

Discrete reflexivity in function spaces*

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Abstract

We study systematically when $C_p(X)$ has a topological property \mathcal{P} if $C_p(X)$ is discretely \mathcal{P} , i.e., the set \overline{D} has \mathcal{P} for every discrete subspace $D \subset C_p(X)$. We prove that it is independent of ZFC whether discrete metrizable-ability of $C_p(X)$ implies its metrizability for a compact space X . We show that it is consistent with ZFC that countable tightness and Lindelöf Σ -property are not discretely reflexive in spaces $C_p(X)$. It is also established that a space X must be countable and discrete if $C_p(X)$ is discretely Čech-complete. If $C_p(X)$ is discretely σ -compact then X has to be finite.

1 Introduction

A topological property \mathcal{P} is called *discretely reflexive* in a class \mathcal{A} if a space X from the class \mathcal{A} has \mathcal{P} if and only if the closure of every discrete subspace of X has \mathcal{P} . Tkachuk proved in [15] that compactness is discretely reflexive in any space; Arhangel'skii and Buzyakova established in [6] that the Lindelöf property is discretely reflexive in spaces of countable tightness.

A systematic study of discrete reflexivity was undertaken by Alas, Tkachuk and Wilson in [1]. They proved that initial κ -compactness, hereditary Lindelöf number and sequential compactness are discretely reflexive in every space. In the same paper Alas, Tkachuk and Wilson considered topological properties that

*Research supported by Promep 912011, Proyecto 12611768 (Mexico) and by CONACyT grant CB-2012-01-178103 (Mexico)

Received by the editors in August 2013.

Communicated by E. Colebunders.

2010 *Mathematics Subject Classification* : Primary: 54C35, 54C05; Secondary: 54G20.

Key words and phrases : function spaces, pointwise convergence topology, tightness, discrete subspaces, discretely reflexive property, spread, character, discretely σ -compact space, discretely Čech-complete space.

are discretely reflexive in compact spaces. They established, among other things, that character and tightness are discretely reflexive in X if X is compact.

Burke and Tkachuk discovered (see [7]) that first countability, countable tightness and Fréchet–Urysohn property are discretely reflexive in countably compact spaces of weight $\leq \omega_1$. It is an open problem of Arhangel’skii (see [4, Problem 14]) whether the Lindelöf property is discretely reflexive for all spaces. Tkachuk and Wilson showed in the paper [18] that paracompactness and the Lindelöf property are both discretely reflexive in GO spaces.

However, most convergence properties are not discretely reflexive even in countable spaces. This can be easily deduced from van Douwen’s result on the existence of a countable maximal space [8, Example 3.3]. The respective example shows that a space X need not be sequential even if every discrete subspace in X is closed and hence X is discretely metrizable. Therefore countable weight, metrizability, first countability, Fréchet–Urysohn property and sequentiality are not discretely reflexive. The same example shows that Čech-completeness is not discretely reflexive either.

It was asked in the book [17] whether the Lindelöf property is discretely reflexive in the spaces $C_p(X)$ (see Problem 4.4.2). This problem is still open but it expresses the general idea that the algebraic structure of $C_p(X)$ could imply better behavior with respect to discrete reflexivity. The present study is done with this idea in mind. We show that Čech-completeness is discretely reflexive in the spaces $C_p(X)$, i.e., \overline{D} is Čech-complete for every discrete set $D \subset C_p(X)$ if and only if X is countable and discrete which is equivalent to $C_p(X)$ being Čech-complete. It turns out that σ -compactness is also discretely reflexive in $C_p(X)$ while this is consistently false for countable tightness, and Lindelöf Σ -property.

We also show that it is independent of ZFC whether metrizability, first countability and countable network weight are discretely reflexive in $C_p(X)$ for a compact space X .

2 Notation and terminology

All spaces are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology; given any point $x \in X$ let $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. If $A \subset X$ then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. The set \mathbb{R} is the real line with its usual topology, $\mathbb{N} = \omega \setminus \{0\}$ and $\mathbb{Q} \subset \mathbb{R}$ is the set of rationals. We will also need the closed interval $\mathbb{I} = [-1, 1] \subset \mathbb{R}$. If X is a space then $\Delta_X = \{(x, x) : x \in X\}$ is its diagonal; we write Δ instead of Δ_X if X is clear. If κ is an infinite cardinal then $A(\kappa)$ is the one-point compactification of a discrete space of cardinality κ . We assume that $A(\kappa) = \kappa \cup \{p\}$ where p is the unique non-isolated point of the space $A(\kappa)$.

For any spaces X and Y the set $C(X, Y)$ consists of continuous functions from X to Y ; if it has the topology induced from Y^X then the respective space is denoted by $C_p(X, Y)$. We write $C(X)$ instead of $C(X, \mathbb{R})$ and $C_p(X)$ instead of $C_p(X, \mathbb{R})$. Given a set $A \subset C_p(X)$ let $\varphi(x)(f) = f(x)$ for any $f \in A$. This gives a continuous map $\varphi : X \rightarrow C_p(A)$ which will be called *the reflection map* with respect to A . If we have a continuous map $\varphi : X \rightarrow Y$ then letting $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y)$, we obtain *the dual map* $\varphi^* : C_p(Y) \rightarrow C_p(X)$.

A map $f : X \rightarrow Y$ is called a *condensation* if it is a continuous bijection; in this case we say that X *condenses onto* Y . Say that a family \mathcal{F} of subsets of a space X is a *network modulo a cover* \mathcal{C} if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is *Lindelöf Σ* (or has the *Lindelöf Σ -property*) if there exists a countable family \mathcal{F} of subsets of X such that \mathcal{F} is a network modulo a compact cover \mathcal{C} of the space X . A space X is *Fréchet–Urysohn*, if for any $A \subset X$ and $x \in \overline{A}$ there exists a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \rightarrow x$.

The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic. A continuous map $f : X \rightarrow Y$ is called *\mathbb{R} -quotient* if a function $g : Y \rightarrow \mathbb{R}$ is continuous if and only if $g \circ f$ is continuous. Given a space X , a family \mathcal{N} of subsets of X is a *network of X* if for every $U \in \tau(X)$ there exists a family $\mathcal{N}' \subset \mathcal{N}$ such that $U = \bigcup \mathcal{N}'$. Furthermore, $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network in } X\}$. The cardinal $nw(X)$ is called the *network weight* of X and the spaces with a countable network are called *cosmic*. A space X is called *ω -monolithic* if every separable subset of X has a countable network. A space X is called *left-separated* if there exists a well order $<$ on X such that the set $\{x \in X : x < a\}$ is closed for any $a \in X$.

