Limit and end functors of dynamical systems via exterior spaces*

J. M. García Calcines L. J. Hernández Paricio
M. T. Rivas Rodríguez

Abstract

In this paper we analyze some applications of the category of exterior spaces to the study of dynamical systems (flows). We study the notion of an absorbing open subset of a dynamical system; i.e., an open subset that contains the "future part" of all the trajectories. The family of all absorbing open subsets is a quasi-filter which gives the structure of an exterior space to the flow. The limit space and end space of an exterior space are used to construct the limit spaces and end spaces of a dynamical system. On the one hand, for a dynamical system two limits spaces $L^{\mathbf{r}}(X)$ and $\bar{L}^{\mathbf{r}}(X)$ are constructed and their relations with the subflows of periodic, Poisson stable points and Ω-limits of X are analyzed. On the other hand, different end spaces are also associated to a dynamical system having the property that any positive semi-trajectory has an end point in these end spaces. This type of construction permits us to consider the subflow containing all trajectories finishing at an end point a. When a runs over the set of all end points, we have an induced decomposition of a dynamical system as a disjoint union of stable (at infinity) subflows.

*Partially supported by Ministerio de Educación y Ciencia grant MTM2009-12081, University of La Rioja project API12/10 and FEDER

Received by the editors in December 2012 - In revised form in April 2013.

Communicated by E. Colebunders.

2010 *Mathematics Subject Classification :* 18B99, 18A40, 37B99, 54H20.

Key words and phrases : Dynamical system, exterior space, exterior flow, limit space functor, end space functor, positively Poisson stable point.

1 Introduction

It is well known the importance of the purely topological behavior of a continuous dynamical system, i.e., an additive action $\mathbb{R} \times X \to X$ in many, radically different in principle, situations (differential equations, non linear analysis, transformation groups, et cetera). On the other hand, exterior homotopy theory ([2, 5, 7, 9]) has also proven to be very useful in the study of topological aspects of several settings such as proper homotopy theory and its numerical invariants or shape theory. In this paper we describe some basic ideas that permit a new treatment of the study of dynamical systems under an exterior approach. The key to establish such a link is the notion of absorbing open region. Given a flow $\mathbb{R} \times X \to X$ on a topological space X, an open set E is said to be $\mathbf{r}$-exterior if the trajectory of every point of X is, from some time on, totally contained in E, that is, for any $x \in X$, there is $t_0 \in \mathbb{R}$ such that $t \cdot x \in E$ for every $t \geq t_0$. This way X has the structure of an exterior space. Using exterior spaces we construct two limit subflows $L^{\mathbf{r}}(X)$ and $\bar{L}^{\mathbf{r}}(X)$ associated with a flow X. One of the important results of our study (see Corollary 6.14) is the following:

If X is a locally compact T_3 flow, then $L^{\mathbf{r}}(X) = P(X)$ and $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega(X)}$; furthermore we have that

$$L^{\mathbf{r}}(X) = P(X) \subset \mathrm{Poisson}(X) \subset \Omega(X) \subset \overline{\Omega(X)} = \bar{L}^{\mathbf{r}}(X),$$

where $P(X)$ is the subflow of periodic points, $\mathrm{Poisson}(X)$ is the subflow of positively Poisson stable points and $\Omega(X) = \bigcup_{x \in X} \omega(x)$ (being $\omega(x)$ the omega limit set of the point $x \in X$). Under the condition of being regular at infinity (see Proposition 3.14), we have that $L^{\mathbf{r}}(X) = \bar{L}^{\mathbf{r}}(X)$ and in some cases we can also ensure that $\bar{L}^{\mathbf{r}}(X)$ is compact. The constructions $L^{\mathbf{r}}, \bar{L}^{\mathbf{r}}$ induce classifications (for flows) of the following type: Two flows X, Y are said to be $\bar{L}^{\mathbf{r}}$-equivalent if there is a flow morphism $f \colon X \to Y$ such that $\bar{L}^{\mathbf{r}}(f)$ is a homotopy equivalence. This will determine equivalence classes of flows having, for instance, the same number of critical points or the 'same type' of periodic trajectories.

Besides, with the use again of exterior spaces, we construct 'slightly different' end spaces $\check{\pi}_0^{\mathbf{r}}(X)$, $\check{c}^{\mathbf{r}}(X)$, $\check{\bar{\pi}}_0{}^{\mathbf{r}}(X)$, $\check{\bar{c}}^{\mathbf{r}}(X)$, which coincide when X is locally path-connected and regular at infinity. In this case, the end spaces have a pro-discrete topology (with additional conditions, a pro-finite topology). The importance of this end space is that each right semi-trajectory of the flow has an end point in this space. This fact allows one to give a set map $\bar{\chi} \colon X \to \check{\bar{c}}^{\mathbf{r}}(X)$ and the corresponding $\bar{\chi}$-decomposition of the flow

$$X = \bigsqcup_{a \in \check{\bar{c}}^{\mathbf{r}}(X)} \bar{X}_{(\mathbf{r},a)}.$$

In general, there will be end points which are not reached by right semi-trajectories and end points of semi-trajectories that are not reached by right semi-trajectories contained in the limit subflow. In this paper, we give sufficient conditions to ensure that an end point can be reached by right semi-trajectories of $\bar{L}^{\mathbf{r}}(X)$ (see Proposition 4.8). These end spaces will be used (not in this work) to construct completions (compactification under some topological and dynamical conditions) of a flow. These completions are related to Freundenthal compactifications [4, 6, 12] and will permit us to apply some nice properties of compact flows to a more general class of topological flows. It is important to observe that applying the results of this paper to the reversed flow we will obtain all the corresponding dual concepts, constructions and properties.

The authors would like to thank the referee for his/her thorough review and useful comments that helped to improve this paper.

2 Preliminaries on exterior spaces and dynamical systems

2.1 The categories of proper and exterior spaces

A continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non-compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions such as loops, homotopy limits and colimits, et cetera. An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps, see [7, 9]. We refer to [2, 5, 8, 10, 11] for further properties and applications of exterior homotopy, and to [18] for a survey of proper homotopy.

Definition 2.1. Let $(X, \mathbf{t})$ be a topological space, where X is the underlying set and $\mathbf{t}$ its topology. An *externology* on $(X, \mathbf{t})$ is a non-empty collection ε (also denoted by $\varepsilon(X)$) of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \mathbf{t}$ and $E \subset U$ then $U \in \varepsilon$. The members of ε are called exterior open subsets. An *exterior space* $(X, \varepsilon, \mathbf{t})$ consists of a space $(X, \mathbf{t})$ together with an externology ε. Given an exterior space $(X, \varepsilon, \mathbf{t})$ it is useful to work with an *exterior base* (or just a *base* of the externology), which is nothing else than a subcollection $\beta \subset \varepsilon$ such that for every $E \in \varepsilon$ there exists $F \in \beta$ with $F \subset E$. A map $f : (X, \varepsilon, \mathbf{t}) \to (X', \varepsilon', \mathbf{t}')$ is said to be an *exterior map* if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.

The category of exterior spaces and maps will be denoted by **E**. Given a space $(X, \mathbf{t}_X)$, we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ or the total exterior space if one takes $\varepsilon = \mathbf{t}_X$. An important example of externology on a given topological space X is the one constituted by the complements of all closed-compact subsets of X, that will be called the cocompact externology and usually written as $\varepsilon^{\mathbf{c}}(X)$. The category of spaces and proper maps can be considered as a full subcategory of the category of exterior spaces via the full embedding $(\cdot)^{\mathbf{c}} : \mathbf{P} \hookrightarrow \mathbf{E}$. The functor $(\cdot)^{\mathbf{c}}$ carries a space X to the exterior space $X^{\mathbf{c}}$ which is provided with the topology of X and the externology $\varepsilon^{\mathbf{c}}(X)$. A map $f\colon X \to Y$ is carried to the exterior map $f^{\mathbf{c}}\colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ given by $f^{\mathbf{c}} = f$. It is easy to check that a continuous map $f : X \to Y$ is proper if and only if $f = f^{\mathbf{c}}\colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ is exterior.

An important role in this paper will be played by the following construction $(\cdot)\bar{\times}(\cdot)$: Let $(X,\varepsilon(X),\mathbf{t}_X)$ be an exterior space, $(Y,\mathbf{t}_Y)$ a topological space and for $y \in Y$ we denote by $(\mathbf{t}_Y)_y$ the family of open neighborhoods of Y at y. We consider on $X \times Y$ the product topology $\mathbf{t}_{X\times Y}$ and the externology $\varepsilon(X\bar{\times}Y)$ given by those $E \in \mathbf{t}_{X\times Y}$ such that for each $y \in Y$ there exist $U_y \in (\mathbf{t}_Y)_y$ and $T^y \in \varepsilon(X)$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X\bar{\times}Y$ in order to avoid a possible confusion with the product externology. This construction gives a functor

$$(\cdot)\bar{\times}(\cdot)\colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}.$$

When Y is a compact space, we have that E is an exterior open subset if and only if it is an open subset and there exists $G \in \varepsilon(X)$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon(X) = \varepsilon^{\mathbf{c}}(X)$, then $\varepsilon(X\bar{\times}Y)$ coincides with $\varepsilon^{\mathbf{c}}(X \times Y)$ the externology of the complements of closed-compact subsets of $X \times Y$. We also note that if Y is a discrete space, then E is an exterior open subset if and only if it is open and for each $y \in Y$ there is $T^y \in \varepsilon(X)$ such that $T^y \times \{y\} \subset E$.

