n$. Hence $f^{n'}(\theta) = f^{n'}(h_{a_1\ldots a_{n'}}(t_i))$, so $t_i = f^{n'}(\theta)$. Therefore $\theta = f^{n'+1}(\theta)$, contradicting the fact that θ is not periodic.

In the case (a), we shall prove, the cylinders of the form $I_{a_1\ldots a_n}$ consist of only one connected component for all n. We use induction, clearly I_{a_1} has only one connected component. We suppose that $I_{a_1\ldots a_n a_{n+1}}$ has more than one connected component, while $I_{a_1\ldots a_n}$ consists of one connected component. Since $I_{a_1\ldots a_n a_{n+1}}$ has more than one connected component, the discontinuity point of h_{a_1} is in the interior of $I_{a_2\ldots a_{n+1}}$ or $I_{a_2\ldots a_{n+1}}$ has more than one connected component. By definition of the maps h_i, θ is the discontinuity point for $1 \le i \le k$. Therefore in the former case $\theta \in I_{a_2\ldots a_{n+1}}$, so $a_2 = a_1$, contradicting the hypothesis on the itinerary of θ. In the later case, when $I_{a_2\ldots a_{n+1}}$ consists of more than one connected component. We repeat the previous argument, so we conclude that $\theta \in I_{a_j\ldots a_{n+1}}$, for some j, with $2 \le j \le n+1$. Hence $a_j = a_1$, contradicting the hypothesis on the itinerary of θ. Therefore $I_{a_1\ldots a_n}$ consists of only one connected component for all n, when the hypothesis (a) is satisfied.

Let λ_n the geodesic in Λ that joins the extreme points of $I_{a_1\ldots a_n}$. By the first remark in this proof, θ is different of the ends points of λ_n, for all n. Since $I_{a_1\ldots a_n}$ consists of nested intervals whose length goes to zero, $\cap_{n\ge 1} I_{a_1\ldots a_n}$ consists of only one point, which is θ. Therefore λ_n converges to θ in the Hausdorff metric. Hence θ is an element of the lamination Λ.

In (b), by hypothesis $\theta \notin I_{a_j\ldots a_{j+l-1}}$, for $l \le j \le l+m$, so $\theta \notin I_{a_j\ldots a_{l+m}}$, then the cylinder $I_{a_l\ldots a_{l+m}}$ consists of only one connected component. Similarly for $I_{(a_l\ldots a_{l+m})^r}$, for any positive r, where

$$(a_l \ldots a_{l+m})^r = \underbrace{a_l \ldots a_{l+m} \ldots a_l \ldots a_{l+m}}_{r \text{ times}}.$$

Since θ is the only discontinuity point of the maps h_i-s, then $h_{a_1\ldots a_{l-1}}(I_{(a_l\ldots a_{l+m})^r})$, consists of only one connected component for all r.

By a similar argument used in the case (a), the geodesics that joins the end points of the interval $I_{a_1\ldots a_{l-1}(a_l\ldots a_{l+m})^r}$, converges to θ in the Hausdorff metric, when r goes to infinity. Hence θ is an element of the lamination Λ. ∎

We introduce a dynamical system (Λ, F), when θ is a degenerate geodesic (in particular when the hypotheses of Proposition 3.3 are satisfied), as follows: Let λ be an element in Λ whose end points are t and t', we define $F(\lambda)$ as the geodesic that joins $f(t)$ and $f(t')$. The map F is continuous since f is continuous. If λ joins $h_{a_1}\cdots h_{a_n}(t_j)$ with $h_{a_1}\cdots h_{a_n}(t_{j+1})$ for $n \ge 1$ then $F(\lambda)$ joins $h_{a_2}\cdots h_{a_n}(t_j)$ with $h_{a_2}\cdots h_{a_n}(t_{j+1})$. If λ joins t_j with t_{j+1}, let us consider $f(t_j) = R_\theta \phi_j(t_j) = R_\theta(0) = \theta$ and similarly $f(t_{j+1}) = \theta$, and according to Proposition 3.3, the point θ is an element of Λ. Hence $F(\Lambda) \subset \Lambda$.