If X is a space and $x \in X$ then let $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap \mathcal{U} = \{x\}\}$ and $\psi(X) = \sup\{\psi(x, X) : x \in X\}$; the cardinal $\psi(X)$ is called the *pseudocharacter* of the space X . Given a space X , the cardinal $s(X)$ called the *spread* of X , is the supremum of cardinalities of discrete subsets of X . As usual, we denote by $d(X)$ the minimal cardinality of a dense subset of X and $hd(X) = \sup\{d(Y) : Y \subset X\}$. Now, $hl(X) = \sup\{l(Y) : Y \subset X\}$ is the hereditary Lindelöf number of X . The cardinal $iw(X) = \min\{\kappa : \text{the space } X \text{ has a weaker topology of weight } \kappa\}$ is called the *i -weight* of X . Given an infinite cardinal κ we say that $t(X) \leq \kappa$ if, for any $A \subset X$ and $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. If θ is a cardinal invariant then $\theta^*(X) = \sup\{\theta(X^n) : n \in \mathbb{N}\}$ for any space X . We say that X is a *strong S -space* if $hd^*(X) = \omega$ and X is not Lindelöf.

The rest of our notation is standard and follows the book [9]. For the references in C_p -theory, see the book [17].

3 Representative discrete subsets of $C_p(X)$

If \mathcal{P} is a hereditary property then, to show that \mathcal{P} is discretely reflexive in $C_p(X)$, it suffices to find a discrete $\Omega \subset C_p(X)$ such that $\overline{\Omega}$ contains a topological copy of $C_p(X)$. In this section we develop some methods of construction of large discrete subsets of $C_p(X)$ to implement this idea.

3.1 Definition. Given a topological property \mathcal{P} , a space X is called *discretely \mathcal{P}* if \overline{D} has \mathcal{P} for any discrete set $D \subset X$.

To construct representative discrete subspaces of $C_p(X)$ we will need the following concept.

3.2 Definition. Given a space X say that a family \mathcal{F} of subsets of X is *concentrated about a set $A \subset X$* if for any $U \in \tau(A, X)$, the cardinality of the family $\{F \in \mathcal{F} : F \cap U = \emptyset\}$ is strictly less than the cardinality of \mathcal{F} .

The following lemma was proved in [16].

3.3 Lemma. *Given a space X suppose that a family \mathcal{F} consists of finite subsets of X and there is $m \in \mathbb{N}$ such that $|F| \leq m$ for all $F \in \mathcal{F}$. Then there exists a finite set $A \subset X$ (called the core of \mathcal{F}) such that for any finite $B \subset X \setminus A$, the family \mathcal{F} is not concentrated about B .*

We start with the following sufficient condition of existence of big discrete subsets in function spaces.

3.4 Lemma. *Given an infinite space X , assume that there exists a discrete subspace $D \subset X \times X$ with $|D| \geq iw(X)$. Then we can find a discrete set $\Omega \subset C_p(X, [-2, 2]) \setminus C_p(X, \mathbb{I})$ such that $|\Omega| \leq iw(X)$ and $C_p(X, \mathbb{I}) \subset \overline{\Omega}$.*

Proof. For any $k \in \mathbb{N}$ denote by M_k the set $\{1, \dots, k\}$ and let $\kappa = iw(X)$. In this proof we will pass several times to a subset $D' \subset D$ with $|D'| = \kappa$. To simplify the notation we will assume each time that $D' = D$ which means that all previous reasoning can be repeated for our smaller set D' . Given a space Z say that sets $P, Q \subset Z$ are *functionally separated* if there exists a function $f \in C(Z, [0, 1])$ such that $f(P) \subset \{0\}$ and $f(Q) \subset \{1\}$.

We have two mutually exclusive possibilities:

- (a) there is a discrete set $D \subset X$ with $|D| = \kappa$;
- (b) there is no discrete set of cardinality κ in X but there exists a discrete subspace $D \subset X \times X$ with $|D| = \kappa$.

Let $\Delta = \Delta_X$; we will simultaneously give a proof for both cases. If case (b) is considered then $D \cap \Delta$ has cardinality $< \kappa$ so we can pass to an appropriate subset of D of cardinality κ to see that we can assume, without loss of generality, that $D \subset X^2 \setminus \Delta$. For any element $d = (d_1, d_2) \in D$ let $K_d = \{d_1, d_2\}$. If case (a) is under consideration then $K_d = \{d\}$.

If we are dealing with case (b) and some $x \in X$ belongs to κ -many elements of $\mathcal{D} = \{K_d : d \in D\}$ then, according to our policy, we can assume that $x \in K_d$ for all $d \in D$ and therefore $D \subset (\{x\} \times X) \cup (X \times \{x\})$. This shows that either $|D \cap (\{x\} \times X)| = \kappa$ or $|D \cap (X \times \{x\})| = \kappa$. Since both sets $X \times \{x\}$ and $\{x\} \times X$ are homeomorphic to X , a discrete space of cardinality κ embeds in X which is a contradiction. As a consequence, if the case (b) is considered, then

- (1) $|\{d \in D : x \in K_d\}| < \kappa$ for every $x \in X$.

Of course, (1) trivially holds if we deal with the case (a). In the case (b) choose, for any $d = (d_1, d_2) \in D$ a pair $\{U_1^d, U_2^d\}$ of open subsets of X such that $\overline{U_1^d} \cap \overline{U_2^d} = \emptyset$ and, for the set $U^d = U_1^d \times U_2^d$, we have $U^d \cap D = \{d\}$. In the case (a) we choose a set $U^d \in \tau(d, X)$ such that $U^d \cap D = \{d\}$.

Since $iw(X) = \kappa$, we can find a base \mathcal{B} of cardinality κ of some Tychonoff topology μ on the set X weaker than $\tau(X)$; let $X' = (X, \mu)$. There is no loss of generality to consider that $B \neq \emptyset$ for any $B \in \mathcal{B}$. From now on the bar denotes the closure in X and all topological properties for which the space is not mentioned are meant to hold in the space X .