This bar construction provides a natural way to define *exterior homotopy* in $\mathbf{E}$. Indeed, if I denotes the closed unit interval, given exterior maps $f, g : X \to Y$, it is said that *f is exterior homotopic to g* if there exists an exterior map $H : X\bar{\times}I \to Y$ (called *exterior homotopy*) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$. The corresponding homotopy category of exterior spaces will be denoted by $\pi\mathbf{E}$. Similarly, the usual homotopy category of topological spaces will be denoted by $\pi\mathbf{Top}$.

2.2 Dynamical Systems and Ω-Limits

Next we recall some elementary concepts about dynamical systems.

Definition 2.2. A *dynamical system* (or *flow*) on a topological space X is a continuous map $\varphi\colon \mathbb{R}\times X \to X$, $\varphi(t,x) = t\cdot x$, such that

(i) $0 \cdot x = x, \forall x \in X$;

(ii) $t \cdot (s \cdot x) = (t+s)\cdot x, \forall x \in X, \forall t, s \in \mathbb{R}$.

A flow on X will be denoted by (X,φ) and when no confusion is possible, we use X for short.

For a subset $A \subset X$, we denote $\mathrm{inv}(A) = \{x \in A \,|\, \mathbb{R}\cdot x \subset A\}$.

Definition 2.3. A subset S of a flow X is said to be *invariant* if $\mathrm{inv}(S) = S$.

Given a flow $\varphi\colon \mathbb{R}\times X \to X$ one has a subgroup $\{\varphi_t\colon X \to X \,|\, t \in \mathbb{R}\}$ of homeomorphisms, $\varphi_t(x) = \varphi(t,x)$, and a family of motions $\{\varphi^x\colon \mathbb{R} \to X \,|\, x \in X\}$, $\varphi^x(t) = \varphi(t,x)$.

Definition 2.4. Given two flows $\varphi\colon \mathbb{R}\times X \to X$, $\psi\colon \mathbb{R}\times Y \to Y$, a *flow morphism* $f\colon (X,\varphi) \to (Y,\psi)$ is a continuous map $f\colon X \to Y$ such that $f(t\cdot x) = t\cdot f(x)$ for every $t \in \mathbb{R}$ and for every $x \in X$.

We note that if $S \subset X$ is invariant, S has a flow structure and the inclusion is a flow morphism. We denote by **F** the category of flows and flow morphisms.

We recall some basic fundamental examples: (1) $X = \mathbb{R}$ with the action $\varphi\colon \mathbb{R} \times X \to X$, $\varphi(t,s) = t+s$. (2) $X = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with $\varphi\colon \mathbb{R} \times X \to X$, $\varphi(t,z) = e^{2\pi i t} z$. (3) $X = \{0\}$ with the trivial action $\varphi\colon \mathbb{R} \times X \to X$ given by $\varphi(t,0) = 0$. In all these cases, the flows only have one trajectory.

Definition 2.5. For a flow X, the *ω-limit set* (or right-limit set, or positive limit set) of a point $x \in X$ is given as follows:

$$\omega(x) = \{y \in X \mid \exists \text{ a net } t_\delta \to +\infty \text{ such that } t_\delta \cdot x \to y\}.$$

If $\overline{A}$ denotes the closure of a subset A of a topological space, we note that the subset $\omega(x)$ admits the alternative definition

$$\omega(x) = \bigcap_{t \geq 0} \overline{[t,+\infty) \cdot x}$$

which has the advantage of showing that $\omega(x)$ is closed.

Definition 2.6. The *Ω-limit set* of a flow X is the following invariant subset:

$$\Omega(X) = \bigcup_{x \in X} \omega(x)$$

Now we introduce the basic notions of critical, periodic and Poisson stable points.

Definition 2.7. Let X be a flow. A point $x \in X$ is said to be a *critical point* (or *rest point*, or *equilibrium point*) if for every $t \in \mathbb{R}$, $t \cdot x = x$. We denote by $C(X)$ the invariant subset of critical points of X.

Definition 2.8. Let X be a flow. A point $x \in X$ is said to be *periodic* if there is $t \in \mathbb{R}$, $t \neq 0$ such that $t \cdot x = x$. We denote by $P(X)$ the invariant subset of periodic points of X.

It is clear that a critical point is a periodic point. Then

$$C(X) \subset P(X).$$

If $x \in X$ is a periodic point but not critical, then there is a real $t \neq 0$ such that $t \cdot x = x$ and t is called a *period* of x. The smallest positive period t_0 of x is called the *fundamental period* of x.

Definition 2.9. Let (X,φ) be a flow. A point $x \in X$ is said to be *positively Poisson stable* if there is a net $t_\delta \to +\infty$ such that $t_\delta \cdot x \to x$; that is, $x \in \omega(x)$. We will denote by $\mathrm{Poisson}(X)$ the invariant subset of positively Poisson stable points of X.

The reader can easily check that

$$P(X) \subset \mathrm{Poisson}(X) \subset \Omega(X).$$

The notions above can be dualized by considering the notions of α-limit set of a point x, negatively Poisson stable points, et cetera.

Remark 2.10. *Observe that when X satisfies the first axiom of countability (for instance, when X is metrizable) we can consider sequences instead of nets in definitions 2.5 and 2.9.*

3 End Spaces and Limit Spaces of an exterior space

In this section we will deal with special limit constructions associated to any exterior space.

3.1 The functors $L, \check{\pi}_0, \check{c} \colon \mathbf{E} \to \mathbf{Top}$

Given an exterior space $X = (X, \varepsilon(X))$, its externology $\varepsilon(X)$ is an inverse system of spaces. Then we define the limit space of $(X, \varepsilon(X))$ as the topological space

$$L(X) = \lim \varepsilon(X).$$

Note that for each $E' \in \varepsilon(X)$ the canonical map $\lim \varepsilon(X) \to E'$ is continuous and factorizes as $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E \to E'$. Therefore the canonical map $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E$ is continuous. Moreover, by the universal property of the limit, the family of maps $\cap_{E \in \varepsilon(X)} E \to E'$, $E' \in \varepsilon(X)$ induces a continuous map $\cap_{E \in \varepsilon(X)} E \to \lim \varepsilon(X)$. This implies that the canonical map $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E$ defines a natural homeomorphism.

We recall that for a topological space Y, $\pi_0(Y)$ denotes the set of path-components of Y and we have a canonical map $Y \to \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$. Similarly, if $c(Y)$ denotes the set of connected components of a space Y, we have a similar quotient map $Y \to c(Y)$. We remark that if Y is locally path-connected (respect., locally connected), then $\pi_0(Y)$ (respect., $c(Y)$) is a discrete space.

It is also interesting to note that for any topological space Y, there exists a canonical commutative diagram of natural maps:

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ \pi_0(Y) \arrow[rr] & & c(Y) \end{tikzcd}$$

Definition 3.1. Given an exterior space $X = (X, \varepsilon(X))$ the *limit space* of X is the topological subspace

$$L(X) = \lim \varepsilon(X) = \cap_{E \in \varepsilon(X)} E.$$

The *end space* of X is the inverse limit

$$\check{\pi}_0(X) = \lim \pi_0 \varepsilon(X) = \lim_{E \in \varepsilon(X)} \pi_0(E)$$

provided with the inverse limit topology of the spaces $\pi_0(E)$.

The *c-end space* of X is the inverse limit

$$\check{c}(X) = \lim c\, \varepsilon(X) = \lim_{E \in \varepsilon(X)} c(E)$$

provided with the inverse limit topology of the spaces $c(E)$. The elements of $\check{\pi}_0(X)$ or $\check{c}(X)$ will be called *end points* of X.

An end point $a \in \check{\pi}_0(X)$ is represented by the filter base

$$\{U_a^E \mid U_a^E \text{ is a path-component of } E, E \in \varepsilon(X)\}.$$

We note that a locally path-connected exterior space $(X, \varepsilon(X))$ induces the following family of exterior spaces

$$\{(X, \varepsilon(X, a)) \mid a \in \check{\pi}_0(X)\}$$

where $\varepsilon(X, a)$ is the externology generated by the filter base

$$\{U_a^E \mid U_a^E \text{ is a path-component of } E, E \in \varepsilon(X)\}.$$

The end points of $\check{c}(X)$ have similar properties.

It is interesting to observe that if X is an exterior space and X is locally path-connected (respect., locally connected), then $\check{\pi}_0(X)$ (respect., $\check{c}(X)$) is a prodiscrete space. On the other hand, given any exterior space $(X, \varepsilon(X))$, we have a canonical commutative diagram of natural maps

$$\begin{tikzcd} & L(X) \arrow[dl, "e_0"'] \arrow[dr, "e"] & \\ \check{\pi}_0(X) \arrow[rr, "\theta"'] & & \check{c}(X) \end{tikzcd}$$

Definition 3.2. Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\pi}_0(X)$ (respect., $a \in \check{c}(X)$) is said to be *e_0-representable* (respect., *e-representable*) if there is $x \in L(X)$ such that $e_0(x) = a$ (respect., $e(x) = a$). Notice that the maps $e_0 \colon L(X) \to \check{\pi}_0(X), e \colon L(X) \to \check{c}(X)$ induce an e_0-decomposition and an e-decomposition

$$L(X) = \bigsqcup_{a \in \check{\pi}_0(X)} L_a^0(X), \quad L(X) = \bigsqcup_{a \in \check{c}(X)} L_a(X)$$

where $L_a^0(X) = e_0^{-1}(a)$ and $L_a(X) = e^{-1}(a)$. These special subsets will be respectively called the *e_0-component* of the end $a \in \check{\pi}_0(X)$ and the *e-component* of the end $a \in \check{c}(X)$ in the limit $L(X)$.