Let δ be any arc in $\mathbb{D}^2$ joining two distinct geodesics of the lamination, such that the intersection of these two geodesics with δ consists only of the end points of δ and the arc δ intersects each element of Λ at most once. We call δ a *transversal arc to* Λ. We slide the arc δ along the geodesics towards the boundary of the disk according the two possible directions in which the geodesics can be oriented. With this procedure we obtain a limit set in the boundary of the disk, C_δ. More precisely: Let λ_1 and λ_2 be the geodesics in Λ that are joined by an arc δ. This defines two disjoint intervals on the circle $J = [b_1, b_2]$ and $J' = [b_1', b_2']$ where b_k, b_k' are the end points of λ_k for $k = 1, 2$. Let λ be a geodesic in the lamination such that its end points lie on the same interval J or J', we remove from $J \cup J'$ the open interval whose extremities are the end points of λ. The set C_δ is obtained in this way when all the geodesics in Λ with end points in J or J' are considered. See Figure 3. If δ is an arc, whose end points are geodesics that join extreme points of the same cylinder, then the set C_δ is obtained as $\cap_{j\geq 0} K_j$, where K_0 is the closure of the original cylinder and each K_j is a union of closed sub-cylinders of K_0. By construction the set C_δ is not empty.

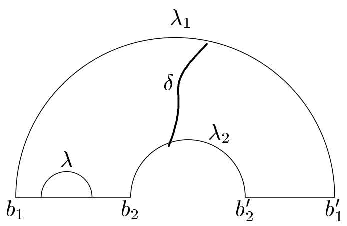

Figure 3: The construction of C_δ.

Let $K_0 = \overline{I_{a_1 \ldots a_m}}$. Since $I_{a_1 \ldots a_m} = \cup_{l=1}^{k} I_{a_1 \ldots a_m l}$, we have

$$K_1 = \overline{I_{a_1 \ldots a_m b_1}} \cup \cdots \cup \overline{I_{a_1 \ldots a_m b_s}},$$

for some $s < k$ and $1 \leq b_i \leq k$. We remark that all the cylinders $I_{a_1 \ldots a_m j}$ consisting of only one connected component are not in K_1. When we consider K_2, we have to see how each of these cylinders $I_{a_1 \ldots a_m b_i}$ are subdivided into cylinders of the form $I_{a_1 \ldots a_m b_i b_{i_2}}$. So $K_2 = \cup_{i=1}^{s} \cup_{i_2=1}^{s_i} \overline{I_{a_1 \ldots a_m b_i b_{i_2}}}$. Some of these cylinders could be sub-divided in the same way as the original cylinder $I_{a_1 \ldots a_m}$, so we say that they are of the same type. In general, we say that two cylinders are of the same type if they are subdivided in a similar manner under this process. If the number of connected components of the family of cylinders $\mathcal{I}$ is bounded, there is a finite number of different types of cylinders. Hence, the set $C_\delta = \cap_{m\geq 1} K_m$, is the fixed point of a graph-directed iterated function system (GIFS). For definition and general properties of GIFS see [8, pp. 47]. If the map f is a regular expanding linear map, then the contracting maps defined by the sub-division process of the cylinders are similarities, so the maps that define the GIFS are similarities. In Section 4, we show in detail the construction of the limit set C_δ for some examples, see also [12, 14]. We have proved the following proposition.

Proposition 3.4. *If the number of the connected components of the family of cylinders $\mathcal{I}$ is bounded. Then the set C_δ is the fixed point of graph directed iterated function system (GIFS). Moreover the maps of this GIFS are similarities when f is a regular expanding linear map.*

Proposition 3.5. *If for some transversal arc δ, the set C_δ is finite. Then the point θ is a f-periodic point.*

Proof. Let $s_1, \ldots, s_m$ be the elements of C_δ and ordered such that $s_i < s_{i+1}$, for $1 \leq i \leq m-1$. So by definition $C_\delta = J \setminus \cup_{i=1}^{m-1} J_i$, where J is the closure of an element of $\mathcal{I}$ and $J_i = (s_i, s_{i+1})$. In particular the extreme points of J belong to C_δ, so s_1 and s_m belong to the closure of J. Let λ_i be the geodesic that joins s_i with s_{i+1}, by the construction of C_δ, λ_i belongs to the lamination Λ. See Figure 4.