We can apply Lemma 3.3 to the space X' to find a finite set $A \subset X$ such that, for any finite $B \subset X \setminus A$, there exists a set $U \in \tau(B, X')$ for which the cardinality

of the set $\{d \in D : K_d \cap \text{cl}_{X'}(U) = \emptyset\}$ is equal to κ . It follows from (1) that only $< \kappa$ -many elements of the family \mathcal{D} meet A so, passing if necessary, to a subset of D of cardinality κ , we can assume, without loss of generality, that $K_d \cap A = \emptyset$ for any $d \in D$. If $A = \emptyset$ then the reasoning is easier so we assume that $A \neq \emptyset$; let $\{a_1, \dots, a_n\}$ be a faithful enumeration of A . Denote by \mathbb{Q}_0 the set $\mathbb{Q} \cap \mathbb{I}$.

Our next step is to consider for every $k \in \mathbb{N}$, the family \mathcal{W}_k of all $3k$ -tuples $(W_1, \dots, W_k, V_1, \dots, V_k, s_1, \dots, s_k) \in \mathcal{B}^{2k} \times \mathbb{Q}_0^k$ such that

- (2) $W_i, V_i \in \mathcal{B}$ for all $i \in M_k$;
- (3) $\overline{V}_i \subset W_i$ and \overline{V}_i is functionally separated from $X \setminus W_i$ for all $i \in M_k$;
- (4) if $W = \bigcup_{i \in M_k} W_i$ then $\overline{W} \cap A = \emptyset$ and $|\{d \in D : \overline{W} \cap K_d = \emptyset\}| = \kappa$;
- (5) the family $\{\overline{W}_i : i \leq k\}$ is disjoint.

It is straightforward that $|\mathcal{W}_k| \leq \kappa$ for any $k \in \omega$ so if $\mathcal{W} = \bigcup\{\mathcal{W}_k : k \in \mathbb{N}\}$ then $|\mathcal{W}| \leq \kappa$. For any element $\xi = (W_1, \dots, W_k, V_1, \dots, V_k, s_1, \dots, s_k)$ of the family \mathcal{W} let $k_\xi = k$, $W[\xi] = \bigcup_{i \leq k} W_i$ and $R_i(\xi) = s_i$ for all $i \leq k$.

Using the property (4) it is easy to construct an injection $\varphi : \mathcal{W} \times \mathbb{Q}_0^n \rightarrow D$ such that $\overline{W[\xi]} \cap K_{\varphi(\xi, r)} = \emptyset$ for any $\xi \in \mathcal{W}$ and $r \in \mathbb{Q}_0^n$.

Fix $\xi \in \mathcal{W}$ and $r = (r_1, \dots, r_n) \in \mathbb{Q}_0^n$; if the case (a) is being considered and $\varphi(\xi, r) = d$ then we can choose a continuous function $f_{\xi, r} : X \rightarrow [-1, 2]$ and a set $H \in \tau(d, X)$ with the following properties:

- (6a) $H \subset U^d$ and $\overline{H} \cap (W[\xi] \cup A) = \emptyset$;
- (7a) $f_{\xi, r}(X \setminus H) \subset \mathbb{I}$ and $f_{\xi, r}(V_i) = \{R_i(\xi)\}$ for all $i \leq k_\xi$;
- (8a) $f_{\xi, r}(d) = 2$ and $f_{\xi, r}(a_i) = r_i$ for all $i \leq n$.

If we are considering the case (b) and $\varphi(\xi, r) = d = (d_1, d_2)$ then we can choose a continuous function $f_{\xi, r} : X \rightarrow [-2, 2]$ and a set $H_i \in \tau(d_i, X)$ for any $i = 1, 2$ with the following properties:

- (6b) $H_i \subset U_i^d$ for $i = 1, 2$ and $(\overline{H}_1 \cup \overline{H}_2) \cap (W[\xi] \cup A) = \emptyset$;
- (7b) $f_{\xi, r}(d_1) = 2$ and $f_{\xi, r}(d_2) = -2$;
- (8b) $f_{\xi, r}(X \setminus H_1) \subset [-2, 1]$ and $f_{\xi, r}(X \setminus H_2) \subset [-1, 2]$;
- (9b) $f_{\xi, r}(a_i) = r_i$ for all $i \leq n$ and $f_{\xi, r}(V_i) = \{R_i(\xi)\}$ for all $i \leq k_\xi$.

It turns out that $C_p(X, \mathbb{I})$ is contained in the closure in the space $C_p(X)$ of the set $\Omega = \{f_{\xi, r} : \xi \in \mathcal{W}, r \in \mathbb{Q}_0^n\} \subset C_p(X, [-2, 2]) \setminus C_p(X, \mathbb{I})$. To prove this, observe first that $|\Omega| \leq \kappa$ so it suffices to show that, for any finite set $B = \{x_1, \dots, x_k\} \subset X \setminus A$ and any $s_1, \dots, s_k, r_1, \dots, r_n \in \mathbb{Q}_0$ there exists $f \in \Omega$ such that $f(x_i) = s_i$ for any $i \leq k$ and $f(a_i) = r_i$ for all $i \leq n$.

Recall that the set A is the core of the family \mathcal{D} in the space X' and therefore we can find sets $W_1, \dots, W_k \in \mathcal{B}$ such that $x_i \in W_i$ for all $i \leq k$, the family $\mathcal{A}' = \{\text{cl}_{X'}(W_i) : i \leq k\}$ is disjoint (and hence the collection $\mathcal{A} = \{\overline{W}_i : i \leq k\}$ is disjoint as well) and there κ -many elements $d \in D$ such that $K_d \cap (\bigcup \mathcal{A}) = \emptyset$.

It is easy to choose $V_i \in \mathcal{B}$ such that $x_i \in V_i$ and the set V_i is functionally separated from $X \setminus W_i$ in X' (and hence in X) for all $i \in M_k$. An immediate consequence is that the $3k$ -tuple $\xi = (W_1, \dots, W_k, V_1, \dots, V_k, s_1, \dots, s_k)$ belongs to \mathcal{W} . If $r = (r_1, \dots, r_n)$ then $f_{\xi, r}(x_i) = s_i$ for all $i \leq k$ and $f_{\xi, r}(a_i) = r_i$ for each $i \leq n$; this proves that $C_p(X, \mathbb{I}) \subset \overline{\Omega}$.