We denote by $e_0L(X)$ and $eL(X)$ the corresponding subsets of representable end points. It is clear that

$$L(X) = \bigsqcup_{a \in e_0L(X)} L_a^0(X), \quad L(X) = \bigsqcup_{a \in eL(X)} L_a(X)$$

and for $b \in eL(X)$ one has that

$$L_b(X) = \bigsqcup_{a \in (\theta^{-1}(b) \cap e_0L(X))} L_a^0(X).$$

Example 3.3. *Let $M \colon \mathbb{R} \to (0, 1)$ be an increasing continuous map such that $\lim_{s \to -\infty} M(s) = 0$ and $\lim_{s \to +\infty} M(s) = 1$ and take $A = \{e^{2\pi i s} \mid s \in \mathbb{R}\}$, $B = \{M(s)e^{2\pi i s} \mid s \in \mathbb{R}\}$. Consider $X = A \cup B \subset \mathbb{C}$ provided with the relative topology* *(observe that X is not locally connected). On the topological space X the flow $\varphi\colon \mathbb{R} \times X \to X$ is given by $\varphi(t, e^{2\pi is}) = e^{2\pi i(t+s)}$, $\varphi(t, M(s)e^{2\pi is}) = M(t+s)e^{2\pi i(t+s)}$. It is clear that this flow has two trajectories A, B. If for each natural number n we denote $B_n = \{M(s)e^{2\pi is} \mid s \geq n\}$, then a base of an externology on X is given by $\{E_n = A \cup B_n \mid n \in \mathbb{N}\}$. Since A, B_n are path-connected and $\overline{B_n} = E_n$ is connected, it follows that $\pi_0(E_n) = \{A, B_n\}$ and $c(E_n) = \{E_n\}$. Therefore*

$$\check{\pi}_0(X) = \{*_A, *_B\}, \quad \check{c}(X) = \{*\}$$

For this example we have $L(X) = A$, the e_0-decomposition

$$L^0_{*_A} = A, \quad L^0_{*_B} = \emptyset$$

and the e-decomposition $L_* = A$. This means that $*_B$ is not e_0-representable.

It is not difficult to check that the functor L preserves homotopies and the functors $\check{\pi}_0, \check{c}$ are invariant by exterior homotopy.

Lemma 3.4. *Suppose that X and Y are exterior spaces and $f, g\colon X \to Y$ exterior maps.*

(i) If $H\colon X\bar{\times} I \to Y$ is an exterior homotopy from f to g, then $L(H) = H|_{L(X)\times I}\colon L(X\bar{\times} I) = L(X) \times I \to L(Y)$ is a homotopy from $L(f)$ to $L(g)$;

(ii) If f is exterior homotopic to g, then $\check{\pi}_0(f) = \check{\pi}_0(g)$ and $\check{c}(f) = \check{c}(g)$.

As a consequence of this lemma one has:

Proposition 3.5. *The functors $L, \check{\pi}_0, \check{c}\colon \mathbf{E} \to \mathbf{Top}$ induce functors*

$$L\colon \pi\mathbf{E} \to \pi\mathbf{Top}, \qquad \check{\pi}_0, \check{c}\colon \pi\mathbf{E} \to \mathbf{Top}.$$

It is interesting to observe that the functor $L\colon \mathbf{E} \to \mathbf{Top}$ admits in a natural way a left adjoint: Given a topological space X, recall that we can consider on X the trivial externology $\varepsilon^{\mathrm{tr}}(X) = \{X\}$. This construction gives the exterior space $X_{\mathrm{tr}} = (X, \varepsilon^{\mathrm{tr}}(X))$ and induces the canonical functor $(\cdot)_{\mathrm{tr}}\colon \mathbf{Top} \to \mathbf{E}$, $X \mapsto X_{\mathrm{tr}}$. The reader can straightforwardly check the following result:

Proposition 3.6. *The functor $(\cdot)_{\mathrm{tr}}\colon \mathbf{Top} \to \mathbf{E}$ is left adjoint to the functor $L\colon \mathbf{E} \to \mathbf{Top}$. Moreover, this pair of adjoint functors induces an adjunction on the homotopy categories: $(\cdot)_{\mathrm{tr}}\colon \pi\mathbf{Top} \to \pi\mathbf{E}$, $L\colon \pi\mathbf{E} \to \pi\mathbf{Top}$.*

3.2 The functors $\bar{L}, \check{\bar{\pi}}_0, \check{\bar{c}}\colon \mathbf{E} \to \mathbf{Top}$

The externology of an exterior space $X = (X, \varepsilon(X))$ and the closure operator of the underlying topological space induce the following inverse system $\bar{\varepsilon}(X) = \{\overline{E} \mid E \in \varepsilon(X)\}$. Using this new inverse system, we can rewrite notions and analogous results of subsection above as follows:

Definition 3.7. Given an exterior space $X = (X, \varepsilon(X))$ the *bar-limit space* of X is the topological subspace

$$\bar{L}(X) = \lim \bar{\varepsilon}(X) = \cap_{E \in \varepsilon(X)} \overline{E}.$$

The *bar-end space* of X is the inverse limit

$$\check{\bar{\pi}}_0(X) = \lim \pi_0 \bar{\varepsilon}(X) = \lim_{E \in \varepsilon(X)} \pi_0(\overline{E})$$

provided with the inverse limit topology of the spaces $\pi_0(\overline{E})$.

The *c-bar-end space* of X is the inverse limit

$$\check{\bar{c}}(X) = \lim c\, \bar{\varepsilon}(X) = \lim_{E \in \varepsilon(X)} c(\overline{E})$$

provided with the inverse limit topology of the spaces $c(\overline{E})$.

Given any exterior space $X = (X, \varepsilon(X))$, we have a canonical diagram of natural maps

$$\begin{tikzcd}
 & \bar{L}(X) \arrow[dl, "\bar{e}_0"'] \arrow[dr, "\bar{e}"] & \\
\check{\bar{\pi}}_0(X) \arrow[rr, "\bar{\theta}"'] & & \check{\bar{c}}(X)
\end{tikzcd}$$

and there canonical natural maps $L(X) \subset \overline{L(X)} \subset \bar{L}(X)$, $\check{\pi}_0(X) \to \check{\bar{\pi}}_0(X)$, $\check{c}(X) \to \check{\bar{c}}(X)$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & L(X) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\check{\pi}_0(X) \arrow[rr] \arrow[ddd] & & \check{c}(X) \arrow[ddd] \\
 & \bar{L}(X) \arrow[ddl] \arrow[ddr] & \\
 & & \\
\check{\bar{\pi}}_0(X) \arrow[rr] & & \check{\bar{c}}(X)
\end{tikzcd}$$

Definition 3.8. Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\bar{\pi}}_0(X)$ (respect., $a \in \check{\bar{c}}(X)$) is said to be $\bar{e}_0$-*representable* (respect., $\bar{e}$-*representable*) if there is $x \in \bar{L}(X)$ such that $\bar{e}_0(x) = a$ (respect., $\bar{e}(x) = a$). The maps $\bar{e}_0 \colon \bar{L}(X) \to \check{\pi}_0(X), \bar{e} \colon \bar{L}(X) \to \check{c}(X)$ induce an $\bar{e}_0$-*decomposition* and an $\bar{e}$-*decomposition*

$$\bar{L}(X) = \bigsqcup_{a \in \check{\bar{\pi}}_0(X)} \bar{L}^0_a(X), \quad \bar{L}(X) = \bigsqcup_{a \in \check{\bar{c}}(X)} \bar{L}_a(X)$$

where $\bar{L}^0_a(X) = \bar{e}_0^{-1}(a)$ (respect., $\bar{L}_a(X) = \bar{e}^{-1}(a)$) will be called the $\bar{e}_0$-*component* (respect., $\bar{e}$-*component*) of the end $a \in \check{\bar{\pi}}_0(X)$ (respect., $a \in \check{\bar{c}}(X)$) in the limit $\bar{L}(X)$.

We denote by $\bar{e}_0\bar{L}(X)$ and $\bar{e}\bar{L}(X)$ the corresponding subsets of representable end points. It is clear that we have a commutative diagram

$$\begin{tikzcd}
 & L(X) \arrow[dl] \arrow[dr] \arrow[dd] & \\
e_0L(X) \arrow[rr] \arrow[dd] & & eL(X) \arrow[dd] \\
 & \bar{L}(X) \arrow[dl] \arrow[dr] & \\
\bar{e}_0\bar{L}(X) \arrow[rr] & & \bar{e}\bar{L}(X)
\end{tikzcd}$$

We also have the following similar results:

Lemma 3.9. *Suppose that X, Y are exterior spaces and $f, g\colon X \to Y$ exterior maps.*

(i) If $H\colon X\bar{\times}I \to Y$ is an exterior homotopy from f to g, then $\bar{L}(H) = H|_{\bar{L}(X)\times I}\colon \bar{L}(X\bar{\times}I) = \bar{L}(X)\times I \to \bar{L}(Y)$ is a homotopy from $\bar{L}(f)$ to $\bar{L}(g)$;

(ii) If f is exterior homotopic to g, then $\check{\bar{\pi}}_0(f) = \check{\bar{\pi}}_0(g)$ and $\check{\bar{c}}(f) = \check{\bar{c}}(g)$.