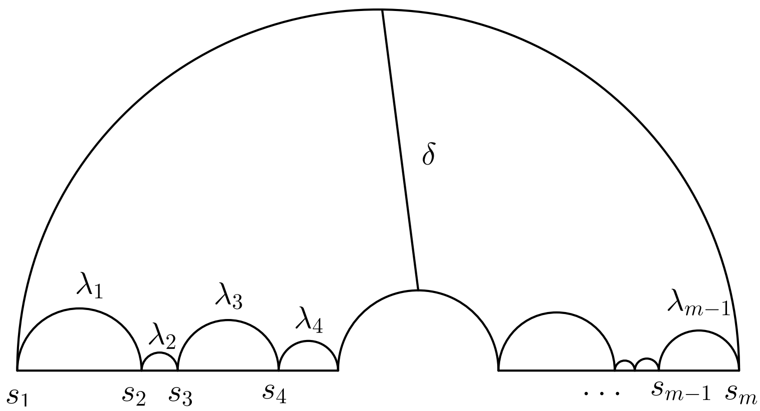

Figure 4: Figure relative to the proof of Proposition 3.5.

Due to the construction of the lamination Λ and the limit set C_δ, we can suppose, without loss of generality, that J is the closure of a cylinder, i.e. $I_{c_1 \ldots c_r}$. According to Proposition 2.1, we have $s_1 = h_{c_1 \ldots c_{r'}}(t_i)$, for some $r' < r$ and $i \in \{1, \ldots, k\}$.

The fact that s_1 is the end point of a cylinder implies that the points s_j, for $j \in \{2, \ldots, m\}$, are end points of cylinders. In particular s_1 and s_2 are the end points of a cylinder contained in $I_{c_1 \ldots c_r}$. So s_1 and s_2 are the end points of a cylinder of the form $I_{c_1 \ldots c_l}$, with $l > r$. By Proposition 2.1, $s_1 = h_{c_1 \ldots c_{l'}}(t_j)$, for some $l' < l$ and $j \in \{1, \ldots, k\}$, and $s_2 = h_{c_1 \ldots c_{l'}}(t_j^-)$. If $r' = l'$ then $s_2 = s_m$, so we can suppose that $r' < l'$. Since the maps h_i-s are the inverses of the map f, we have: $f^{l'}(s_1) = t_j$, so $f^{l'+1}(s_1) = f(t_j) = \theta$, similarly $f^{r'+1}(s_1) = \theta$. Hence $f^{l'-r'}(\theta) = \theta$, so θ is a periodic point of f. ∎

We conjecture that if θ is a f-periodic point then the set C_δ is countable.

For a given δ, a transversal arc to Λ, we define $\mu(\delta) = \mathcal{M}_{s_0}(C_\delta)$ where $\mathcal{M}_{s_0}$ is the s_0-Hausdorff measure and s_0 is the Hausdorff dimension of C_δ. When f is a regular expanding linear map, the sets C_δ are the fixed points of GIFS consisting of similarities, and the GIFS satisfies the open set condition (see [8] for the definition). Due to the self-similarity of C_δ, its Hausdorff dimension is independent of δ.

The domain of F can be extended to the set of equivalence classes of transverse curves to the lamination Λ. Given δ and δ' two transverse curves to Λ, we say that $\delta \sim \delta'$ if the end points of each curve lie in the same pair of distinct geodesics, and

$C_\delta = C_{\delta'}$. Therefore $\mu(\delta) = \mu(\delta')$. We extend the definition of the map F to the transverse curves to Λ and their equivalence classes. The curve $F(\delta)$ is defined as a transverse curve only to all $F(\lambda)$ where λ are the geodesics transverse to δ.