Fix any element $\xi = (W_1, \dots, W_k, V_1, \dots, V_k, s_1, \dots, s_k) \in \mathcal{W}$ and a point $r = (r_1, \dots, r_n) \in \mathbb{Q}_0^n$. For the case (b) consider the point $d = (d_1, d_2) = \varphi(\xi, r)$;

the set $B_{\xi,r} = \{f \in C_p(X) : f(d_1) > 1 \text{ and } f(d_2) < -1\}$ is open in $C_p(X)$ and contains the function $f_{\xi,r}$ so it suffices to establish the equality $B_{\xi,r} \cap \Omega = \{f_{\xi,r}\}$. Assume toward a contradiction, that $f_{\eta,t} \in B_{\xi,r}$ for some $\eta \in \mathcal{W}$ and $t \in \mathbb{Q}_0^n$ such that $(\xi, r) \neq (\eta, t)$; then $a = (a_1, a_2) = \varphi(\eta, t) \neq d$.

We have $f_{\eta,t}(d_1) > 1$ and $f_{\eta,t}(d_2) < -1$ which, together with (6b)–(8b), implies that $d_1 \in U_1^a$ and $d_2 \in U_2^a$; this shows that $d \in U^a$ contradicting the equality $U^a \cap \Omega = \{a\}$. Consequently, $B_{\xi,r} \cap \Omega = \{f_{\xi,r}\}$ and hence Ω is a discrete subspace such that $C_p(X, \mathbb{I}) \subset \overline{\Omega}$. We leave to the reader the evident simplification of the above reasoning to show that the set Ω is also discrete in the case (a). ■

3.5 Theorem. *Assume that κ is an infinite cardinal and $D \subset X \times X$ is a discrete subset of cardinality κ . Then for any set $A \subset C_p(X, \mathbb{I})$ with $|A| \leq \kappa$, there exists a discrete set $\Omega \subset C_p(X, [-2, 2])$ such that $|\Omega| \leq \kappa$ and $A \subset \overline{\Omega} \setminus \Omega$.*

Proof. For every point $d = (d_1, d_2) \in D$ we can find sets $U_1^d, U_2^d \in \tau(X)$ such that $(U_1^d \times U_2^d) \cap D = \{d\}$. There exist functions $f_1^d, f_2^d \in C_p(X, \mathbb{I})$ such that $f_i^d(d_i) = 1$ and $f_i^d(X \setminus U_i^d) \subset \{0\}$ for each $i \in \{1, 2\}$. It is clear that the set $B = A \cup \{f_i^d : d \in D, i \in \{1, 2\}\}$ has cardinality at most κ .

Let $\varphi : X \rightarrow C_p(B)$ be the reflection map. By [17, Problem 163] the dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ is an embedding of $C_p(Y)$ in $C_p(X)$ and it is easy to check that $B \subset \varphi^*(C_p(Y))$.

For every function $g \in B$ there is a unique function $p_g \in C_p(Y)$ such that $\varphi^*(p_g) = g$. It is not difficult to see that $B' = \{p_g : g \in B\} \subset C_p(Y, \mathbb{I})$. If we can find a discrete set $\Omega' \subset C_p(Y, [-2, 2])$ such that $|\Omega'| \leq \kappa$ and $B' \subset \overline{\Omega'} \setminus \Omega'$ then the set $\Omega = \varphi^*(\Omega')$ will be as promised because φ^* is an embedding and $\varphi^*(C_p(Y, [-2, 2])) \subset C_p(X, [-2, 2])$.

For every $d = (d_1, d_2) \in D$ let $\xi(d) = (\varphi(d_1), \varphi(d_2)) \in Y \times Y$. It is standard to check that our choice of the set B guarantees that $\xi|_D$ is an injection and $D' = \{\xi(d) : d \in D\}$ is a discrete subset of $Y \times Y$ with $|D'| = |D| = \kappa$. We have $w(Y) \leq w(C_p(B)) = |B| \leq \kappa$ which shows that $|D'| \geq w(Y) \geq iw(Y)$ and hence Lemma 3.4 is applicable to find a discrete set $\Omega' \subset C_p(Y, [-2, 2])$ such that $|\Omega'| \leq iw(Y) \leq \kappa$ and $C_p(Y, \mathbb{I}) \subset \overline{\Omega'} \setminus \Omega'$. Finally, it follows from $B' \subset C_p(Y, \mathbb{I})$ that $B' \subset \overline{\Omega'} \setminus \Omega'$. ■

3.6 Lemma. *If X is a space with $d(X) \leq \kappa$ then there exists a discrete subset $D \subset X \times (A(\kappa) \setminus \{p\})$ such that $X \times \{p\} \subset \overline{D}$. Here p is the unique non-isolated point of $A(\kappa)$.*

Proof. Take a dense set $A \subset X$ such that $|A| \leq \kappa$ and choose a disjoint family $\{S_\alpha : \alpha < \kappa\}$ of countably infinite subsets of κ . Let $\{a_\alpha : \alpha < \kappa\}$ be an enumeration (possibly with repetitions) of the set A .

The set $D = \bigcup \{\{a_\alpha\} \times S_\alpha : \alpha < \kappa\} \subset X \times (A(\kappa) \setminus \{p\})$ is discrete because the projection of $X \times A(\kappa)$ onto $A(\kappa)$ maps D injectively into $A(\kappa) \setminus \{p\}$. Besides, the point (a_α, p) is in the closure of the set $\{a_\alpha\} \times S_\alpha$ for every $\alpha < \kappa$ so $A \times \{p\} \subset \overline{D}$ and hence $X \times \{p\} \subset \overline{D}$ as promised. ■

4 Convergence properties in $C_p(X)$

It is not difficult to convince ourselves that van Douwen's example of a countable maximal space (see [8, Example 3.3]) shows that practically all convergence properties fail to be discretely reflexive. We will see in this section that the situation is not so simple in function spaces.

4.1 Example. If we assume the Continuum Hypothesis, then there exists a compact separable non-metrizable space X such that \overline{D} is second countable for any discrete $D \subset C_p(X)$. Therefore, under CH, first countability, metrizability, second countability, countable network weight and the Lindelöf Σ -property are not discretely reflexive in the spaces $C_p(X)$ even if X is compact.

Proof. It follows from Theorem 2.4 of [19] that under CH there exists a locally countable locally compact uncountable space Y such that $hd(Y^n) \leq \omega$ for any $n \in \mathbb{N}$. Let X be the one-point compactification of the space Y and denote by p the unique point of the set $X \setminus Y$. It is easy to see that every compact subspace of Y is countable and therefore $|X \setminus U| \leq \omega$ for any $U \in \tau(p, X)$. This implies that X is functionally countable, i.e., every continuous second countable image of X is countable.