As a consequence of this lemma we have:

Proposition 3.10. *The functors $\bar{L}, \check{\bar{\pi}}_0, \check{\bar{c}}\colon \mathbf{E} \to \mathbf{Top}$ induce functors*

$$\bar{L}\colon \pi\mathbf{E} \to \pi\mathbf{Top}, \quad \check{\bar{\pi}}_0, \check{\bar{c}}\colon \pi\mathbf{E} \to \mathbf{Top}.$$

In Proposition 3.6 a left adjoint has been constructed for the functor L. Nevertheless in the case of functor $\bar{L}$ we have the following alternative result whose proof is a simple checking:

Proposition 3.11. *Suppose that X and Y are exterior spaces and X satisfies that for every $E \in \varepsilon(X)$, $\overline{E} = X$. Then we have the following canonical injective map*

$$Hom_{\mathbf{E}}(X, Y) \to Hom_{\mathbf{Top}}(X_{\mathbf{t}}, \bar{L}(Y)),$$

where $X_{\mathbf{t}}$ denotes the underlying topological space of X. Therefore, if $\mathbf{E}_{\mathrm{den}}$ denotes the full subcategory of exterior spaces X satisfying that for every $E \in \varepsilon(X)$, $\overline{E} = X$, then $\bar{L}\colon \mathbf{E}_{\mathrm{den}} \to \mathbf{Top}$ is a faithful functor.

3.3 Topological and exterior properties and canonical maps

Let $X = (X, \varepsilon(X))$ be an exterior space and consider $\bar{\varepsilon}(X) = \{\overline{E} \mid E \in \varepsilon(X)\}$.

Definition 3.12. An exterior space $X = (X, \varepsilon(X))$ is said to be *regular at infinity* (respect., *locally compact at infinity*) if for every $E \in \varepsilon(X)$, there exists $E' \in \varepsilon(X)$ such that $\overline{E'} \subset E$ (respect., $\overline{E'}$ is compact and $\overline{E'} \subset E$).

Obviously, locally compact at infinity implies regular at infinity.

Example 3.13. *As an example of a regular at infinity exterior space we can take any Hausdorff locally compact space X provided with its cocompact externology. Now, if K is a compact subset of X and we take the externology given by all the neighborhoods of K in X, then we obtain a new exterior space which is locally compact at infinity. Next, we describe an exterior space which is not regular at infinity: Consider the following planar differential system*

$$\frac{d\alpha}{dt} = f(\alpha, \beta), \quad \frac{d\beta}{dt} = af(\alpha, \beta)$$

where f is a C^1-function such that $f(\alpha, \beta) > 0$, $f(\alpha + 1, \beta) = f(\alpha + 1, \beta + 1) = f(\alpha, \beta + 1)$ and a is an irrational fixed number. This system induces a flow on the torus $X = S^1 \times S^1$, which satisfies that each trajectory is dense in X. Take the externology $\varepsilon(X)$ constituted by those open subsets E such that for any $x \in X$ there is $t_0 \in \mathbb{R}$ such that $t \cdot x \in E$, for all $t \geq t_0$. Then one can check that the exterior space $(X, \varepsilon(X))$ is not regular at infinity.

The proofs of the next two propositions are straightforward and left to the reader.

Proposition 3.14. *If an exterior space $X = (X, \varepsilon(X))$ is regular at infinity, then*

(i) $L(X) = \bar{L}(X)$;

(ii) $\check{\pi}_0(X) = \check{\bar{\pi}}_0(X)$, $e_0L(X) = \bar{e}_0\bar{L}(X)$;

(iii) $\check{c}(X) = \check{\bar{c}}(X)$, $eL(X) = \bar{e}\bar{L}(X)$.

Proposition 3.15. *Suppose that $X = (X, \varepsilon(X))$ is an exterior space.*

(i) If X is locally path-connected, then $\check{\pi}_0(X) = \check{c}(X)$, $e_0L(X) = eL(X)$;

(ii) If X is locally path-connected and regular at infinity, then $\check{\pi}_0(X) = \check{\bar{\pi}}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$, $e_0L(X) = \bar{e}_0\bar{L}(X) = eL(X) = \bar{e}\bar{L}(X)$.

Proposition 3.16. *If an exterior space $X = (X, \varepsilon(X))$ is locally compact at infinity, then $L(X) = \bar{L}(X)$ is compact.*

Proof. Since X is regular at infinity, by Proposition 3.14, $L(X) = \bar{L}(X)$. Take $E_0 \in \varepsilon(X)$ such that $\overline{E_0}$ is compact; then the closed subset $\bar{L}(X)$ satisfies $\bar{L}(X) \subset \overline{E_0}$. Therefore $L(X) = \bar{L}(X)$ is compact. ∎

Theorem 3.17. *Let $X = (X, \varepsilon(X))$ be an exterior space and suppose that X is locally path-connected and locally compact at infinity. Then,*

(i) $L(X) = \bar{L}(X)$ *is compact;*

(ii) $e_0L(X) = \bar{e}_0\bar{L}(X) = eL(X) = \bar{e}\bar{L}(X) = \check{\pi}_0(X) = \check{\bar{\pi}}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$*(any end point is representable by a point of the limit);*

(iii) $\check{\pi}_0(X) = \check{\bar{\pi}}_0(X) = \check{c}(X) = \check{\bar{c}}(X)$ *is a profinite compact space;*

(iv) If $a \in \check{\pi}_0(X) = \check{\check{\pi}}_0(X) = \check{c}(X) = \check{\check{c}}(X)$, *then* $L_a(X) = \bar{L}_a(X) = L_a^0(X) = \bar{L}_a^0(X)$ *is a non-empty continuum.*

Proof. As a consequence of Propositions 3.15 and 3.16, it follows (i), $\check{\pi}_0(X) = \check{\check{\pi}}_0(X) = \check{c}(X) = \check{\check{c}}(X)$ and $e_0L(X) = \bar{e}_0\bar{L}(X) = eL(X) = \bar{e}\bar{L}(X)$.

Now take $E_0 \in \varepsilon(X)$ such that $\overline{E_0}$ is compact. Then $\bar{\varepsilon}'(X) = \{\overline{E} \mid E \in \varepsilon(X), \overline{E} \subset \overline{E_0}\}$ is cofinal and we have $\bar{L}(X) = \cap_{\overline{E} \in \bar{\varepsilon}'(X)} \overline{E}$ and $\check{\check{c}}(X) = \lim_{\overline{E} \in \bar{\varepsilon}'(X)} c(\overline{E})$.

Note that any end a can be represented by $\{F\}_{F \in c(\overline{E}), \overline{E} \in \bar{\varepsilon}'(X)}$. Since F is a non-empty component of $\overline{E} \subset \overline{E_0}$, it follows that F is closed (F is a continuum). We also have that the family of closed subset $\{F\}_{F \in c(\overline{E}), \overline{E} \in \bar{\varepsilon}'(X)}$ satisfies the finite intersection property. Since $\overline{E_0}$ is compact, one has that $L_a(X) = \bigcap_{F \in c(\overline{E}), \overline{E} \in \bar{\varepsilon}'(X)} F$ is a non-empty continuum (see Theorem 6.1.20 in [3]). Therefore this end is representable by points of the limit space. Since the map $e\colon L(X) \to \check{\pi}_0(X)$ is continuous and $L(X)$ is compact it follows that $\check{\pi}_0(X)$ is compact. Moreover, since $\check{\pi}_0(X) \cong \lim \pi_0(E)$ is prodiscrete, taking into account that $\check{\pi}_0(X)$ is compact, we have that $\check{\pi}_0(X)$ is a profinite compact space. ∎

Remark 3.18. *For an ANR exterior space X, under some topological conditions the shape of the limit space is determined by the resolution* $\varepsilon(X)$. *Some applications of shape theory to dynamical systems can be seen in [15, 17].*

4 The category of exterior flows

We are going to consider the exterior space $\mathbb{R} \equiv (\mathbb{R}, \mathbf{r})$, where $\mathbf{r}$ is the following externology:

$$\mathbf{r} = \{U \mid U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (n, +\infty) \subset U\}.$$

Note that a base for $\mathbf{r}$ is given by $\mathcal{B}(\mathbf{r}) = \{(n, +\infty) | n \in \mathbb{N}\}$.

The exterior space $\mathbb{R}$ plays an important role in the definition of *exterior flow* below. Such notion mixes the structures of dynamical system and that of exterior space:

Definition 4.1. Let M be an exterior space, $M_{\mathbf{t}}$ denote the underlying topological space and $M_{\mathbf{d}}$ denote the set M provided with the discrete topology. An *exterior flow* is a continuous flow $\varphi\colon \mathbb{R} \times M_{\mathbf{t}} \to M_{\mathbf{t}}$ such that $\varphi\colon \mathbb{R} \bar{\times} M_{\mathbf{d}} \to M$ is exterior and for any $t \in \mathbb{R}$, the map $F_t\colon M \bar{\times} I \to M$, $F_t(x, s) = \varphi(ts, x)$, $s \in I, x \in M$, is also exterior.

An *exterior flow morphism* of exterior flows $f\colon M \to N$ is a flow morphism such that f is exterior. We will denote by **EF** the category of exterior flows and exterior flow morphisms.