Proposition 3.6. *Let $f : I \to I$ be a regular expanding linear map, such that θ not f-periodic; and $F : \Lambda \to \Lambda$ its induced map on the lamination Λ. Then F has the property $F_*\mu = (\alpha_1^{s_0} + \cdots + \alpha_k^{s_0})\mu$, where $F_*\mu(\delta) := \mu(F^{-1}(\delta))$, and δ is a transverse arc to Λ.*

Proof. By definition $F_*(\mu(\delta)) = \mu(F^{-1}(\delta)) = \mathcal{M}_{s_0}(C_{F^{-1}}(\delta))$. And by construction the limit set $C_\delta = \cap_{j\geq 0}(K_j)$, so $f^{-1}(C_\delta) = \cap_{j\geq 0} f^{-1}(K_j)$ and $C_{F^{-1}(\delta)} = f^{-1}(C_\delta)$. Since f is k to 1 and its inverse branches are the maps h_i, we have

$$\mathcal{M}_{s_0}(C_{F^{-1}(\delta)}) = \mathcal{M}_{s_0}(f^{-1}(C_\delta)) = \mathcal{M}_{s_0}(\bigcup_{j=1}^{k} h_j(C_\delta)) = \sum_{j=1}^{k} \mathcal{M}_{s_0}(h_j(C_\delta)) =$$

$$= (\alpha_1^{s_0} + \cdots + \alpha_k^{s_0})\mathcal{M}_{s_0}(C_\delta) = (\alpha_1^{s_0} + \cdots + \alpha_k^{s_0})\mu(\delta).$$

Therefore $F_*\mu = (a_1^{s_0} + \cdots + a_k^{s_0})\mu$. ∎

If $0 < s_0 < 1$ then $(\alpha_1^{s_0} + \cdots + \alpha_k^{s_0}) > 1$ since $\sum_{i=1}^k \alpha_i = 1$, $0 < \alpha_i < 1$; so we can think F as an expanding map the lamination Λ.

From these results, we have proved the following theorem:

Theorem 3.1. *Let $f : I \to I$ be a regular expanding map, such that θ satisfies the hypotheses of Proposition 3.3. Then there exists a geodesic lamination Λ on the disk with transversal measure μ and a continuous map $F : \Lambda \to \Lambda$. Moreover, if f is a regular expanding linear map and s_0 is the Hausdorff dimension of a limit set on the boundary of the disk obtained by any transverse arc to Λ, satisfying $0 < s_0 < 1$, then there is $\rho > 1$ such that $F_*\mu = \rho\mu$.*

4 Examples

Example 1: Let $\alpha_1 = \alpha^2$, $\alpha_2 = \alpha_3 = \alpha\beta$ and $\alpha_4 = \beta^2$, with $\alpha + \beta = 1$, and $\alpha, \beta > 0$; and $\theta = 1 - \alpha^2/2$. We define $\phi_i(t) := (t - \sum_{j=0}^{i-1} \alpha_j)/\alpha_i$, for $1 \leq i \leq 4$, and $\alpha_0 = 0$. This example was studied in detail in [14]. The point θ is not periodic for f since its itinerary is $1\overline{3} = 13333\ldots$. The Hausdorff dimension of the limit sets C_δ is s_0, the solution of the equation $\alpha^{2s} + \beta^{2s} = 1$. Moreover $0 < \mathcal{M}_{s_0}(C_\delta) < \infty$. In [14], it was shown that there is an space-filling curves associated to this expanding map. The lamination of this example, with $\alpha = (-1 + \sqrt{5})/2$ and $\beta = \alpha^2$, is shown in Figure 1.