Given a countable set $A \subset C_p(X)$ let $\varphi : X \rightarrow C_p(A)$ be the reflection map. If $Z = \varphi(X)$ then the dual map $\varphi^* : C_p(Z) \rightarrow C_p(X)$ is a closed embedding of the space $C_p(Z)$ in $C_p(X)$ (see [17, Problem 163]). It is easy to see that $A \subset E = \varphi^*(C_p(Z))$ and hence $\overline{A} \subset E$. As a consequence, $w(\overline{A}) \leq w(E) = w(C_p(Z)) = |Z| \leq \omega$; this proves that the closure of every countable subset of $C_p(X)$ is second countable. We also have $hl^*(C_p(X)) = hd^*(X) = \omega$ and therefore $s(C_p(X)) \leq hl(C_p(X)) \leq \omega$, i.e., every discrete subset of $C_p(X)$ is countable. Thus, \overline{D} is second countable for any discrete $D \subset C_p(X)$ so $C_p(X)$ is discretely second countable space which fails to be first countable because $\chi(C_p(X)) = |X| > \omega$. The space $C_p(X)$ is not Lindelöf Σ because X is not ω -monolithic being separable and non-metrizable (see [3, Theorem IV.9.8]). ■

4.2 Theorem. Assume $MA + \neg CH$ and suppose that X is compact.

(a) if \overline{D} is first countable for any discrete set $D \subset C_p(X)$ then X is countable and hence $C_p(X)$ is second countable;

(b) if \overline{D} has a countable network for any discrete set $D \subset C_p(X)$ then X is metrizable and hence $C_p(X)$ has a countable network.

In other words, $MA + \neg CH$ implies that metrizability, first countability, second countability, and countable network weight are discretely reflexive in $C_p(X)$ for any compact space X .

Proof. Let prove first that in both cases, the space X has to be metrizable. If $w(X) > \omega$ and we have either (a) or (b), then there exists a continuous onto map $\varphi : X \rightarrow Y$ such that $w(Y) = \omega_1$ (see [3, Proposition IV.8.11]). Since $C_p(Y)$ embeds in $C_p(X)$, the space $C_p(Y)$ is also discretely first countable (or discretely cosmic respectively). If there exists an uncountable discrete set $D \subset Y \times Y$ then Lemma 3.4 is applicable to see that $C_p(Y, \mathbb{I}) \subset \overline{\Omega}$ for some discrete set $\Omega \subset C_p(Y)$. Therefore $C_p(Y, \mathbb{I})$ is first countable (or cosmic respectively) and hence so is $C_p(Y)$ being embeddable in $C_p(Y, \mathbb{I})$. As an immediate consequence, $|Y| = \chi(C_p(Y)) \leq \omega$

(or $w(Y) = nw(Y) = nw(C_p(Y)) = \omega$ respectively) and hence Y is metrizable which is a contradiction.

It is a theorem of Szentmiklossy [14] that under $\text{MA} + \neg\text{CH}$, every compact space of countable spread is perfectly normal. Thus, if $s(Y \times Y) = \omega$ then we can apply Szentmiklossy's result to conclude that $Y \times Y$ is perfectly normal and hence Δ_Y is a G_δ -subset of $Y \times Y$. This implies that Y is metrizable (see [11, Corollary 7.6]), which is again a contradiction, showing that X must be metrizable, i.e., $w(X) \leq \omega$ and hence we proved (b).

For the case (a) observe that any infinite space has an infinite discrete subspace so there is a countably infinite discrete subspace $D \subset X \times X$ and therefore we can apply Lemma 3.4 again to conclude that $C_p(X, \mathbb{I}) \subset \overline{\Omega}$ for some discrete set $\Omega \subset C_p(X)$. Therefore $C_p(X, \mathbb{I})$ is first countable and hence so is $C_p(X)$ being embeddable in $C_p(X, \mathbb{I})$. This implies that $|X| = \chi(C_p(X)) \leq \omega$. ■

4.3 Corollary. *If \mathcal{P} is a topological property from the list $\mathbb{L} = \{\text{metrizability, first countability, second countability, existence of a countable network}\}$ then it is independent of ZFC whether \mathcal{P} is discretely reflexive in spaces $C_p(X)$ for a compact X .*

4.4 Proposition. *If X is an ω -monolithic compact space and $\psi(\overline{D}) = \omega$ for any discrete $D \subset C_p(X)$ then X is metrizable and hence $\psi(C_p(X)) = \omega$. In other words, countable pseudocharacter is reflexive in $C_p(X)$ for compact ω -monolithic spaces X .*

Proof. If X is not metrizable then there exists a continuous onto map $\varphi : X \rightarrow Y$ where $w(Y) = \omega_1$. Since $C_p(Y)$ embeds in $C_p(X)$, the closure of every discrete subspace of $C_p(Y)$ also has countable pseudocharacter. It is easy to see that Y is also ω -monolithic and hence non-separable.

If there exists an uncountable discrete set $D \subset Y \times Y$ then Lemma 3.4 is applicable to see that $C_p(Y, \mathbb{I}) \subset \overline{\Omega}$ for some discrete set $\Omega \subset C_p(Y)$. Therefore the space $C_p(Y, \mathbb{I})$ has countable pseudocharacter and hence so has $C_p(Y)$ being embeddable in $C_p(Y, \mathbb{I})$. This implies $d(Y) = \psi(C_p(Y)) \leq \omega$; therefore Y is separable and hence metrizable which is a contradiction.

Therefore we can assume that $s(Y \times Y) \leq \omega$; this, together with ω -monolithicity of the space $Y \times Y$ implies that $hl(Y \times Y) = \omega$ (see Proposition 7 of [5]) and hence the diagonal of the space Y is a G_δ -subset of $Y \times Y$. As a consequence, Y is metrizable by [11, Corollary 7.6], a contradiction. ■

4.5 Proposition. *Assume that X^n is Lindelöf for any $n \in \mathbb{N}$. Then*

- (a) *the Lindelöf property is reflexive in $C_p(X)$;*
- (b) *the Fréchet–Urysohn property is reflexive in $C_p(X)$.*

Proof. Observe first that have $t(C_p(X)) = \omega$ by [17, Problem 149]); if $C_p(X)$ is discretely Lindelöf then we can apply Proposition 3.2 of [6] to see that it is Lindelöf. This settles (a).

(b) Suppose that $C_p(X)$ is discretely Fréchet–Urysohn; since it embeds in $C_p(X, \mathbb{I})$, it suffices to prove that $C_p(X, \mathbb{I})$ is Fréchet–Urysohn. Given a set $A \subset C_p(X, \mathbb{I})$ and $f \in \overline{A}$ we can find a countable set $B \subset A$ such that $x \in \overline{B}$.