Given an exterior flow $(M, \varphi) \in \mathbf{EF}$, one also has a flow $(M_{\mathbf{t}}, \varphi) \in \mathbf{F}$. This gives a forgetful functor

$$(\cdot)_{\mathbf{t}}\colon \mathbf{EF} \to \mathbf{F}.$$

Now given a continuous flow $X = (X, \varphi)$, an open $N \in \mathbf{t}_X$ is said to be $\mathbf{r}$-*exterior* if for any $x \in X$ there is $T^x \in \mathbf{r}$ such that $T^x \cdot x \subset N$. It is easy to check that the family of $\mathbf{r}$-exterior subsets of X is an externology, denoted by $\varepsilon^{\mathbf{r}}(X)$, which gives an exterior space $X^{\mathbf{r}} = (X, \varepsilon^{\mathbf{r}}(X))$ such that $\varphi\colon \mathbb{R}\bar{\times}X_{\mathbf{d}} \to X^{\mathbf{r}}$ is exterior and $F_t\colon X^{\mathbf{r}}\bar{\times}I \to X^{\mathbf{r}}$, $F_t(x,s) = \varphi(ts,x)$, is also exterior for every $t \in \mathbb{R}$. Therefore $(X^{\mathbf{r}}, \varphi)$ is an exterior flow which is said to be the *exterior flow associated to* X. When there is no possibility of confusion, $(X^{\mathbf{r}}, \varphi)$ will be briefly denoted by $X^{\mathbf{r}}$. Then we have a functor

$$(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{EF}.$$

Note that for a flow (X, φ), if E is an open subset such that $\overline{E}$ is compact, then E is an $\mathbf{r}$-exterior subset if and only if $\overline{E}$ is an "absorbing region" in the sense of Definition 1.4.2 in [1].

The forgetful functor and the given constructions of exterior flows are related as follows:

Proposition 4.2. *The functor* $(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{EF}$ *is left adjoint to the forgetful functor* $(\cdot)_{\mathbf{t}}\colon \mathbf{EF} \to \mathbf{F}$. *Moreover* $(\cdot)_{\mathbf{t}}\,(\cdot)^{\mathbf{r}} = \mathrm{id}$ *and* $\mathbf{F}$ *can be considered as a full subcategory of* $\mathbf{EF}$ *via* $(\cdot)^{\mathbf{r}}$.

Proof. Let X be in $\mathbf{F}$ and M be in $\mathbf{EF}$. If $f\colon X^{\mathbf{r}} \to M$ is a morphism in $\mathbf{EF}$, then it is clear that $f\colon X = (X^{\mathbf{r}})_{\mathbf{t}} \to M_{\mathbf{t}}$ is a morphism in $\mathbf{F}$. Now take $g\colon X \to M_{\mathbf{t}}$ a morphism in $\mathbf{F}$ and $E \in \varepsilon(M)$. Given any $x \in X$ one has $g(x) \in M$ and, taking into account that M is an exterior flow, there exists $T^{g(x)}$ such that $T^{g(x)} \cdot g(x) \subset E$. This implies that $T^{g(x)} \cdot x \subset g^{-1}(E)$. Therefore $g^{-1}(E) \in \varepsilon(X^{\mathbf{r}}) = \varepsilon^{\mathbf{r}}(X)$. ∎

4.1 End Spaces and Limit Spaces of an exterior flow

In section 3 we have defined the end and limit spaces of an exterior space. In particular, since any exterior flow X is an exterior space, we can consider the end spaces $\check{\pi}_0(X)$, $\check{c}(X)$, $\check{\bar{\pi}}_0(X)$, $\check{\bar{c}}(X)$ and the limit spaces $L(X)$, $\bar{L}(X)$. Notice that one has the following properties:

Proposition 4.3. *Suppose that* $X = (X, \varphi)$ *is an exterior flow. Then*

(i) The spaces $L(X), \bar{L}(X)$ *are invariant;*

(ii) There are trivial flows induced on $\check{\pi}_0(X)$, $\check{c}(X)$, $\check{\bar{\pi}}_0(X)$ *and* $\check{\bar{c}}(X)$.

Proof. (i) We have that $L(X) = \cap_{E\in\varepsilon(X)}E$. Note that for any $s \in \mathbb{R}$, $\varphi_s(E) \in \varepsilon(X)$ if and only if $E \in \varepsilon(X)$. Then $\varphi_s(L(X)) = \varphi_s(\cap_{E\in\varepsilon(X)}E) = \cap_{E\in\varepsilon(X)}\varphi_s(E) = \cap_{E\in\varepsilon(X)}E = L(X)$. In a similar way, it can be checked that $\bar{L}(X)$ is also invariant.

(ii) For any $s \in \mathbb{R}$, consider the exterior homotopy $F_s\colon X\bar{\times}I \to X$, $F_s(x,t) = \varphi(ts,x)$, from id_X to φ_s. By Lemma 3.4, it follows that $\mathrm{id} = \check{\pi}_0(\varphi_s)$. Therefore the induced action is trivial. In the other cases the proof is similar using Lemma 3.9. ∎

As an immediate consequence one has functors $L, \check{\pi}_0, \check{c}, \bar{L}, \check{\bar{\pi}}_0, \check{\bar{c}}\colon \mathbf{EF} \to \mathbf{F}$.

Proposition 4.4. *The functors* $L, \check{\pi}_0, \check{c}, \bar{L}, \check{\check{\pi}}_0, \check{\check{c}} \colon \mathbf{EF} \to \mathbf{F}$ *induce functors*

$$L, \bar{L} \colon \pi\mathbf{EF} \to \pi\mathbf{F}, \quad \check{\pi}_0, \check{c}, \check{\check{\pi}}_0, \check{\check{c}} \colon \pi\mathbf{EF} \to \mathbf{F},$$

where the homotopy categories are constructed in a canonical way.

4.2 The end point of a trajectory and the induced decompositions of an exterior flow

For an exterior flow X, one has that each trajectory has an end point given as follows: Given $x \in X$ and $E \in \varepsilon(X)$, there is $T^x \in \mathbf{r}$ such that $T^x \cdot x \subset E$. We can suppose that T^x is path-connected and therefore so is $T^x \cdot x$. This way there is a unique path-component $\chi_0(x, E)$ (respect., component $\chi(x, E)$) of E such that $T^x \cdot x \subset \chi_0(x, E) \subset E$ (respect., $T^x \cdot x \subset \chi(x, E) \subset E$). This gives set maps $\chi_0(\cdot, E) \colon X \to \pi_0(E)$ and $\chi_0 \colon X \to \check{\pi}_0(X)$ (respect., $\chi(\cdot, E) \colon X \to c(E)$ and $\chi \colon X \to \check{c}(X)$) such that the following diagram commutes:

$$\begin{tikzcd} & L(X) \arrow[d] \arrow[ddl, "e_0"'] \arrow[ddr, "e"] & \\ & X \arrow[dl, "\chi_0"] \arrow[dr, "\chi"'] & \\ \check{\pi}_0(X) \arrow[rr, "\theta"] & & \check{c}(X) \end{tikzcd}$$

These maps permit us to divide a flow in simpler subflows.

Definition 4.5. Let X be an exterior flow. We will consider $X^0_a = (\chi_0)^{-1}(a)$, $a \in \check{\pi}_0(X)$ and $X_b = \chi^{-1}(b)$, $b \in \check{c}(X)$. The invariant spaces X^0_a and X_b will be called the χ_0-*basin at* $a \in \check{\pi}_0(X)$ and the χ-*basin at* $b \in \check{c}(X)$, respectively.

The maps χ_0 and χ induce the following partitions of X in simpler subflows

$$X = \bigsqcup_{a \in \check{\pi}_0(X)} X^0_a, \quad X = \bigsqcup_{b \in \check{c}(X)} X_b$$

that will be called respectively, the χ_0-*decomposition* and the χ-*decomposition* of the exterior flow X.

Similarly, given $x \in X$, if $\chi_0(x, E)$ is the path-component of E such that $T^x \cdot x \subset \chi_0(x, E) \subset E$, then we also have that $T^x \cdot x \subset \chi_0(x, E) \subset \bar{\chi}_0(x, \overline{E}) \subset \overline{E}$, where $\bar{\chi}_0(x, \overline{E})$ is the unique path-component of $\overline{E}$ containing $T^x \cdot x$. In the same way as above, we have maps $\bar{\chi}_0(\cdot, \overline{E}) \colon X \to \pi_0(\overline{E})$ and $\bar{\chi}_0 \colon X \to \check{\check{\pi}}_0(X)$. Analogously, we obtain set maps $\bar{\chi}(\cdot, \overline{E}) \colon X \to c(\overline{E})$ and $\bar{\chi} \colon X \to \check{\check{c}}(X)$ such that the following diagram commutes:

$$\begin{tikzcd} & \bar{L}(X) \arrow[d] \arrow[ddl, "\bar{e}_0"'] \arrow[ddr, "\bar{e}"] & \\ & X \arrow[dl, "\bar{\chi}_0"] \arrow[dr, "\bar{\chi}"'] & \\ \check{\check{\pi}}_0(X) \arrow[rr, "\bar{\theta}"] & & \check{\check{c}}(X) \end{tikzcd}$$

Remark 4.6. *It is important to note that the maps $\chi_0, \chi, \bar{\chi}_0, \bar{\chi}$ need not be continuous.*

As in the case above we can consider the corresponding $\bar{\chi}_0$-*basin* and $\bar{\chi}$-*basin*, denoted by $\bar{X}^0_a = (\bar{\chi}_0)^{-1}(a)$, $\bar{X}_a = \bar{\chi}^{-1}(a)$, respectively, and their induced decompositions. We also note that the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "\chi_0"'] \arrow[dr, "\bar{\chi}_0"] \arrow[ddl, "\chi"] \arrow[ddr, "\bar{\chi}"'] & \\
\check{\pi}_0(X) \arrow[rr] \arrow[d] & & \check{\bar{\pi}}_0(X) \arrow[d] \\
\check{c}(X) \arrow[rr] & & \check{\bar{c}}(X)
\end{tikzcd}$$

Definition 4.7. Let X be an exterior flow. An end point $a \in \check{\pi}_0(X)$ is said to be χ_0-*representable* (similarly for $\chi, \bar{\chi}_0, \bar{\chi}$) if there is $x \in X$ such that $\chi_0(x) = a$.