Example 2: Let $\alpha_i = 1/3$, for $i \in \{1, 2, 3\}$, and $\theta = 5/9$. As in Example 1, $\phi_i(t) := (t - \sum_{j=0}^{i-1} \alpha_j)/\alpha_i$, for $i = 1, 2, 3$, and $\alpha_0 = 0$. It can be easily checked that the itinerary of θ is $2\overline{1}$, hence it is not periodic. The graph of this expanding map, is shown in Figure 5. Let $I_{a_1 \ldots a_n}$ be a cylinder, using induction on n, can be checked that there are three different types of cylinder, according how are decomposed into sub-cylinders. The cylinders could have one or two connected

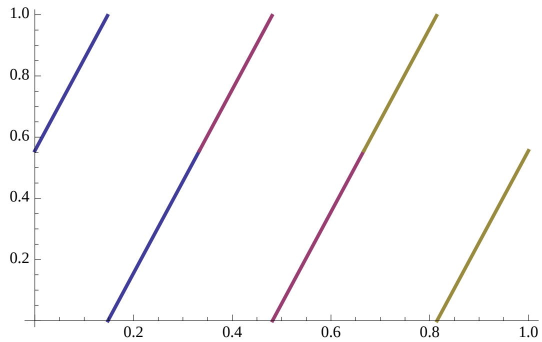

Figure 5: Graph of the expanding map of Example 2.

component. In order to understand the cylinders and how are decomposed into sub-cylinders we give the following model. Let

$$0 \leq x_1 < y_1 < y_2 < x_2 \leq x_3 < y_3 < x_4 < 1, \tag{1}$$

with the relations:

$$\frac{|y_1 - x_1| + |x_4 - y_3|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{3},$$

$$\frac{|y_2 - x_2| + |y_3 - x_3|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{3},$$

$$\frac{|y_2 - y_1|}{|x_2 - x_1| + |x_4 - x_3|} = \frac{1}{3}.$$

We consider $[x_1, x_2) \cup [x_3, x_4)$ as a cylinder. There are the following possibilities:

- If $x_2 = x_3$, i.e. the cylinder consists of one interval. We say that the cylinder is of *type I*.
- If $x_2 \neq x_3$, i.e. the cylinder consists of two intervals. Here we allow cyclic permutations of the intervals. See Figure 6.

  We say that:

  - it is of *type II-1* if the intervals are ordered in $[0, 1)$ as in (1).
  - it is of *type II-2* if the order is as $0 \leq x_3 < y_3 < x_4 < x_1 < y_1 < y_2 < x_2$.

If $I_{a_1 \ldots a_n}$ is of type I then its decomposition into sub-cylinders is as follows:

$$I_{a_1 \ldots a_n 1} = [y_2, y_3), \quad I_{a_1 \ldots a_n 2} = [x_1, y_1) \cup [y_3, x_4), \quad I_{a_1 \ldots a_n 3} = [y_1, y_2);$$

where $I_{a_1 \ldots a_n 3}$ and $I_{a_1 \ldots a_n 1}$ are of type I, $I_{a_1 \ldots a_n 2}$ is of type II-2.

If $I_{a_1 \ldots a_n}$ is of type II-1 then

$$I_{a_1 \ldots a_n 1} = [y_2, x_2) \cup [x_3, y_3), \quad I_{a_1 \ldots a_n 2} = [x_1, y_1) \cup [y_3, x_4), \quad I_{a_1 \ldots a_n 3} = [y_1, y_2);$$

where $I_{a_1 \ldots a_n 3}$ is of type I, $I_{a_1 \ldots a_n 1}$ is of type II-1 and $I_{a_1 \ldots a_n 2}$ of type II-2.

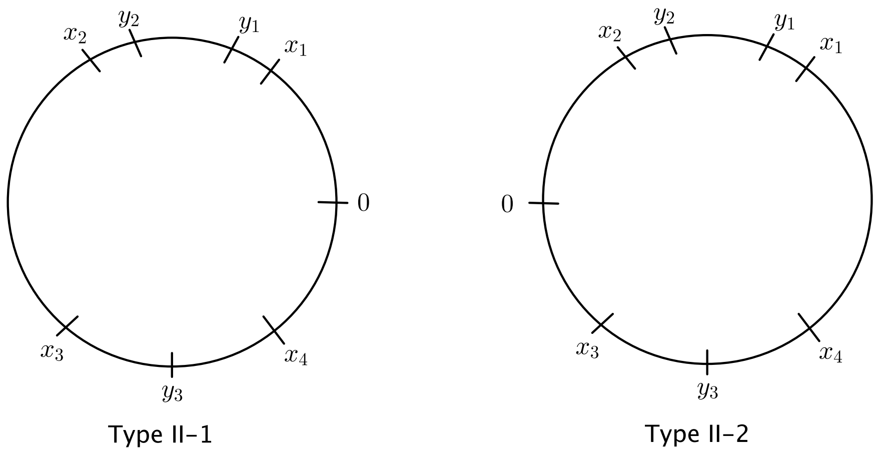

Figure 6: Different types of cylinders in Example 2.