Since every infinite space has an infinite discrete subspace, we can find a countably infinite discrete set $D \subset X \times X$. By Theorem 3.5, there exists a discrete set $\Omega \subset C_p(X)$ such that $B \subset \overline{\Omega}$ and therefore $B \cup \{f\} \subset \overline{D}$. The space

$\overline{\Omega}$ is Fréchet–Urysohn so $B \cup \{f\}$ is also Fréchet–Urysohn; this shows that there exists a sequence $S = \{f_n : n \in \omega\} \subset B$ which converges to f . Therefore the sequence $S \subset A$ witnesses the Fréchet–Urysohn property of the space $C_p(X)$. ■

4.6 Corollary. *If X is compact then both Lindelöfness and the Fréchet–Urysohn property are discretely reflexive in $C_p(X)$.*

4.7 Example. It is consistent with ZFC that there exists a countably compact non-compact space X such that \overline{D} has countable network for any discrete set $D \subset C_p(X)$. In particular, $t(\overline{D}) = \omega$ for any discrete $D \subset C_p(X)$ but $t(C_p(X)) > \omega$, i.e., it is consistent with ZFC that tightness is not discretely reflexive in spaces $C_p(X)$.

Proof. Juhasz proved in [12, 4.10] that it is consistent with ZFC that there exists a countably compact strong S -space X . Since X is not Lindelöf, we have $t(C_p(X)) > \omega$. On the other hand, $s(C_p(X)) \leq hl(C_p(X)) \leq hd^*(X) = \omega$ which shows that every discrete subset of $C_p(X)$ is countable. It follows from countable compactness of X that \overline{A} has a countable network for any countable $A \subset C_p(X)$ (see Proposition II.6.2 and Theorem II.6.8 of [3]). Therefore $nw(\overline{D}) \leq \omega$ and hence $t(\overline{D}) \leq \omega$ for any discrete $D \subset C_p(X)$. Thus, tightness is not discretely reflexive in $C_p(X)$. ■

5 Discrete Čech-completeness in $C_p(X)$

It is a classical theorem of Lutzer and McCoy that X must be countable and discrete if $C_p(X)$ is Čech-complete (see [13, Theorem 8.6]). The main result of this section is to show that the same conclusion can be obtained if we assume that \overline{D} is Čech-complete for every discrete $D \subset C_p(X)$.

5.1 Lemma. *If $C_p(X)$ is discretely Čech-complete then the closure of every countable subset of $C_p(X)$ is metrizable and Čech-complete.*

Proof. Given a countable set $A \subset C_p(X)$ let $\varphi : X \rightarrow C_p(A)$ be the reflection map. If $Y = \varphi(X)$ then the dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ is an embedding of $C_p(Y)$ in $C_p(X)$ (see [17, Problem 163]). It is easy to see that we have the inclusion $A \subset Z = \varphi^*(C_p(Y))$. Apply Proposition 0.4.9 of [3] to find an \mathbb{R} -quotient map $\xi : X \rightarrow X'$ and a condensation $\pi : X' \rightarrow Y$ such that $\pi \circ \xi = \varphi$.

The dual map $\xi^* : C_p(X') \rightarrow C_p(X)$ embeds $C_p(X')$ in $C_p(X)$ and the set $Q = \xi^*(C_p(X'))$ is closed in $C_p(X)$ by [17, Problem 163]. Therefore Q is discretely Čech-complete and hence so is $C_p(X')$. Observe that X' condenses onto Y and $w(Y) \leq \omega$ so $d(C_p(X')) = iw(X') \leq \omega$ by [17, Problem 173], i.e., $C_p(X')$ is separable.

Fix a point $a \in X'$ and consider the set $E = \{f \in C_p(X') : f(a) = 0\}$. Note that $C_p(X')$ is homeomorphic to $E \times \mathbb{R}$ by [17, Problem 182] and E is separable being a continuous image of $C_p(X')$. The set $A(\omega)$ is a convergent sequence so it embeds in \mathbb{R} and hence $E \times A(\omega)$ is a closed subset of $C_p(X')$ so it is discretely Čech-complete.

Apply Lemma 3.6 to find a discrete subspace of $D \subset E \times A(\omega)$ such that $E \times \{p\} \subset \overline{D}$. The set \overline{D} is Čech-complete, so the space $E \times \{p\}$ is Čech-complete as well being a closed subspace of \overline{D} . Therefore the space E is Čech-complete and hence so is $C_p(X')$ because Čech-completeness is countably productive. Next apply [17, Problem 265] to see that X' countable and discrete which shows that $C_p(X')$ is second countable.

Finally observe that $A \subset Z \subset Q$ and hence $\overline{A} \subset \overline{Z} \subset \overline{Q} = Q$ is a closed subset of $Q \simeq C_p(X')$ so it is metrizable and Čech-complete. ■

5.2 Lemma. *Suppose that $C_p(X)$ is discretely Čech-complete and $\varphi : X \rightarrow Y$ is a continuous onto map such that $w(Y) \leq \omega$. Then there exists an \mathbb{R} -quotient map $\xi : X \rightarrow X'$ and a condensation $\pi : X' \rightarrow Y$ such that the space X' is countable, discrete and $\pi \circ \xi = \varphi$.*

Proof. Apply Proposition 0.4.9 of [3] to find an \mathbb{R} -quotient map $\xi : X \rightarrow X'$ and a condensation $\pi : X' \rightarrow Y$ such that $\pi \circ \xi = \varphi$. Observe that $C_p(X')$ embeds in $C_p(X)$ as a closed subspace so it must be discretely Čech-complete. Since X' condenses onto Y , we have $d(C_p(X')) = iw(X') = \omega$ which shows that $C_p(X')$ is separable and hence Čech-complete by Lemma 5.1. This implies that X' is countable and discrete (see Problem 265 of [17]). ■

5.3 Lemma. *Suppose that $C_p(X)$ is discretely Čech-complete. Then*

- (a) *X is functionally countable, i.e., every second countable continuous image of X is countable;*
- (b) *every countable set $A \subset X$ is closed and discrete in X .*

Proof. It follows from Lemma 5.2 that any second countable continuous image Y of the space X is a continuous image of a countable space so $|Y| \leq \omega$, i.e., we settled (a).