Denote by $\chi_0(X)$ the space of χ_0-representable end points (similarly, $\chi(X)$, $\bar{\chi}_0(X)$, $\bar{\chi}(X)$). Since the χ-decompositions of X are compatible with the e-decompositions of the limit subspace, we have the following commutative diagram of representable end points:

$$\begin{tikzcd}
e_0L(X) \arrow[rr] \arrow[dr] \arrow[dd] & & eL(X) \arrow[dr] \arrow[dd] & \\
& \chi_0(X) \arrow[rr] \arrow[dd] & & \chi(X) \arrow[dd] \\
\bar{e}_0\bar{L}(X) \arrow[rr] \arrow[dr] & & \bar{e}\bar{L}(X) \arrow[dr] & \\
& \bar{\chi}_0(X) \arrow[rr] & & \bar{\chi}(X)
\end{tikzcd}$$

Proposition 4.8. *Let X be an exterior flow. Then*

(i) $\omega(x) \subset \bar{L}_{\bar{\chi}(x)}(X)$, *for any* $x \in X$;

(ii) If $a \in \check{\bar{c}}(X)$ is $\bar{\chi}$-representable (that is, $\bar{X}_a = \bar{\chi}^{-1}(a) \neq \emptyset$) and there exists $x \in \bar{X}_a$ such that $\omega(x) \neq \emptyset$, then a is $\bar{e}$-representable.

Proof. (i) If $y \in \omega(x)$, then $y \in \cap_{T \in \mathbf{r}} \overline{T \cdot x}$. On the other hand, given $E \in \varepsilon(X)$, if $\bar{\chi}(a, \overline{E})$ is the connected component of $\overline{E}$ determined by a, since $\bar{\chi}(x) = a$, there is $T \in \mathbf{r}$ such that $\overline{T \cdot x} \subset \bar{\chi}(a, \overline{E})$. Therefore $y \in \bar{\chi}(a, \overline{E})$ for every $E \in \varepsilon(X)$ and $y \in \bar{L}(X)$. This implies that $\bar{e}(y) = a$ for any $y \in \omega(x)$, and $\omega(x) \subset \bar{L}_{\bar{\chi}(x)}(X)$. (ii) follows from (i). ∎

Proposition 4.9. *Let X be an exterior flow and denote $\gamma^+(x) = \{t \cdot x \mid t \geq 0\}$ for $x \in X$. If $\overline{\gamma^+(x)} \cap \bar{L}(X) \neq \emptyset$, then $\bar{\chi}(x)$ is $\bar{e}$-representable.*

Proof. We note that $\overline{\gamma^+(x)} = \gamma^+(x) \cup \omega(x)$. If $x \in \bar{L}(X)$, then $\bar{\chi}(x) = \bar{e}(x)$, so $\bar{\chi}(x)$ is $\bar{e}$-representable. If $x \notin \bar{L}(X)$, then $\gamma^+(x) \cap \bar{L}(X) = \emptyset$ and $\overline{\gamma^+(x)} \cap \bar{L}(X) = \omega(x)$. Then $\omega(x) \neq \emptyset$ and, by Proposition 4.8 above, $\bar{\chi}(x)$ is $\bar{e}$-representable. ∎

5 End and Limit spaces of a flow via exterior flows

Recall that we have considered the functor:

$$(\cdot)^{\mathbf{r}} \colon \mathbf{F} \to \mathbf{EF}$$

and the functors:

$$L, \check{\pi}_0, \check{c}, \bar{L}, \check{\bar{\pi}}_0, \check{\bar{c}} \colon \mathbf{EF} \to \mathbf{F}.$$

Therefore we can consider the composites:

$$L^{\mathbf{r}} := L(\cdot)^{\mathbf{r}},\ \check{\pi}_0^{\mathbf{r}} := \check{\pi}_0(\cdot)^{\mathbf{r}},\ \check{c}^{\mathbf{r}} := \check{c}(\cdot)^{\mathbf{r}},\ \bar{L}^{\mathbf{r}} := \bar{L}(\cdot)^{\mathbf{r}},\ \check{\bar{\pi}}_0{}^{\mathbf{r}} := \check{\bar{\pi}}_0(\cdot)^{\mathbf{r}},\ \check{\bar{c}}^{\mathbf{r}} := \check{\bar{c}}(\cdot)^{\mathbf{r}}$$

to obtain functors $L^{\mathbf{r}}, \check{\pi}_0^{\mathbf{r}}, \check{c}^{\mathbf{r}}, \bar{L}^{\mathbf{r}}, \check{\bar{\pi}}_0{}^{\mathbf{r}}, \check{\bar{c}}^{\mathbf{r}} \colon \mathbf{F} \to \mathbf{F}$.

In this way, given a flow X, we have the end spaces $\check{\pi}_0^{\mathbf{r}}(X) = \check{\pi}_0(X^{\mathbf{r}})$, $\check{c}^{\mathbf{r}}(X) = \check{c}(X^{\mathbf{r}})$, the limit space $L^{\mathbf{r}}(X) = L(X^{\mathbf{r}})$, the bar-end spaces $\check{\bar{\pi}}_0{}^{\mathbf{r}}(X) = \check{\bar{\pi}}_0(X^{\mathbf{r}})$, $\check{\bar{c}}^{\mathbf{r}}(X) = \check{\bar{c}}(X^{\mathbf{r}})$ and the bar-limit space $\bar{L}^{\mathbf{r}}(X) = \bar{L}(X^{\mathbf{r}})$.

Similarly, using the associated exterior flow $X^{\mathbf{r}}$, we denote

$$X^0_{(\mathbf{r},a)} = (\chi_0)^{-1}(a), \quad a \in \check{\pi}_0^{\mathbf{r}}(X)$$

$$X_{(\mathbf{r},a)} = \chi^{-1}(a), \quad a \in \check{c}^{\mathbf{r}}(X)$$

The maps χ_0, χ induce the following partitions of X in simpler subflows

$$X = \bigsqcup_{a \in \check{\pi}_0^{\mathbf{r}}(X)} X^0_{(\mathbf{r},a)}, \quad X = \bigsqcup_{a \in \check{c}^{\mathbf{r}}(X)} X_{(\mathbf{r},a)}$$

that will be called respectively, the χ_0-decomposition and the χ-decomposition of the flow X. Now take

$$\bar{X}^0_{(\mathbf{r},a)} = (\bar{\chi}_0)^{-1}(a), \quad a \in \check{\bar{\pi}}_0{}^{\mathbf{r}}(X)$$

$$\bar{X}_{(\mathbf{r},a)} = \bar{\chi}^{-1}(a), \quad a \in \check{\bar{c}}^{\mathbf{r}}(X)$$

the maps $\bar{\chi}_0, \bar{\chi}$ induce the $\bar{\chi}_0$-decomposition and the $\bar{\chi}$-decomposition of the flow X :

$$X = \bigsqcup_{a \in \check{\bar{\pi}}_0{}^{\mathbf{r}}(X)} \bar{X}^0_{(\mathbf{r},a)}, \quad X = \bigsqcup_{a \in \check{\bar{c}}^{\mathbf{r}}(X)} \bar{X}_{(\mathbf{r},a)}.$$

It is interesting to consider the following equivalence of categories: Given any flow $\varphi \colon \mathbb{R} \times X \to X$, one can consider the *reversed flow* $\varphi' \colon \mathbb{R} \times X \to X$ defined by $\varphi'(r, x) = \varphi(-r, x)$, for every $(r, x) \in \mathbb{R} \times X$. The correspondence, $(X, \varphi) \to (X, \varphi')$, gives rise to a functor

$$(\cdot)' \colon \mathbf{F} \to \mathbf{F}$$

which is an equivalence of categories and verifies $(\cdot)'(\cdot)' = \mathrm{id}$. Using the composites

$$L^{\mathbf{l}} := (\cdot)' L^{\mathbf{r}}(\cdot)',\ \check{\pi}_0^{\mathbf{l}} := (\cdot)' \check{\pi}_0^{\mathbf{r}}(\cdot)',\ \check{c}^{\mathbf{l}} := (\cdot)' \check{c}^{\mathbf{r}}(\cdot)'$$

$$\bar{L}^{\mathbf{1}} := (\cdot)'\bar{L}^{\mathbf{r}}(\cdot)', \ \check{\bar{\pi}}_0{}^{\mathbf{1}} := (\cdot)'\check{\bar{\pi}}_0{}^{\mathbf{r}}(\cdot)', \ \check{\bar{c}}^{\mathbf{1}} := (\cdot)'\check{\bar{c}}^{\mathbf{r}}(\cdot)'$$

we obtain new functors $L^{\mathbf{1}}, \check{\pi}_0^{\mathbf{1}}, \check{c}^{\mathbf{1}}, \bar{L}^{\mathbf{1}}, \check{\bar{\pi}}_0{}^{\mathbf{1}}, \check{\bar{c}}^{\mathbf{1}} : \mathbf{F} \to \mathbf{F}$ and the decompositions

$$X = \bigsqcup_{a \in \check{\pi}_0^{\mathbf{1}}(X)} X^0_{(\mathbf{1},a)}, \quad X = \bigsqcup_{a \in \check{c}^{\mathbf{1}}(X)} X_{(\mathbf{1},a)}$$