If $I_{a_1 \ldots a_n}$ is of type II-2 then

$$I_{a_1 \ldots a_n 1} = [x_3, y_3) \cup [y_2, x_2), \quad I_{a_1 \ldots a_n 2} = [y_3, x_4) \cup [x_1, y_1), \quad I_{a_1 \ldots a_n 3} = [y_1, y_2);$$

where $I_{a_1 \ldots a_n 3}$ is of type I, $I_{a_1 \ldots a_n 1}$ is of type II-2 $I_{a_1 \ldots a_n 2}$ is of type II-1.

We use this information to understand the construction of the limit set C_δ. Let δ be a transverse arc whose end points are in the geodesics that joins the extreme points of a cylinder $I_{a_1 \cdots a_n}$, of type II-1, see Figure 7. The study of the case of type II-2 is done in a similar manner. So $I_{a_1 \cdots a_n} = [x_1, x_2) \cup [x_3, x_4)$. Let K_0 be the closure of $I_{a_1 \cdots a_n}$, i.e. $K_0 = [x_1, x_2] \cup]x_3, x_4]$. In the first step of the construction of the set C_δ, we remove from K_0 the open set (y_1, y_2). So $K_1 = [x_1, y_1] \cup [y_2, x_2] \cup [x_3, x_4]$, the set K_1 consists of the closure of the cylinders $I_{a_1 \cdots a_n 1}$ and $I_{a_1 \cdots a_n 2}$, which are of type $II - 1$ and $II - 2$. This process defines the map $\Psi_{1,1}$ as the map that sends $I_{a_1 \cdots a_n}$, of type II-1, into $I_{a_1 \cdots a_n 1}$, of type II-1, using the maps h_1, h_2 and h_3. Similarly we get $\Psi_{1,2}$. Booth of then contract the distance by a factor $1/3$. We repeat this process to these new cylinders. Hence we obtained the set C_δ as the attractor of the graph directed iterated function systems, defined by the graph, shown in Figure 8. Since the maps $\Psi_{i,j}$ contract the distance by a factor of $1/3$, then the Hausdorff dimension of C_δ is $\log 2 / \log 3$. Furthermore it can be proved using standard techniques that $0 < \mathcal{M}_{s_0}(C_\delta) < \infty$.

This analysis can be extended for any k odd. In this case we consider $\alpha_i = 1/k$, for $1 \leq i \leq k$, and $\theta = (k^2 - k - 1)/k^2$. It can be checked that the itinerary of θ is $(k-1)\overline{(k-2)}$. The corresponding limit set C_δ has Hausdorff dimension $\log 2 / \log k$.

Example 3: Let α be the real root of the polynomial $x^3 + x^2 + x - 1$. Let $\alpha_j = \alpha^j$, for $1 \leq j \leq 3$ and $\theta = (\alpha + \alpha^2)/2$. As before, we define $\phi_i(t) := (t - \sum_{j=0}^{i-1} \alpha_j)/\alpha_i$, for $i = 1, 2, 3$, and $\alpha_0 = 0$. The itinerary of θ is $1\overline{12}$, which satisfies the hypothesis (b) of Proposition 3.3. Some dynamical properties of this map were studied in [3]. The geodesic lamination was introduced and studied in detail in [12]. The lamination is shown in Figure 10. In this example the dimension of the limit sets is calculated in detail and the value for s_0 is $\log \rho / \log \alpha$, where ρ is the real

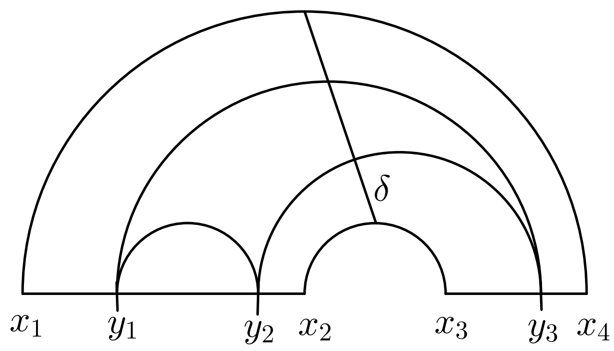

Figure 7: The construction of the set C_δ in Example 2.