To prove (b) take any countable set $A \subset X$ and assume that there exists a point $x \in \overline{A} \setminus A$. It is easy to find a continuous map $\varphi : X \rightarrow Y$ such that $w(Y) \leq \omega$ and $\varphi|_{(A \cup \{x\})}$ is injective. Lemma 5.2 guarantees that we can find a discrete countable space X' together with a continuous map $\xi : X \rightarrow X'$ and an injection $\pi : X' \rightarrow Y$ such that $\varphi = \pi \circ \xi$. It follows from our choice of φ that $\xi|_{(A \cup \{x\})}$ is an injection and hence $\xi(x) \notin \xi(A)$; however, $x \in \overline{A}$ implies that $\xi(x) \in \overline{\xi(A)} = \xi(A)$ which is a contradiction. Therefore every countable set is closed in X so if $A \subset X$ is countable then all subsets of A are closed in X , i.e., A is closed and discrete. ■

5.4 Lemma. *Assume that there exists a space X such that $C_p(X)$ is discretely Čech-complete but not Čech-complete. Then it is possible to find a space X' such that $C_p(X')$ is discretely Čech-complete, fails to be Čech-complete and $d(C_p(X')) \leq \omega_1$.*

Proof. Since $C_p(X)$ is discretely Čech-complete but not Čech-complete, it is not separable by Lemma 5.1 so we can find a left-separated subspace $A \subset C_p(X)$ with $|A| = \omega_1$. Let $\varphi : X \rightarrow C_p(A)$ be the reflection map. If $Y = \varphi(X)$ then the dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ is an embedding of $C_p(Y)$ in $C_p(X)$ (see [17, Problem 163]). It is easy to see that $A \subset Z = \varphi^*(C_p(Y))$. Apply Proposition

0.4.9 of [3] to find an \mathbb{R} -quotient map $\xi : X \rightarrow X'$ and a condensation $\pi : X' \rightarrow Y$ such that $\pi \circ \xi = \varphi$.

The dual map $\xi^* : C_p(X') \rightarrow C_p(X)$ embeds $C_p(X')$ in $C_p(X)$ and the set $Q = \xi^*(C_p(X'))$ is closed in $C_p(X)$ by [17, Problem 163]. Therefore Q is discretely Čech-complete and hence so is $C_p(X')$. Observe that X' condenses onto Y and $w(Y) \leq \omega_1$ so $d(C_p(X')) = iw(X') \leq \omega_1$ by [17, Problem 173]. It follows from $A \subset Z \subset Q$ that Q is not hereditarily separable so $C_p(X')$ is not hereditarily separable as well. This implies that $C_p(X')$ is not second countable so it is not Čech-complete by [17, Problem 265]. ■

5.5 Theorem. *If $C_p(X)$ is discretely Čech-complete then it is Čech-complete and hence X is discrete and countable.*

Proof. If this is not true then Lemma 5.4 shows that we can assume, without loss of generality, that X is a counterexample and $iw(X) = d(C_p(X)) \leq \omega_1$. If there exists an uncountable discrete $D \subset X \times X$ then we can apply Lemma 3.4 to convince ourselves that $C_p(X, \mathbb{I}) \subset \overline{\Omega}$ for some discrete set $\Omega \subset C_p(X)$. The set $\overline{\Omega}$ is Čech-complete and hence so is $C_p(X, \mathbb{I})$ being closed in $\overline{\Omega}$. Therefore X is discrete by [17, Problem 287].

If X is uncountable then $C_p(X) = \mathbb{R}^X$ and hence \mathbb{R}^{ω_1} embeds in $C_p(X)$ as a closed subspace. However, Lemma 5.1 implies that $\mathbb{R}^{\omega_1} = C_p(D(\omega_1))$ is not discretely Čech-complete because it is separable and non-metrizable. Here $D(\omega_1)$ is a discrete space of cardinality ω_1 . This contradiction shows that X must be countable.

Now assume that the space $X \times X$ has no uncountable discrete subset, i.e., $s(X \times X) \leq \omega$. It follows from Lemma 5.3 that $X \times X$ is ω -monolithic (we actually have a much stronger property, namely that every countable subset of $X \times X$ is closed and discrete) so Proposition 7 of the paper [5] can be applied to see that $X \times X$ is hereditarily Lindelöf and hence Δ_X is a G_δ -subset of X . Applying Theorem 2.1.8 of [2] we conclude that $iw(X) \leq \omega$ and hence $C_p(X)$ is separable. Now Lemma 5.1 shows that $C_p(X)$ is metrizable and Čech-complete whence X is discrete and countable by [17, Problem 265]. ■

It is an old theorem (see, e.g., [3, Theorem I.2.1]) that σ -compactness of $C_p(X)$ implies that X is finite. It turns out that it is possible to weaken the assumption to discrete σ -compactness of $C_p(X)$ to obtain the same conclusion.

5.6 Theorem. *The space $C_p(X)$ is discretely σ -compact if and only if X is finite.*

Proof. Take any $\varphi \in C_p(X)$ and let $Y = \varphi(X)$. Apply Proposition 0.4.9 of [3] to find an \mathbb{R} -quotient map $\xi : X \rightarrow Y'$ and a condensation $\pi : Y' \rightarrow Y$ such that $\pi \circ \xi = \varphi$. The dual map $\xi^* : C_p(Y') \rightarrow C_p(X)$ is an embedding and the set $Z = \xi^*(C_p(Y'))$ is discretely σ -compact being closed in $C_p(X)$. Therefore $C_p(Y')$ is also discretely σ -compact. Since Y' condenses onto Y and $w(Y) \leq \omega$, we have $iw(Y') \leq \omega$. Any infinite space has an infinite discrete subspace so we can find an infinite discrete set $D \subset Y' \times Y'$.

It follows from $|D| \geq \omega \geq iw(Y')$ that we can apply Lemma 3.4 to see that there is a discrete $\Omega \subset C_p(Y')$ such that $C_p(Y', \mathbb{I}) \subset \overline{\Omega}$ and hence $C_p(Y', \mathbb{I})$ is

σ -compact being closed in $\overline{\Omega}$. By [17, Problem 396], the space Y' is discrete and therefore $C_p(Y') = \mathbb{R}^\lambda$ where $\lambda = |Y'|$. If Y' is infinite then \mathbb{R}^ω embeds in \mathbb{R}^λ as a closed subset so \mathbb{R}^ω is discretely σ -compact.