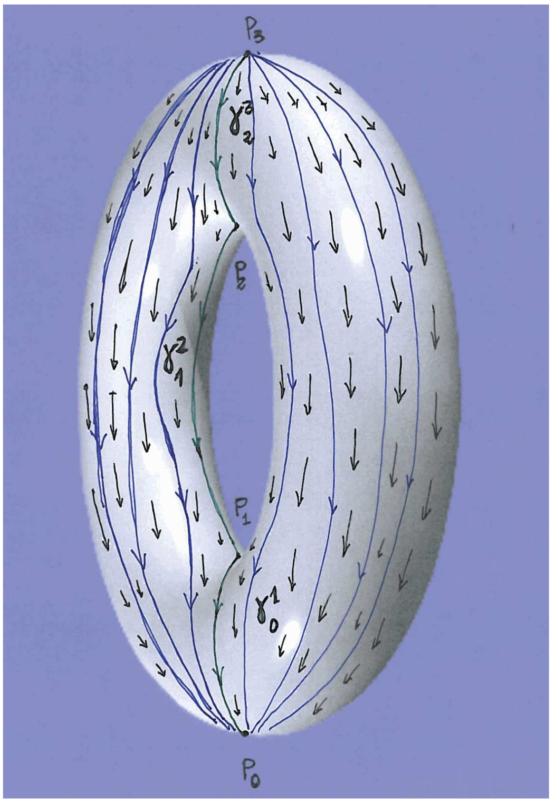
$$X = \bigsqcup_{a \in \check{\bar{\pi}}_0{}^{\mathbf{1}}(X)} \bar{X}^0_{(\mathbf{1},a)}, \quad X = \bigsqcup_{a \in \check{\bar{c}}^{\mathbf{1}}(X)} \bar{X}_{(\mathbf{1},a)}.$$

Remark 5.1. *All decompositions above can be considered as generalizations for a continuous flow of the disjoint union of "stable" (or "unstable" for the dual case) submanifolds of a differentiable flow(see [14], [15], [16]).*

We note that the decompositions of a flow X are compatible with decompositions of limit subspaces.

For a Morse function [13] $f\colon M \to \mathbb{R}$, where M is a compact T_2 Riemannian manifold, one has that the opposite of the gradient of the f induces a flow with a finite number of critical points. In this case, we have that M is locally path-connected and the flow is locally compact at infinity. Then we have all the properties obtained by Theorem 3.17. For instance we can take the height function of a 2-torus:

Example 5.2. *Let $\varphi\colon \mathbb{R} \times (S^1 \times S^1) \to S^1 \times S^1$ be the flow induced by the opposite of the gradient of the height function with four critical points:*

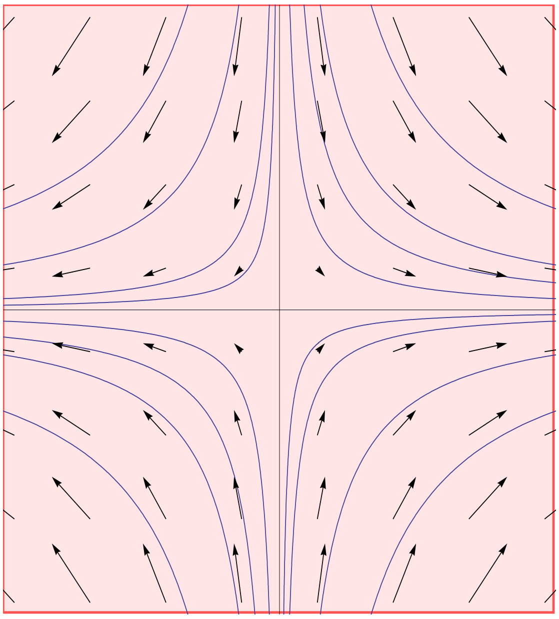


In this example, the limit space, end space and decomposition of the flow and the reverse flow are given in the following table:

$L^{\mathbf{r}} = \{P_0, P_1, P_2, P_3\}$	$\check{\pi}_0^{\mathbf{r}} = \{P_0, P_1, P_2, P_3\}$
$L^{\mathbf{r}}_{P_i} = \{P_i\}$	
$X_{(\mathbf{r},P_3)} = \{P_3\}$	$X_{(\mathbf{r},P_2)} = \{P_2\} \cup \gamma_2^3 \cup \tilde{\gamma}_2^3$
$X_{(\mathbf{r},P_1)} = \{P_1\} \cup \gamma_1^2 \cup \tilde{\gamma}_1^2$	$X_{(\mathbf{r},P_0)} = (S^1 \times S^1) \setminus \bigcup_{i=1}^3 X_{(\mathbf{r},P_i)}$
$L^{\mathbf{l}} = \{P_0, P_1, P_2, P_3\}$	$\check{\pi}_0^{\mathbf{l}} = \{P_0, P_1, P_2, P_3\}$
$L^{\mathbf{l}}_{P_i} = \{P_i\}$	
$X_{(\mathbf{l},P_0)} = \{P_0\}$	$X_{(\mathbf{l},P_1)} = \{P_1\} \cup \gamma_0^1 \cup \tilde{\gamma}_0^1$
$X_{(\mathbf{l},P_2)} = \{P_2\} \cup \gamma_1^2 \cup \tilde{\gamma}_1^2$	$X_{(\mathbf{l},P_3)} = (S^1 \times S^1) \setminus \bigcup_{i=0}^2 X_{(\mathbf{l},P_i)}$

Now we consider a flow induced by a linear differential equation on $\mathbb{R}^2$ that also induces a new flow on the Alexandrov one-point compactification $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Example 5.3. *Consider on S^2 the flow induced by $\varphi(t,(u_1,u_2)) = (e^{t\lambda_1}u_1, e^{t\lambda_2}u_2)$, $u_1, u_2 \in \mathbb{R}$, $\varphi(t,\infty) = \infty$, $(\lambda_1 > 0, \lambda_2 < 0)$*



The limit spaces, end spaces and decomposition of the flow (S^2, ϕ) (as well as the reverse flow) are given in the following table:

$L^{\mathbf{r}} = \{0, \infty\}$	$\check{\pi}_0^{\mathbf{r}} = \{0, \infty\}$
$L^{\mathbf{r}}_0 = \{0\}$	$L^{\mathbf{r}}_\infty = \{\infty\}$
$X_{(\mathbf{r},0)} = \{0\} \times \mathbb{R}$	$X_{(\mathbf{r},\infty)} = ((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cup \{\infty\}$
$L^{\mathbf{l}} = \{0, \infty\}$	$\check{\pi}_0^{\mathbf{l}} = \{0, \infty\}$
$L^{\mathbf{l}}_0 = \{0\}$	$L^{\mathbf{l}}_\infty = \{\infty\}$
$X_{(\mathbf{l},0)} = \mathbb{R} \times \{0\}$	$X_{(\mathbf{l},\infty)} = (\mathbb{R} \times (\mathbb{R} \setminus \{0\})) \cup \{\infty\}$

6 Relations between limit and end spaces of a flow and its dynamical properties

6.1 Periodic points

The relation of the limit space of a flow or an exterior flow and the subflow of periodic points is analyzed in the following results:

Lemma 6.1. *If X is an exterior flow, then $P(X) \subset L(X)$. In particular, if X is a flow, then $P(X) \subset L^{\mathbf{r}}(X)$.*

Proof. Take x a periodic point and $E \in \varepsilon(X)$ arbitrary. Then there exists $T \in \mathbf{r}$ such that $T \cdot x \subset E$. Since x is periodic, $T \cdot x = \mathbb{R} \cdot x$ and taking into account that $x \in \mathbb{R} \cdot x$, we have that that $x \in E$. ∎

Lemma 6.2. *Let X be a flow and suppose that X is a T_1-space. Then, for every $x \in X$ the following statements are equivalent:*

(i) *x is a non-periodic point;*

(ii) *$X \setminus \{x\}$ is an $\mathbf{r}$-exterior subset of X.*

Proof. In order to prove (i) implies (ii) take $y \in X$; if the trajectory of y is different of the trajectory of x, then for every $T \in \mathbf{r}$, $T \cdot y \subset X \setminus \{x\}$. If y is in the trajectory of x, considering that x is not periodic, one can find $T \in \mathbf{r}$ such that $T \cdot y \subset X \setminus \{x\}$. Then, one has that $X \setminus \{x\} \in \varepsilon^{\mathbf{r}}(X)$. Conversely, suppose that x is a periodic point. By Lemma 6.1 above $X \setminus \{x\}$ is not $\mathbf{r}$-exterior. ∎

Using these two lemmas we obtain the following result.

Theorem 6.3. *Let X be a flow and suppose that X is a T_1-space. Then*

$$P(X) = L^{\mathbf{r}}(X).$$

Proof. Let $x \in X \setminus P(X)$. Then, by Lemma 6.2, one has that $X \setminus \{x\} \in \varepsilon^{\mathbf{r}}(X)$ and

$$P(X) = X \setminus \Big(\bigcup_{x \notin P(X)} \{x\} \Big) = \bigcap_{x \notin P(X)} X \setminus \{x\} \supset \bigcap_{E \in \varepsilon^{\mathbf{r}}(X)} E = L^{\mathbf{r}}(X).$$

Now the result follows from Lemma 6.1. ∎

Taking into account the theorem above, if X is flow and X is a T_1 space, then we also have that

$$L^{\mathbf{r}}(X) = P(X) \subset \text{Poisson}(X) \subset \Omega(X) \subset X.$$

6.2 Limit spaces and invariant sets

Lemma 6.4. *Given a flow $\varphi\colon \mathbb{R} \times X \to X$ and $A \subset X$ we have that*

$$\mathrm{inv}(A) = \bigcap_{t\in\mathbb{R}} \varphi_t(A).$$

Proof. If $x \in \mathrm{inv}(A)$, then $\mathbb{R} \cdot x \subset A$. Notice that $x = \varphi_t(\varphi_{-t}(x)) \in \varphi_t(A)$ so $x \in \bigcap_{t\in\mathbb{R}} \varphi_t(A)$. Conversely, if $x \in \bigcap_{t\in\mathbb{R}} \varphi_t(A)$, then for any $t \in \mathbb{R}$ there is $a_t \in A$ such that $x = \varphi_t(a_t)$. This implies that $\varphi_{-t}(x) = a_t \in A$ for all t, and therefore $x \in \mathrm{inv}(A)$. ∎

Then, by Proposition 4.3, we obtain:

Proposition 6.5. *If X is an exterior flow, then*

(i) $L(X) = \lim_{E\in\varepsilon(X)} E = \lim_{E\in\varepsilon(X)} \mathrm{inv}(E)$;

(ii) $\bar{L}(X) = \lim_{E\in\varepsilon(X)} \overline{E} = \lim_{E\in\varepsilon(X)} \mathrm{inv}(\overline{E})$.