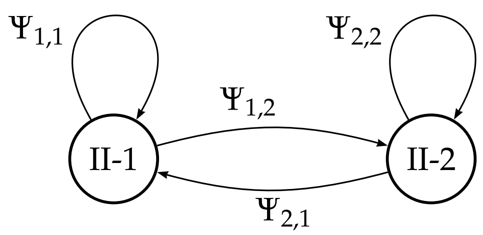

Figure 8: The graph-directed IFS that defines the limit set C_δ of Example 2.

root smaller than one, in absolute value, of the polynomial $x^4 + 2x^3 - 1$, s_0 is approximately 0.5466. This construction is related to the Rauzy fractal, for details see [1, 12, 2]. This construction can be generalized to k maps, i.e. $\{\phi_1, \ldots, \phi_k\}$. Where $\alpha_j = \alpha^j$, for $1 \leq j \leq k$, and α is the real root smaller than one, in absolute value of the polynomial $x^k + x^{k-1} + \cdots + x - 1$, for details see [12].

Example 4: Here we will consider a non-linear version of Example 3. Let $\phi_i : I_i \to I$ given by

$$\phi_i(t) = \frac{\exp(t - \sum_{j=0}^{i-1} \alpha_j) - 1}{\exp(\alpha_i) - 1},$$

where $\alpha_i = \alpha^i$, for $1 \leq i \leq 3$ and $\alpha_0 = 0$, as in Example 3. Numerical evidence, suggests that the point θ is not periodic, and part of its itinerary is $1122231121\ldots$. The graph of the expanding map is shown in Figure 11 and the corresponding geodesic lamination is shown in Figure 10.

Example 5: Here we will consider another linear example, but θ periodic. Let $\alpha_i = 1/3$, for $i \in \{1, 2, 3\}$, and $\theta = 1/6$, whose itinerary is $\overline{123}$. It can be easily checked in this example that θ is not a degenerate geodesic. The corresponding lamination is shown in Figure 12.

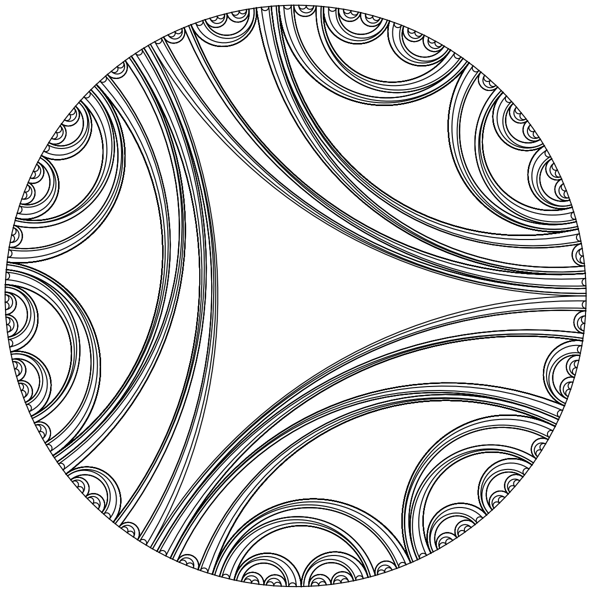

Figure 9: Geodesic lamination of Example 2.

5 Open problems and some questions

1. Is it true that if θ is a periodic point for the map f, then the limit set C_δ is countable, for all transversal arcs δ?