Note that the space $A(\omega)$ embeds in \mathbb{R} as a closed subset and therefore the space $A(\omega) \times \mathbb{R}^\omega$ is a closed subset of $\mathbb{R} \times \mathbb{R}^\omega \simeq \mathbb{R}^\omega$ which shows that $A(\omega) \times \mathbb{R}^\omega$ is discretely σ -compact as well. Denote by p the unique non-isolated point of $A(\omega)$ and apply Lemma 3.6 to see that $\mathbb{R}^\omega \times \{p\} \subset \overline{D}$ for some discrete $D \subset A(\omega) \times \mathbb{R}^\omega$. Since $\mathbb{R}^\omega \times \{p\}$ is closed in \overline{D} , it must be σ -compact which is a contradiction.

Therefore Y' is finite and hence so is Y , i.e., we proved that $\varphi(X)$ is finite for any $\varphi \in C_p(X)$. Now, Lemma 2.6 of [10] shows that X must be finite. ■

6 Open problems

Discrete reflexivity in function spaces turned out to be an interesting topic with a potential to provide new information about classical convergence properties in $C_p(X)$. We hope that the following list of open questions can convince the reader that there are still a lot of nice facts to be discovered.

6.1 Question. *Does there exist in ZFC a non-compact space X such that $nw(X) > \omega$ while $nw(\overline{D}) \leq \omega$ for any discrete set $D \subset C_p(X)$?*

6.2 Question. *Does there exist in ZFC a non-compact uncountable space X such that $\chi(\overline{D}) \leq \omega$ for any discrete set $D \subset C_p(X)$?*

6.3 Question. *Suppose that \overline{D} is normal for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ be normal?*

6.4 Question. *Suppose that the set \overline{D} is hereditarily normal for any discrete $D \subset C_p(X)$. Must the space $C_p(X)$ be normal?*

6.5 Question. *Suppose that \overline{D} has countable pseudocharacter for any discrete set $D \subset C_p(X)$. Must the space X be separable, or, equivalently, is it true that $\psi(C_p(X)) = \omega$?*

6.6 Question. *Suppose that \overline{D} has the Fréchet–Urysohn property for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ have the Fréchet–Urysohn property?*

6.7 Question. *Assume that \overline{D} is sequential for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ have the Fréchet–Urysohn property?*

6.8 Question. *Assume that \overline{D} is a k -space for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ have the Fréchet–Urysohn property?*

6.9 Question. *Assume that \overline{D} is realcompact for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ be realcompact?*

6.10 Question. *Assume that \overline{D} is Lindelöf for any discrete set $D \subset C_p(X)$. Must the space $C_p(X)$ be realcompact?*

6.11 Question. *Is it consistent with ZFC that countable tightness is discretely reflexive in spaces $C_p(X)$?*

6.12 Question. *Is there a ZFC example of a compact space X such that \overline{D} has the Lindelöf Σ -property for any discrete subspace $D \subset C_p(X)$ but the space $C_p(X)$ is not Lindelöf Σ ?*

6.13 Question. *Suppose that X is compact and \overline{D} is K -analytic for any discrete subspace $D \subset C_p(X)$. Must the space $C_p(X)$ be K -analytic?*

6.14 Question. *Suppose that the subspace \overline{D} is Lindelöf for any discrete set $D \subset C_p(X) \times C_p(X)$. Must $C_p(X)$ be Lindelöf?*

6.15 Question. *Given a space X , denote by Δ the diagonal of the space $C_p(X)$ and assume that $(C_p(X) \times C_p(X)) \setminus \Delta$ is discretely Lindelöf. Must $C_p(X)$ be Lindelöf? Must X be separable?*

References

- [1] O. Alas, V.V. Tkachuk, R.G. Wilson, *Closures of discrete sets often reflect global properties*, *Topology Proc.*, **25**(2000), 27–44.
- [2] A.V. Arhangel'skii, *The structure and classification of topological spaces and cardinal invariants (in Russian)*, *Uspehi Matem. Nauk*, **33:6**(1978), 29–84.
- [3] A.V. Arhangel'skii, *Topological Function Spaces*, Kluwer Academic Publishers, Mathematics and Its Applications 78, Dordrecht 1992.
- [4] A.V. Arhangel'skii, *A generic theorem in the theory of cardinal invariants of topological spaces*, *Comment. Math. Univ. Carolin.*, **36:2**(1995), 303–325.
- [5] A.V. Arhangel'skii, *On Lindelöf property and spread in C_p -theory*, *Topology Appl.*, **74:1**(1996), 83–90.
- [6] A.V. Arhangel'skii, R.Z. Buzyakova, *On linearly Lindelöf and strongly discretely Lindelöf spaces*, *Proc. Amer. Math. Soc.*, **127:8**(1999), 2449–2458.
- [7] D. Burke, V.V. Tkachuk, *Discrete reflexivity and complements of the diagonal*, *Acta Math. Hungar.* **139**(2013), 120–133.
- [8] E.K. van Douwen, *Applications of maximal topologies*, *Topology Appl.*, **51**(1993), 125–139.
- [9] R. Engelking, *General Topology*, PWN, Warszawa, 1977.
- [10] D. Guerrero Sánchez, *Closure-preserving covers in function spaces*, *Comment. Math. Univ. Carolinae*, **51:4**(2012), 693–703.
- [11] R.E. Hodel, *Cardinal Functions I*, *Handbook of Set-Theoretic Topology*, ed. by K. Kunen and J.E. Vaughan, North Holland, Amsterdam, 1984, 1–61.

- [12] I. Juhász, *HFD and HFC type spaces, with applications*, *Topol. Appl.*, **126**(2002), 217–262.
- [13] D.J. Lutzer, R.A. McCoy, *Category in function spaces, I*, *Pacific J. Math.*, **90:1**(1980), 145–168.
- [14] Z. Szentmiklossy, *S-spaces and L-spaces under Martin's axiom*, *Coll. Math. Soc. Janos Bolyai*, **23**, Budapest, 1978, 1139–1145.
- [15] V.V. Tkachuk, *Spaces that are projective with respect to classes of mappings*, *Trans. Moscow Math. Soc.*, **50**(1988), 139–156.
- [16] V.V. Tkachuk, *Function spaces and d -separability*, *Quaestiones Math.*, **28**(2005), 409–424.
- [17] V.V. Tkachuk, *A C_p -theory Problem Book. Topological and Function Spaces*, Springer, New York, 2011.
- [18] V.V. Tkachuk, R.G. Wilson, *Discrete reflexivity in GO spaces*, submitted for publication.
- [19] S. Todorčević, *Partition Problems in Topology*, *Contemporary Math.*, **84**, Amer. Math. Soc., Providence, RI, 1989.

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