Next we give a characterization of the points lying in the difference $\bar{L}(X) \setminus L(X)$. Its proof is routine and left to the reader. Here Fr(.) denotes the frontier (or boundary) operator.

Proposition 6.6. *Let X be an exterior flow and $x \in \bar{L}(X)$. Then $x \in \bar{L}(X) \setminus L(X)$ if and only if there exist $E \in \varepsilon(X)$ and $t \in \mathbb{R}$ such that $t \cdot x \in \mathrm{Fr}(E)$.*

Note that propositions above can be applied to $L^{\mathbf{r}}(X)$ and $\bar{L}^{\mathbf{r}}(X)$ for a continuous flow X.

6.3 Limits and Ω-limits

In the following result, we analyse the relationship between the Ω-limit and the bar-limit induced by an externology.

Lemma 6.7. *If X is an exterior flow, then*

$$\Omega(X) \subset \bar{L}(X)$$

Proof. If $E \in \varepsilon(X)$, then for every $x \in X$ there exists $T \in \mathbf{r}$ such that $T \cdot x \subset E$ and therefore $\overline{T \cdot x} \subset \overline{E}$. By definition this implies that $\omega(x) \subset \bar{L}(X)$ for every $x \in X$. Hence $\Omega(X) \subset \bar{L}(X)$. ∎

And now some technical results.

Proposition 6.8. *Let X be an exterior flow, $\varepsilon(X)$ its externology and $x \in X$. Then there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x \in \varepsilon(X)$ if and only if $x \notin \bar{L}(X)$.*

Proof. If $X \setminus \overline{V}_x \in \varepsilon(X)$, then, taking into account that $V_x \cap (X \setminus \overline{V}_x) = \emptyset$, we have that $x \notin \overline{X \setminus \overline{V}_x}$. Since $X \setminus \overline{V}_x \in \varepsilon(X)$, it follows that $x \notin \bigcap_{E \in \varepsilon(X)} \overline{E} = \bar{L}(X)$.

Conversely, if $x \notin \bar{L}(X)$, then there exists $E \in \varepsilon(X)$ such that $x \notin \overline{E}$. If Int(.) denotes the the interior operator, then taking $V_x = X \setminus \overline{E} = \mathrm{Int}(X \setminus E)$ we have that $X \setminus \overline{V}_x = \mathrm{Int}(X \setminus V_x) = \mathrm{Int}(\overline{E}) \supset E$. Consequently, $X \setminus \overline{V}_x \in \varepsilon(X)$. ∎

Corollary 6.9. *Let X be an exterior flow, $\varepsilon(X)$ its externology and $x \in X$. If there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x \in \varepsilon(X)$, then $x \notin \overline{\Omega(X)}$.*

Proof. It is a consequence of Proposition 6.8 and Lemma 6.7. ∎

Corollary 6.10. *Let X be a flow and $x \in X$. If there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x$ is $\mathbf{r}$-exterior, then $x \notin \overline{\Omega(X)}$.*

Lemma 6.11. *Let X be a flow and X is a locally compact regular space. If $x \notin \overline{\Omega(X)}$, then there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x$ is $\mathbf{r}$-exterior.*

Proof. Suppose that $x \notin \overline{\Omega(X)}$. Since X is locally compact, there is a compact neighborhood K at x such that $K \cap \Omega(X) = \emptyset$. Take $y \in X$ and assume that for every $T \in \mathbf{r}$, $T \cdot y \cap K \neq \emptyset$. Then there is a sequence $t_n \to +\infty$ such that $t_n \cdot y \in K$. Being K compact, one can take a subsequence $t_{n_i} \to +\infty$ such that $t_{n_i} \cdot y \to u \in K$. This fact implies that $u \in K \cap \omega(y) \subset K \cap \Omega(X)$, which is a contradiction. Therefore, there is T such that $T \cdot y \cap K = \emptyset$. By the regularity of X there exists $V_x \in (\mathbf{t}_X)_x$ such that $\overline{V}_x \subset K$ and $X \setminus \overline{V}_x$ is $\mathbf{r}$-exterior. ∎

Corollary 6.12. *Let X be a flow. If X is a locally compact regular space, then $\bar{L}^{\mathbf{r}}(X) \subset \overline{\Omega(X)}$.*

Proof. If $x \notin \overline{\Omega(X)}$, by the lemma above, there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x$ is $\mathbf{r}$-exterior. By Proposition 6.8, it follows that $x \notin \bar{L}^{\mathbf{r}}(X)$. ∎

By the corollary above and Lemma 6.7, we obtain the following result.

Theorem 6.13. *Let X be a flow. If X is a locally compact regular space, then $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega(X)}$.*

Corollary 6.14. *Let X be a flow. If X is a locally compact T_3 space, then*

$$L^{\mathbf{r}}(X) = P(X) \subset \mathrm{Poisson}(X) \subset \Omega(X) \subset \overline{\Omega(X)} = \bar{L}^{\mathbf{r}}(X).$$

Proof. This is a consequence of Theorems 6.3 and 6.13. ∎

References

[1] V. AFRAIMOVICH, S. B. HSU. *Lectures on Chaotic Dynamical Systems*, AMS/IP studies in advanced mathematics, vol. 28, 2003.

[2] A. DEL RÍO, L.J. HERNÁNDEZ AND M.T. RIVAS RODRÍGUEZ. *S-Types of global towers of spaces an exterior spaces*, Appl. Cat. Struct. **17** no. 3, 287-301, (2009).

[3] R. ENGELKING. *General topology*, Sigma Series in Pure Mathematics, vol 6, Heldelmann Verlag Berlin, 1989.

[4] J.I. EXTREMIANA, L.J. HERNÁNDEZ AND M.T. RIVAS. *Sobre el infinito de los espacios*, P. of the workshop on proper homotopy theory, Univ La Rioja, 23-69, 1993.

[5] J.I. EXTREMIANA, L.J. HERNÁNDEZ AND M.T. RIVAS. *Postnikov factorizations at infinity*, Top. and its Appl., **153**, 370-393, (2005).

[6] H. FREUDENTHAL. *Über die Enden topologisher Räume und Gruppen*, Math. Zeith., **53**, 692-713, (1931).

[7] J.M. GARCÍA CALCINES, M. GARCÍA PINILLOS AND L.J. HERNÁNDEZ. *A closed simplicial model category for proper homotopy and shape theories*, Bull Aus. Math. Soc. **57** no. 2, 221–242, (1998).

[8] J.M. GARCÍA CALCINES AND L.J. HERNÁNDEZ. *Sequential homology*, Top. and its Appl. **114** /2 , 201-225, (2001).

[9] J.M. GARCÍA CALCINES, M. GARCÍA PINILLOS AND L.J. HERNÁNDEZ. *Closed simplicial model structures for exterior and proper homotopy*, Appl. Cat. Struct. **12**, no. 3, 225–243, (2004).

[10] M. GARCÍA PINILLOS, L.J. HERNÁNDEZ PARICIO AND M.T. RIVAS RODRÍGUEZ. *Exact sequences and closed model categories*, Appl. Cat. Struct, **18**, no. 4, 343–375, (2010).

[11] L.J. HERNÁNDEZ. *Application of simplicial M-sets to proper homotopy and strong shape theories*, Transactions of the AMS, **347**, no. 2, 363-409, (1995).

[12] B. KERÉKJÁRTÓ, *Vorlesungen uber Topologie*, vol. 1, Springer-Verlag, 1923.

[13] J. MILNOR. *Morse Theory*, Princeton University Press, 1963.

[14] J. W. ROBBIN AND D. SALAMON. *Dynamical systems and shape theory*, Ergod. Th. and Dynam. Sys., 8 (Charles Conley Memorial Volume), 375-393, (1988).

[15] J. M. SANJURJO. *Morse equations and unstable manifolds of isolated invariant sets*, Nonlinearity **16** 1435-1448, (2003).

[16] S. SMALE. *Differentiable dynamical systems*, Bull. Amer. Math. Soc., 1967.

[17] J.J. SÁNCHEZ GABITES. *Dynamical systems and shapes*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A: Matemáticas (RACSAM), **102**, no. 1, 127-160, (2008).

[18] T.PORTER. *Proper homotopy theory*, Handbook of Algebraic Topology, Chapter 3, pp. 127-167, 1995.

Departamento de Matemática Fundamental
Universidad de La Laguna
38271 LA LAGUNA.
email:jmgarcal@ull.es

Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 LOGROÑO.
email:luis-javier.hernandez@unirioja.es, maria-teresa.rivas@unirioja.es