2. Given a transversal arc δ; for which values of θ, C_δ is a Cantor set with $0 < \mathcal{M}_s(C_\delta) < \infty$?

3. Let α_j, $1 \leq j \leq k$, as considered previously, i.e. $0 < \alpha_j < 1$ and $\sum_{j=1}^{k} \alpha_j = 1$, and $\theta \in (0,1)$. Let f_θ be the linear expanding map associated to R_θ and the linear maps $\phi_j : I_j \to I$, $\phi_j(t) := (t - \sum_{i=1}^{j-1})/\alpha_j$. Let s_θ be the Hausdorff dimension of the limit sets of the transversal to the lamination Λ. We would like to know the behaviour of the function $\theta \mapsto s_\theta$.

4. Does exist a regular expanding (linear) map with θ non-periodic and not a degenerate geodesic?

5. Characterization of the degenerate geodesics.

6. Give necessary and sufficient conditions on the map f, so that the elements of $\mathcal{I}$ have a bounded number of connected components.

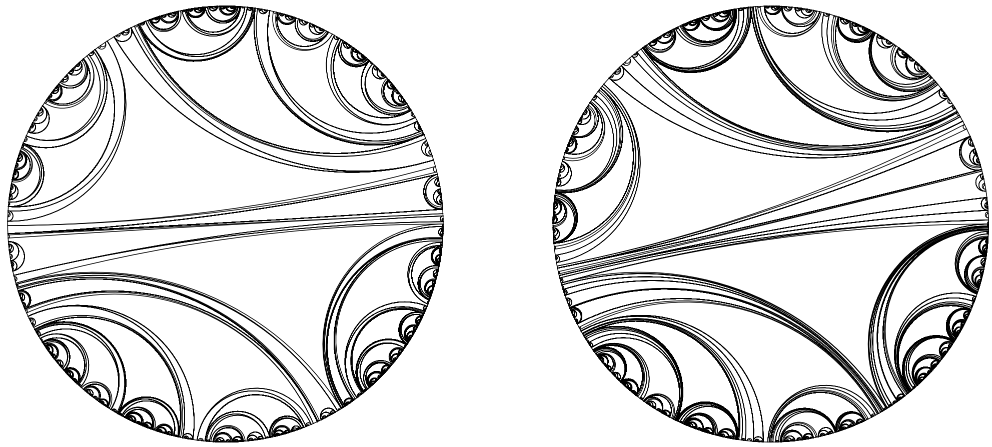

Figure 10: Geodesic lamination of Example 3 (left) and Example 4 (right).

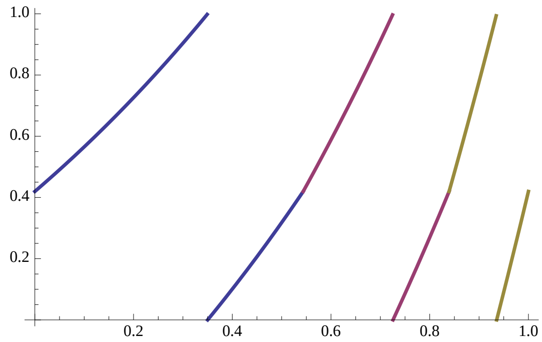

Figure 11: Graph of the expanding map of Example 4.

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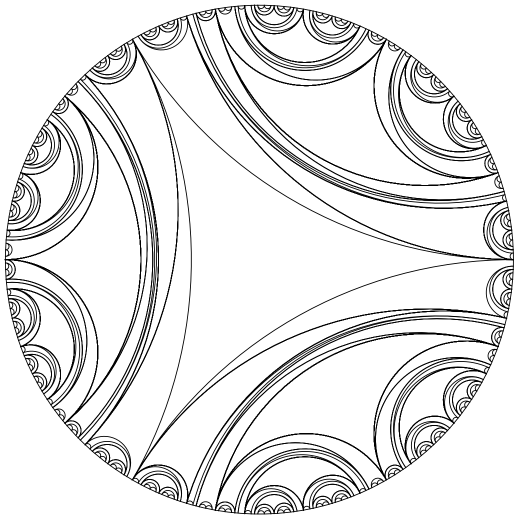

Figure 12: Geodesic lamination of Example 5, where θ is periodic.

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